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A short proof of Eilenberg and Moore's theorem*

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Abstract: In this paper we give a short and simple proof the following theorem of S. Eilenberg and J.C. Moore: the only injective object in the category of groups is the trivial group.

Keywords: Group, injective, fundamental group, covering space
MSC (2000): 20J15, 14H30

First we recall one necessary definition.
Definition. An object I in a category $\mathfrak{C}$ is called injective if for any monomorphism $K \to L$, and any map $K \to I$, there is a map $L \to I$ such that the diagram

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d] & L \arrow[dl, dashed] \\ & I & \end{tikzcd}$$

commutes.

Theorem (S. Eilenberg, J. C. Moore)**.** The only injective object in the category of groups is the trivial group.

The reader can find the original proof in [1]. A different proof is due to Fred Cohen but it was never published. The proof presented in this paper is shorter and, in our view, easier than the ones mentioned above. We will need the following lemma. It follows from

* This work first appeared as a part of the author's Ph.D. dissertation [3]
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a classical proof of the fact that the free group on two letters contains the free group on countably many letters [2]. The point of this lemma is to exhibit a specific injection which makes the proof of the theorem an easy calculation.

Lemma. Let $F[a, b]$ denote the free group on letters a and b. Then the group homomorphism

$$F[a, b] \to F[c, d]$$

given by

$$a \mapsto c$$
$$b \mapsto dcd^{-1}$$

is an injection.

Proof (of Lemma)**.** Consider the covering space of a bouquet of two circles shown in Figure 1 (each point of degree 4 in the covering space is mapped into the basepoint of the base space; each loop is mapped onto loop **c** in the bouquet; each vertical segment between two loops is mapped onto loop **d**).

Classes of loops **a** and **b** shown in Figure 2 generate a subgroup $F[a, b]$ of the fundamental group of the covering space, which is free since π_1 of a covering projection is injective, and π_1 of the codomain is the free group on two generators. Consequently, $F[a, b]$ is free; moreover, the injection $i : F[a, b] \to F[c, d]$ is given by

$$i(a) = c,$$
$$i(b) = dcd^{-1}.$$

□

Remark. The covering space is homotopy equivalent to a bouquet of countably many circles, and hence its fundamental group is isomorphic to the free group on countably many generators.

Proof (of Theorem)**.** Suppose group G in injective, and let $x \in G$ be any element. Let the homomorphism $f : F[a, b] \to G$ be given by

$$f(a) = 1$$
$$f(b) = x$$

and let $i : F[a, b] \to F[c, d]$ be as in the Lemma. Then there exists a homomorphism $g : F[c, d] \to G$ such that the diagram

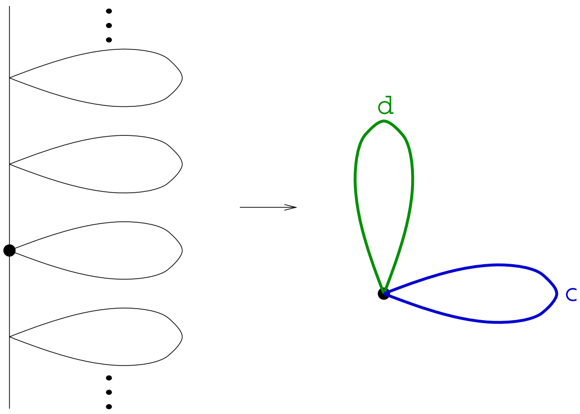

Fig. 1 Covering space of a bouquet of two circles

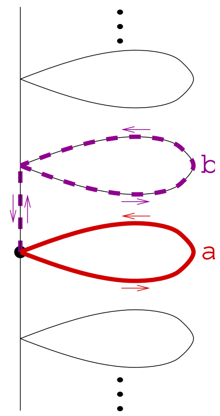

Fig. 2 Two loops whose classes generate subgroup $F[a,b]$ of the fundamental group of the covering space

$$\begin{tikzcd}
1 \arrow[r] & F[a,b] \arrow[r, "i"] \arrow[d, "f"'] & F[c,d] \arrow[dl, dashed, "g"] \\
 & G &
\end{tikzcd}$$

commutes. Then we have

$$g(c) = g(i(a)) = f(a) = 1,$$

and

$$x = f(b) = g(i(b)) = g(dcd^{-1}) = g(d)g(c)g(d^{-1}) = g(d)1(g(d))^{-1} = 1,$$

i.e. any element of G is the identity element. $\square$

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