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A SHORT PROOF OF SCHOENBERG'S THEOREM

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ABSTRACT. Using positive semidefiniteness of Laplace transforms, we give a short and simple proof of Schoenberg's theorem characterising radially symmetric positive semidefinite functions on a Hilbert space. A slight generalisation of this theorem is also given.

In his paper *Metric spaces and completely monotone functions* [3], I. J. Schoenberg raises the question about the connection between the class of Fourier transforms of (finite, nonnegative) measures in Euclidean spaces and the class of Laplace transforms of (finite, nonnegative) measures on the half-line $\mathbf{R}_+ = [0, \infty)$. He states: "In spite of the entirely different analytical character of these two classes, a certain kinship was to be expected for the following two reasons:

1. In both classes the defining kernel is the exponential function.

2. The less formal reason of the similarity of the closure properties of both classes, for both classes are convex, i.e. $a_1 f_1 + a_2 f_2$ $(a_1 \geqslant 0, a_2 \geqslant 0)$ belongs to the class if f_1 and f_2 belong to it, multiplicative, i.e., also $f_1 \cdot f_2$ belongs to the class, and finally closed with respect to ordinary convergence to a continuous limit function." The answer Schoenberg could give to the above question was the remarkable result that to each continuous function f: $\mathbf{R}_+ \to \mathbf{C}$ with the property that $f \circ |\cdot|_n$ is positive semidefinite on $\mathbf{R}^n$ for all n ($|\cdot|_n$ denoting the Euclidean norm) there exists a finite nonnegative measure on $\mathbf{R}_+$ with Laplace transform $f(\sqrt{t})$ [3, Theorem 2]. Positive semidefiniteness of a mapping g: $\mathbf{R}^n \to \mathbf{C}$ has the meaning that the kernel $K(x,y) = g(x - y)$ is positive semidefinite. The proof of this theorem, even that given in the more recent book of Donoghue [1, pp. 201–206], however is rather complicated and technical in nature. But there is a further common feature of Fourier and Laplace transforms seemingly unknown until quite recently: Laplace transforms, too, are characterised essentially by positive semidefiniteness. More precisely, a function f: $\mathbf{R}_+^p \to \mathbf{C}$ is the Laplace transform of a finite nonnegative measure on the Borel sets of $\mathbf{R}_+^p$ if and only if f is continuous, bounded and positive semidefinite in the sense that $\sum_{i=1}^k \sum_{j=1}^k \alpha_i \alpha_j f(t_i + t_j) \geqslant 0$ for all $(\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k$, $(t_1, \ldots, t_k) \in (\mathbf{R}_+^p)^k$, and $k \in N$ [2, Satz 1]. Using this we give a new proof of

THEOREM 1 (SCHOENBERG). *A continuous function f: $\mathbf{R}_+ \to \mathbf{C}$ has the property that $f \circ |\cdot|_n$ is positive semidefinite on $\mathbf{R}^n$ for all $n \in N$ if and only if there exists a finite nonnegative measure μ on $\mathbf{R}_+$ such that*

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$$f(t) = \int_0^\infty \exp(-t^2\lambda)d\mu(\lambda), \quad \textit{for all } t \in \mathbf{R}_+.$$

PROOF. One direction, of course, is easy. Hence, let us assume that $f \circ |\cdot|_n$ are positive semidefinite for all n. In view of the above all we have to show is that $f(\sqrt{t})$ is positive semidefinite on $\mathbf{R}_+, f$ being bounded by $f(0)$. Let $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$ and $t_1, \ldots, t_k \in \mathbf{R}_+$ be given. Fix any $n \in N$ and let $e_1, e_2, \ldots, e_{kn}$ be the standard orthonormal basis vectors in $\mathbf{R}^{kn}$. Put $x_{im} := \sqrt{t_i}\, e_{(i-1)n+m}$, $1 \leqslant i \leqslant k$, $1 \leqslant m \leqslant n$, and $\beta_{im} := \alpha_i/n$. Then

$$\begin{aligned} 0 &\leqslant \sum_{i,m} \sum_{i',m'} \beta_{im}\beta_{i'm'} f(|x_{im} - x_{i'm'}|_{kn}) \\ &= \frac{1}{n^2}\left[(n^2 - n)\sum_{i=1}^k \alpha_i^2 f(\sqrt{2t_i}) + nf(0)\sum_{i=1}^k \alpha_i^2 + n^2 \sum_{i \neq j} \alpha_i\alpha_j f(\sqrt{t_i + t_j})\right] \\ &= \sum_{i=1}^k \sum_{j=1}^k \alpha_i\alpha_j f(\sqrt{t_i + t_j}) + \frac{1}{n}\left[f(0)\sum_{i=1}^k \alpha_i^2 - \sum_{i=1}^k \alpha_i^2 f(\sqrt{2t_i})\right]. \end{aligned}$$

If n now tends to infinity, this finishes the proof.

It should perhaps be remarked that the only deeper result involved into the proof of Satz 1 in [2] is the solution of Hausdorff's moment problem. Our proof being simple in nature, we are able to give a generalization of Theorem 1.

Observe that $f \circ |\cdot|_n$ is positive semidefinite for all n if and only if $f(\|x\|)$ is positive semidefinite on a real (infinite dimensional) Hilbert space H. (In this way Schoenberg formulated his result.) Let now $H_1, \ldots, H_p$ be a finite sequence of real Hilbert spaces. Then we can state

THEOREM 2. *A continuous function $f: \mathbf{R}_+^p \to \mathbf{C}$ has the property that $f(\|x^1\|, \ldots, \|x^p\|)$ is positive semidefinite on $H_1 \times \cdots \times H_p$ if and only if there exists a finite nonnegative measure μ on $\mathbf{R}_+^p$ such that*

$$f(t_1, \ldots, t_p) = \int_{\mathbf{R}_+^p} \exp\left(-\sum_{i=1}^p t_i^2\lambda_i\right) d\mu(\lambda_1, \ldots, \lambda_p)$$

for all $t = (t_1, \ldots, t_p) \in \mathbf{R}_+^p$.

PROOF. Again one direction is easy. Hence let $f(\|x^1\|, \ldots, \|x^p\|)$ be positive semidefinite on $H_1 \times \cdots \times H_p$. Then f is bounded and we are left with its positive semidefiniteness. Let $\alpha_1, \ldots, \alpha_p \in \mathbf{R}$ and $t^1, \ldots, t^k \in \mathbf{R}_+^p$ be given, $t^j = (t_1^j, \ldots, t_p^j)$, and let $e_1^q, e_2^q, \ldots$ be orthonormal vectors in H_q, $1 \leqslant q \leqslant p$. Put $x_{im}^q := \sqrt{t_q^i}\, e_{(i-1)n+m}$, $1 \leqslant i \leqslant k$, $1 \leqslant m \leqslant n$, $1 \leqslant q \leqslant p$, and $\beta_{im} := \alpha_i/n$. Then

$$\begin{aligned} 0 &\leqslant \sum_{i,m} \sum_{i',m'} \beta_{im}\beta_{i'm'} f(\|x_{im}^1 - x_{i'm'}^1\|, \ldots, \|x_{im}^p - x_{i'm'}^p\|) \\ &= \sum_{i=1}^k \sum_{j=1}^k \alpha_i\alpha_j f\left(\sqrt{t_1^i + t_1^j}, \ldots, \sqrt{t_p^i + t_p^j}\right) \\ &\quad + \frac{1}{n}\left[f(0)\sum_{i=1}^k \alpha_i^2 - \sum_{i=1}^k \alpha_i^2 f\left(\sqrt{2t_1^i}, \ldots, \sqrt{2t_p^i}\right)\right]. \end{aligned}$$

Hence $f(\sqrt{t_1}, \ldots, \sqrt{t_p})$ is bounded, continuous and positive semidefinite and again an appeal to Satz 1 in [2] finishes the proof.

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