A SHORT PROOF OF THE λ_g-CONJECTURE WITHOUT GROMOV-WITTEN THEORY: HURWITZ THEORY AND THE MODULI OF CURVES

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Abstract. We give a short and direct proof of the λ_g-Conjecture. The approach is through the Ekedahl-Lando-Shapiro-Vainshtein theorem, which establishes the "polynomiality" of Hurwitz numbers, from which we pick off the lowest degree terms. The proof is independent of Gromov-Witten theory.

We briefly describe the philosophy behind our general approach to intersection numbers and how it may be extended to other intersection number conjectures.

1. Introduction

1.1. **Background.** The λ_g-Conjecture, now a theorem, states that

Theorem 1.1 (The λ_g-Conjecture). *For $n, g \geq 1$,*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{b_1} \cdots \psi_n^{b_n} \lambda_g = \binom{2g-3+n}{b_1, \dots, b_n} c_g,$$

where $\sum_{i=1}^n b_i = 2g-3+n$, $b_1, \dots, b_n \geq 0$ and c_g is a constant that depends only on g.

As usual, $\overline{\mathcal{M}}_{g,n}$ is the (compact) moduli space of stable n-pointed genus g curves, $\psi_1, \dots, \psi_n$ are (complex) codimension 1 classes corresponding to the n marked points, and λ_k is the (complex codimension k) kth Chern class of the Hodge bundle. The constant c_g can be obtained from the $n = 1$ case, giving $c_g = \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2} \lambda_g = \langle \tau_{2g-2} \lambda_g \rangle_g$, and throughout the paper c_g is used to denote this particular value. For a summary of necessary facts about the moduli space of curves, the reader is referred to [V]. We shall assume background about $\overline{\mathcal{M}}_{g,n}$ in the Introduction, but the proof of the λ_g-Conjecture that is presented does not require any knowledge of these notions.

The λ_g-Conjecture can be interpreted as a description of the top intersections in the tautological cohomology ring of the moduli space $\mathcal{M}_{g,n}^c$ of curves of compact type (curves whose Jacobian is compact, or equivalently, whose dual graph is a tree). As such, it is part of a family of four problems. Pandharipande has outlined a philosophy that we should expect the "tautological cohomology rings" of various moduli spaces to satisfy a "Gorenstein" property, *i.e.* that the top degree term of the ring is one-dimensional, and that the multiplication map into it should be a perfect pairing, see [P, §1]. Three spaces mentioned there are the moduli space of stable curves $\mathcal{M}_{g,n}$, $\mathcal{M}_{g,n}^c$, and the moduli space of smooth curves $\mathcal{M}_g$ (or, better, the moduli space of pointed curves with "rational tails" $\mathcal{M}_{g,n}^{rt}$). In each case, the one-dimensionality is known (see [GV1, FabP2, GV3], for example).

The top intersections in this ring are determined in each case by top intersections of ψ-classes by work of Faber (based on earlier work of Mumford). Then, parallel to Pandharipande's Gorenstein predictions, there are "intersection-number" predictions determining the full ring structure. These are the following: **i)** the case of $\overline{\mathcal{M}}_{g,n}$ is *Witten's Conjecture* (Kontsevich's theorem), which now has a number of very different and very enlightening proofs; **ii)** the case of $\mathcal{M}_{g,n}^c$ is the λ_g-*Conjecture*;

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iii) the case of $\mathcal{M}_g$ (or $\mathcal{M}_{g,n}^{rt}$) is *Faber's intersection number conjecture.* To these we add a fourth case that seems to be of the same flavor: **iv)** the case of a conjectural compactified universal Picard variety over $\overline{\mathcal{M}}_{g,n}$ (related to double Hurwitz numbers, described in [GJV2]) yields a generating series with similar behavior (see [GJV2, SZ]), which we shall discuss more in Section 5.2.

Our proof of the λ_g-Conjecture is through the Ekedahl-Lando-Shapiro-Vainshtein formula [ELSV2], that establishes the "polynomiality" of the Hurwitz numbers, and by identifying the Hodge integral in the λ_g-Conjecture as a coefficient in the lowest degree terms in this polynomial. The proof is short, direct and requires no Gromov-Witten theory. There are already several proofs of the λ_g-Conjecture, and these will be discussed in Section 1.3.

Our method of proof can be extended to give a proof of Faber's intersection number conjecture (for up to 3 points, [GJV3]). Comments on the philosophy behind this are made in Section 5.

1.2. **Preliminaries.**

1.2.1. *The Join-cut Equation.* The Hurwitz numbers H^g_α count connected, branched covers of $\mathbb{P}^1$ by a non-singular genus g curve, with branching over $\infty \in \mathbb{P}^1$ corresponding to a partition $\alpha \vdash d$ (these branch points are ordered), and with simple branching $(1^{d-2}\,2)$ above $r = d+n+2g-2$ other points, where $n = l(\alpha)$, the number of parts in α. Hurwitz [H] observed that $d!H^g_\alpha$ counts the number of factorizations of an arbitrary permutation in the conjugacy class $\mathcal{C}_\alpha$ of $\mathfrak{S}_d$ with cycles of lengths $\alpha_1, \ldots, \alpha_n$, into an ordered, transitive product of r transpositions in $\mathfrak{S}_d$ (such a product is *transitive* if the group generated by the factors acts transitively on $\{1, \ldots, d\}$).

Ordered factorizations are amenable, in principle, to combinatorial techniques. The action of a transposition on the disjoint cycles of a permutation can be analyzed by observing that either the transposition joins an i-cycles and a j-cycle to make an $(i+j)$-cycle, or it cuts an $(i+j)$-cycle into an i-cycle and a j-cycle. In this join-cut process, an i-cycle is annihilated by the operator $i\partial/\partial p_i$ and is created by the operator p_i (regarded as pre-multiplication by p_i) acting on the *genus series*

$$H = \sum_{g\geq 0, n\geq 1} H^g_n x^g,$$

where H^g_n is the *Hurwitz series*, given by

$$H^g_n(z, \mathbf{p}) = \sum_{d\geq 1} \sum_{\substack{\alpha\vdash d,\\ l(\alpha)=n}} |\mathcal{C}_\alpha| \frac{H^g_\alpha}{r!} p_\alpha z^d,$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $p_\alpha = p_{\alpha_1} \cdots p_{\alpha_n}$. It follows immediately from this construction that the genus series satisfies the *Join-cut Equation* (see [GJVai]):

$$(1) \quad \left(z\frac{\partial}{\partial z} + 2x\frac{\partial}{\partial x} - 2 \;+\; \sum_{i\geq 1} p_i \frac{\partial}{\partial p_i}\right) H = \frac{1}{2}\sum_{i,j\geq 1}\left(ijx p_{i+j}\frac{\partial^2 H}{\partial p_i \partial p_j} + ijp_{i+j}\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j} + (i+j)p_i p_j \frac{\partial H}{\partial p_{i+j}}\right),$$

where the first two operators on the right hand side give the cycle-type after a *join* and the third operator gives the cycle-type after a *cut*. Because of transitivity, there are two cases of joins. The first operator is a join of two cycles within a *single* transitive factorization, while the second operator is a join of two cycles, one from each of *two* disjoint transitive ordered factorizations.

1.2.2. *The Genus Expansion Ansatz.* The background to our proof is an observation about Hurwitz numbers H^g_α. For fixed $n = l(\alpha)$ and g, with $n, g \geq 1$ or $n \geq 3, g = 0$, it was conjectured that

$$H^g_\alpha = r! \prod_{i=1}^{n} \left(\frac{\alpha_i^{\alpha_i}}{\alpha_i!} \right) P_{g,n}(\alpha_1, \ldots, \alpha_n), \tag{2}$$

for some symmetric polynomial $P_{g,n}$ in the α_i, with terms of total degrees between $2g - 3 + n$ and $3g - 3 + n$. This important property is essentially the *Polynomiality Conjecture* of [GJ2, Conj. 1.2] (the connection is made in [GJV1]). The Polynomiality Conjecture was settled by Ekedahl, Lando, M. Shapiro, and Vainshtein, who proved the remarkable ELSV-formula [ELSV1, ELSV2]. (For a proof in the context of Gromov-Witten theory, see [GV2], and also [GV3].) In the present notation, the ELSV-formula states that

$$P_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \cdots + (-1)^g \lambda_g}{(1 - \alpha_1 \psi_1) \cdots (1 - \alpha_n \psi_n)}. \tag{3}$$

Equation (3) should be interpreted as follows: formally invert the denominator as a geometric series; select the terms of codimension $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$; and "intersect" these terms on $\overline{\mathcal{M}}_{g,n}$. The formula therefore yields

$$P_{g,n} = \sum_{\substack{b_1 + \cdots + b_n + k = 3g-3+n, \\ b_i \geq 0,\ 0 \leq k \leq g}} (-1)^k \langle \tau_{b_1} \cdots \tau_{b_n} \lambda_k \rangle_g \alpha_1^{b_1} \cdots \alpha_n^{b_n}, \tag{4}$$

where we have used the Witten symbol (from Gromov-Witten theory)

$$\langle \tau_{b_1} \cdots \tau_{b_n} \lambda_k \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{b_1} \cdots \psi_n^{b_n} \lambda_k,$$

and note that

$$\langle \tau_{b_1} \cdots \tau_{b_n} \lambda_k \rangle_g = 0 \tag{5}$$

unless $b_1 + \cdots + b_n = 3g - 3 + n - k$.

Then substituting (4) into (2), we obtain the *Genus Expansion Ansatz* for the Hurwitz series (see Thm. 2.5 of [GJV1] for details), namely

$$H^g_n = \frac{1}{n!} \sum_{\substack{b_1, \ldots, b_n \geq 0, \\ 0 \leq k \leq g}} (-1)^k \langle \tau_{b_1} \cdots \tau_{b_n} \lambda_k \rangle_g \prod_{i=1}^{n} \phi_{b_i}(z, \mathbf{p}) \tag{6}$$

for $g, n \geq 1$ and for $g = 0$, $n \geq 3$, where

$$\phi_i(z, \mathbf{p}) = \sum_{m \geq 1} \frac{m^{m+i}}{m!} p_m z^m, \quad i \geq 0.$$

This should be interpreted as just a re-writing of the ELSV formula.

1.2.3. *Our approach to the λ_g-Conjecture.* The second observation about $P_{g,n}(\alpha)$ (recall that the first is that it is a polynomial) is that its lowest total degree (this is $2g - 3 + n$) part appears to have the form

$$(\alpha_1 + \cdots + \alpha_n)^{2g-3+n} c_g,$$

where c_g is a constant depending only upon g. This assertion is equivalent to the λ_g-Conjecture by (4) and is the form of the result that we prove.

We require only two properties of the Hurwitz series, namely that it satisfies the Join-cut Equation (1) and that it has the Genus Expansion Ansatz (6). To obtain a characterization of the left hand side of Theorem 1.1 in terms of an operator acting on the Hurwitz series H^g_n we transform the latter in a series of three steps:

(i) symmetrization of the Hurwitz series and the Join-cut Equation;
(ii) change of variables to obtain a polynomial; and
(iii) determination of the full (to be defined later) terms of minimum degree in this polynomial.

In Section 2, we apply this transformation to the Genus Expansion Ansatz for the Hurwitz series. In our main result of this section, Theorem 2.1, we prove that each Witten symbol whose evaluation is the subject of the λ_g-Conjecture is the coefficient of a unique monomial in the transformed Hurwitz series. In Section 3, we apply this transformation to the Join-cut Equation (1) for the Hurwitz series. In our main result of this section, Theorem 3.2, we prove that a genus generating series for the transformed Hurwitz series satisfies a simple partial differential equation. We then solve this partial differential equation in Theorem 3.3. Finally, in Section 4, we prove the λ_g-Conjecture by comparing the results obtained in Sections 2 and 3.

We note in passing that the transformations we apply in this paper are also used in [GJV3] in which we are able to prove (up to 3 parts) the Faber intersection number conjecture (see [Fab]). In the latter Faber case, we apply the steps **(i)** and **(ii)** of the transformations applied in the present paper, but for step **(iii)**, in the Faber case, we consider terms of *maximum* degree rather than the minimum degree (on a *different* polynomial). This philosophy will be discussed in Section 5.

In the Appendix, we indicate how our approach can be used to obtain the generating series of intersection numbers that are close to "minimum" in the sense that has been described above, and we exhibit the explicit series in a few cases.

1.3. **Previous proofs of the λ_g-Conjecture.** The λ_g-Conjecture was first proved in Faber and Pandharipande's landmark paper [FabP1]. Their approach was to use localization on the space of stable maps to $\mathbb{P}^1$ to obtain relations among these intersection numbers. They then showed that the λ_g-Conjecture's prediction satisfied these relations. Finally, they proved that the relations uniquely determined the predictions of the λ_g-Conjecture by establishing the invertibility of a large matrix whose entries are counts of various partitions; this requires seven pages of explicit calculation.

A second proof is as follows. Getzler and Pandharipande showed that the λ_g-Conjecture is a formal consequence of the Virasoro Conjecture for $\mathbb{P}^1$ [GeP, Thm. 3], by showing that it satisfies a recursion arising from the Virasoro Conjecture, and then showing that the recursion has a unique solution. The Virasoro Conjecture for $\mathbb{P}^1$ was then shown in two ways. It was proved for all curves by Okounkov and Pandharipande [OP]. Also, Givental has announced a proof of the Virasoro Conjecture for Fano toric varieties [Giv]. The details have not yet appeared, but Y.-P. Lee and Pandharipande are writing a book [LP] supplying them. These proofs of the Virasoro Conjecture in important cases are among the most significant results in Gromov-Witten theory, and this method of proof of the λ_g-Conjecture seems somewhat circuitous. (Much of this paragraph also applies to Faber's intersection number conjecture.)

Liu, Liu, and Zhou gave a new proof in [LLZ2] as a consequence of the Mariño-Vafa formula [MVaf], which was proposed by the physicists Mariño and Vafa and proved by Liu, Liu, and Zhou in [LLZ1]. This Gromov-Witten-theoretic proof is quite compact.

2. Transformation of the Genus Expansion Ansatz

In this section, we transform the Hurwitz series H_n^g through the Genus Expansion Ansatz (6) by constructing the operator to extract the intersection number of Theorem 1.1.

2.1. **Step 1 – Symmetrization.** For the first step of our transformation, we *symmetrize* the Hurwitz series using the linear symmetrization operator Ξ_n, given by

$$\Xi_n \, p_\alpha z^{|\alpha|} = \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)}^{\alpha_1} \cdots x_{\sigma(n)}^{\alpha_n}, \quad n \geq 1,$$

if $l(\alpha) = n$ (with $\alpha = (\alpha_1, \ldots, \alpha_n)$), and 0 otherwise. Thus, applying Ξ_n to (6) we obtain, for $n, g \geq 1$ and $n \geq 3, g \geq 0$,

$$\Xi_n \, H_n^g = \frac{1}{n!} \sum_{\substack{b_1, \ldots, b_n \geq 0, \\ 0 \leq k \leq g}} (-1)^k \langle \tau_{b_1} \cdots \tau_{b_n} \lambda_k \rangle_g \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} \phi_{b_i}(x_{\sigma(i)}), \tag{7}$$

where

$$\phi_i(x) = \phi_i(x, \mathbf{1}) = \sum_{m \geq 1} \frac{m^{m+i}}{m!} x^m.$$

We note that

$$\phi_i(x) = \left(x \frac{d}{dx} \right)^{i+1} w(x), \tag{8}$$

where

$$w(x) = \sum_{m \geq 1} m^{m-1} \frac{x^m}{m!}$$

is the (exponential) generating series for the number m^{m-1} of trees with m vertices, labelled from 1 to m, and having a single vertex which is further distinguished (for example, by painting it red). Such trees are termed vertex-labelled rooted trees, and we shall refer to $w(x)$ as the *rooted tree series*. It is the unique formal power series solution of the (transcendental) functional equation (see *e.g.* [GJ] §3.3.10)

$$w = xe^w \tag{9}$$

(which we shall refer to as the *rooted tree equation*).

2.2. **Step 2 – change of variables.** We next consider a change of variables for the symmetrized Hurwitz series. Consider $y(x) = (1 - w(x))^{-1}$. Then

$$y(x) = 1 + \sum_{m \geq 1} \frac{m^m}{m!} x^m = 1 + \phi_0(x), \tag{10}$$

which can be seen most easily perhaps from (11) below. Let $w_j = w(x_j)$ and $y_j = y(x_j)$, $j = 1, \ldots, n$, and let C be an operator, applied to a formal power series in $x_1, \ldots, x_n$, that changes variables, from the indeterminates $x_1, \ldots, x_n$ to $y_1, \ldots, y_n$. Thus, from (10), to carry out C we substitute $x_j = g(y_j - 1)$, where g is the compositional inverse of ϕ_0. In general, this will not yield a formal power series in $y_1, \ldots, y_n$, but when we apply C to $\Xi_n \, H_n^g$, we do obtain a formal power series (in fact, for each fixed n, g it is a polynomial) as we prove below.

First we prove some properties of C. Differentiating the rooted tree equation (9), we obtain the operator identity

$$x_j \frac{d}{dx_j} = \frac{w_j}{1 - w_j} \frac{d}{dw_j}. \tag{11}$$

But $dy_j = y_j^2 dw_j$, so we have the operator identities

$$\mathsf{C} \, \frac{x_j \partial}{\partial x_j} = (y_j^3 - y_j^2) \frac{\partial}{\partial y_j} \, \mathsf{C}, \qquad \mathsf{C} \, w_j \frac{\partial}{\partial w_j} = (y_j^2 - y_j) \frac{\partial}{\partial y_j} \, \mathsf{C}, \tag{12}$$

where when we apply C to expressions involving w_j, we interpret w_j as $w(x_j)$. From (8), (11) and (12), we also obtain

$$\mathsf{C}\,\phi_i(x_j)\left((y_j^3-y_j^2)\frac{\partial}{\partial y_j}\right)^i(y_j-1),\qquad \text{for } i\geq 0. \tag{13}$$

Now (5), (7) and (13) together enable us to obtain a polynomial expression for $\mathsf{C}\,\Xi_n\,H_n^g$. The fact that this is unique, and hence that the application of C to $\Xi_n\,H_n^g$ is well-defined for formal power series, follows immediately from the fact that the non-negative powers of the rooted tree series $w(x)$ are linearly independent, as formal power series in x.

2.3. **Step 3 – full terms of minimum total degree.** The final step in the transformation of the Hurwitz series is to identify a particular subset of terms. We say that a monomial $y_1^{i_1}\cdots y_n^{i_n}$ is *full* if $i_1,\ldots,i_n\geq 1$. Let $\mathsf{F}_k\,f$ be the subseries of a series f in $y_1,\ldots,y_n$ consisting of the full terms of total degree k. Thus, for example, from (13), we immediately obtain

$$\mathsf{F}_{i+1}\,\mathsf{C}\,\phi_i(x_j)=(-1)^i i!\,y_j^{i+1}, \tag{14}$$

by induction on $i\geq 0$, and

$$\mathsf{F}_k\,\mathsf{C}\,\phi_i(x_j)=0,\quad i\geq 0,\ \ k<i+1. \tag{15}$$

In addition, when applied to $\mathsf{C}\,\Xi_n\,H_n^g$, let M denote $\mathsf{F}_{2g-3+2n}$.

Let m_β denote the monomial symmetric function, where we allow 0 parts in β, and write $\beta\vdash_0 d$ to indicate that β, with parts equal to 0 allowed, is a partition of d. As usual, $l(\beta)$ is the number of parts of β (including the parts equal to 0).

Theorem 2.1. *Let* $\mathbf{y}=(y_1,\ldots,y_n)$*. For* $n,g\geq 1$ *and* $n\geq 3, g=0$*,*

$$\mathsf{M}\,\mathsf{C}\,\Xi_n\,H_n^g=y_1\cdots y_n(-1)^{3g-3+n}\sum_{\substack{\beta\vdash_0 2g-3+n,\\ l(\beta)=n}}\langle\tau_{\beta_1}\cdots\tau_{\beta_n}\lambda_g\rangle_g\,\beta_1!\cdots\beta_n!\,m_\beta(\mathbf{y}),$$

where $\beta=(\beta_1,\ldots,\beta_n)$*, and*

$$\mathsf{F}_k\,\mathsf{C}\,\Xi_n\,H_n^g=0,\quad \textit{for}\ \ k<2g-3+2n. \tag{16}$$

Proof. We apply $\mathsf{M}\,\mathsf{C}$ to the symmetrized Genus Expansion Ansatz (7), so from (5), (14) and (15), we obtain

$$\mathsf{M}\,\mathsf{C}\,\Xi_n\,H_n^g=\frac{1}{n!}\sum_{\substack{b_1,\ldots,b_n\geq 0,\\ b_1+\cdots+b_n=2g-3+n}}(-1)^g\langle\tau_{b_1}\cdots\tau_{b_n}\lambda_g\rangle_g\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n(-1)^{b_i}b_i!\,y_{\sigma(i)}^{b_i+1}.$$

But we have

$$\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n y_{\sigma(i)}^{b_i}=|\mathrm{Aut}\,\beta|\,m_\beta(\mathbf{y}),$$

where β is the partition (with 0 allowed as parts) whose parts are $b_1,\ldots,b_n$, reordered and $\mathrm{Aut}\,\beta$ is the subgroup of $\mathfrak{S}_n$ preserving $(b_1,\ldots,b_n)$ (through its permutation action on the coordinates). The first part follows by changing the range of summation from $b_1,\ldots,b_n$ to β. The second part follows immediately from (7) and (15). □

Note that (16) implies that there are no full terms in the series $\mathsf{C}\,\Xi_n\,H_n^g$ whose total degree is less than $2g-3+2n$. Thus we say that $\mathsf{M}\,\mathsf{C}\,\Xi_n\,H_n^g$ consists of the full terms of *minimum* total degree in $\mathsf{C}\,\Xi_n\,H_n^g$ (though we understand that this is informal, since it assumes that the full terms of total degree $2g-3+2n$ are not identically zero).

An aside. Theorem 5.1 of [GJVai], which is not used in this paper, concerns the terms of *maximum* total degree when we apply C since it gives an upper limit for the total degree. This should be corrected. The total degree of the terms is in fact less than or equal to $3m-6+3g$, *not* $2m-5+6g$, as was incorrectly given there.

3. Transformation of the Join-cut Equation

We now apply the operator $\mathsf{M}\,\mathsf{C}\,\Xi_n$ to the Join-cut Equation (1) to derive a partial differential equation for the generating series for $\mathsf{M}\,\mathsf{C}\,\Xi_n\, H_n^g$. The following notation is needed for applying the symmetrization operator Ξ_n and stating the equation.

For $i,j\geq 0$, $i+j\leq n$, let $\underset{i,j}{\overset{x}{\mathsf{sym}}}$ be the mapping, applied to a series in $x_1,\ldots,x_n$, given by

$$\underset{i,j}{\overset{x}{\mathsf{sym}}}\, f(x_1,\ldots,x_n)=\sum_{\mathcal{R},\mathcal{S},\mathcal{T}} f(\mathbf{x}_{\mathcal{R}},\mathbf{x}_{\mathcal{S}},\mathbf{x}_{\mathcal{T}}),$$

where the sum is over all ordered partitions $(\mathcal{R},\mathcal{S},\mathcal{T})$ of $\{1,\ldots,n\}$, where $\mathcal{R}=\{x_{r_1},\ldots,x_{r_i}\}$, $\mathcal{S}=\{x_{s_1},\ldots,x_{s_j}\}$, $\mathcal{T}=\{x_{t_1},\ldots,x_{t_{n-i-j}}\}$ and $(\mathbf{x}_{\mathcal{R}},\mathbf{x}_{\mathcal{S}},\mathbf{x}_{\mathcal{T}})=(x_{r_1},\ldots,x_{r_i},x_{s_1},\ldots,x_{s_j},x_{t_1},\ldots,x_{t_{n-i-j}})$, and where $r_1<\ldots<r_i$, $s_1<\ldots<s_j$, and $t_1<\ldots<t_{n-i-j}$. If i or j is equal to 0, then we may suppress them by writing $\underset{2}{\overset{x}{\mathsf{sym}}}$ for $\underset{2,0}{\overset{x}{\mathsf{sym}}}$, for example.

The following result gives an expression for the result of applying the symmetrization operator Ξ_n to the Join-cut Equation for the Hurwitz series.

Theorem 3.1 (see [GJVai] Thm. 4.4)**.** *The series $\Xi_n\, H_n^g$ satisfy the partial differential equation*

$$\left(\sum_{i=1}^{n} w_i\frac{\partial}{\partial w_i}+n+2g-2\right)\Xi_n\, H_n^g(x_1,\ldots,x_n)=T_1+T_2+T_3+T_4,$$

where

$$\begin{aligned}
T_1 &= \tfrac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i\partial}{\partial x_i}\,\frac{x_{n+1}\partial}{\partial x_{n+1}}\;\Xi_n\, H_{n+1}^{g-1}(x_1,\ldots,x_{n+1})\right)\Bigg|_{x_{n+1}=x_i},\\
T_2 &= \underset{1,1}{\overset{x}{\mathsf{sym}}}\,\frac{w_2}{1-w_1}\,\frac{1}{w_1-w_2}\,\frac{x_1\partial}{\partial x_1}\,\Xi_n\, H_{n-1}^{g}(x_1,x_3,\ldots,x_n),\\
T_3 &= \sum_{k=3}^{n}\underset{1,k-1}{\overset{x}{\mathsf{sym}}}\left(\frac{x_1\partial}{\partial x_1}\,\Xi_n\, H_k^0(x_1,\ldots,x_k)\right)\left(\frac{x_1\partial}{\partial x_1}\,\Xi_n\, H_{n-k+1}^{g}(x_1,x_{k+1},\ldots,x_n)\right),\\
T_4 &= \tfrac{1}{2}\sum_{\substack{1\leq k\leq n,\\ 1\leq a\leq g-1}}\underset{1,k-1}{\overset{x}{\mathsf{sym}}}\left(\frac{x_1\partial}{\partial x_1}\,\Xi_n\, H_k^a(x_1,\ldots,x_k)\right)\left(\frac{x_1\partial}{\partial x_1}\,\Xi_n\, H_{n-k+1}^{g-a}(x_1,x_{k+1},\ldots,x_n)\right),
\end{aligned}$$

for $n,g\geq 1$, with initial condition $\Xi_n\, H_0^g=0$ for $g\geq 1$.

Here we shall only consider Theorem 3.1 for $n\geq 1, g\geq 2$ and $n\geq 2, g=1$, and note that for this range of values, $\Xi_n\, H_i^0$ only arises in this equation for $i\geq 3$. In the statement of the result, a meaning is attached to $(w_i-w_j)^{-1}$ for $1\leq i<j\leq n$ by imposing the total order $w_1\prec\ldots\prec w_n$, and then defining $(w_i-w_j)^{-1}=-w_j^{-1}(1-w_i/w_j)^{-1}$. This then defines a formal power series ring in w_1 with coefficients that are formal Laurent series in $w_2,\ldots,w_n$ (see Xin [X]).

We now consider the partial differential equation for a genus generating series $\Omega_n(y_1,\ldots,y_n;t)$, which arises by applying $\mathsf{M}\,\mathsf{C}$ to the symmetrized Join-cut Equation given in Theorem 3.1. For this purpose, let $\mathsf{O}\,f$ and $\mathsf{E}\,f$ denote, respectively, the *odd* and *even* subseries of the formal power series f in the indeterminate t.

Theorem 3.2. *Let*

$$\Omega_n(\mathbf{y};t) = \sum_{g\geq 1} \frac{(-1)^{3g-3+n}}{c_g} \mathsf{M}\,\mathsf{C}\,\Xi_n\, H_n^g \frac{t^{2g-3+n}}{(2g-3+n)!}, \quad n \geq 1.$$

Then, for $n \geq 2$, we have the partial differential equation

$$(n-1)\frac{\partial}{\partial t}\Omega_n(\mathbf{y};t) = \underset{1,1}{\overset{y}{\mathsf{sym}}}\, \frac{y_1^3 y_2}{y_1 - y_2}\frac{\partial}{\partial y_1}\Omega_{n-1}(y_1, y_3, \ldots, y_n; t),$$

with initial condition $\Omega_1(y_1;t) = \mathsf{E}\, \frac{y_1}{1-y_1 t}$.

Proof. We begin by applying C to Theorem 3.1, and note that

$$\mathsf{C}\,\frac{w_2}{1-w_1}\frac{1}{w_1-w_2} = y_1^2\frac{y_2-1}{y_1-y_2}.$$

Let $\Delta_j^y = (y_j^3 - y_j^2)\frac{\partial}{\partial y_j}$. Then this result, together with (12), transforms the equation in Theorem 3.1 into a partial differential equation for $\mathsf{C}\,\Xi_n\, H_n^g$ given by

$$\left(\sum_{i=1}^n y_i(y_i-1)\frac{\partial}{\partial y_i} + n + 2g - 2\right)\mathsf{C}\,\Xi_n\, H_n^g(y_1,\ldots,y_n) = T_1' + T_2' + T_3' + T_4', \tag{17}$$

where $n \geq 2, g = 1$ or $n \geq 1, g \geq 2$, and

$$\begin{aligned}
T_1' &= \tfrac{1}{2}\sum_{i=1}^n \left(\Delta_i^y\, \Delta_{n+1}^y\, \mathsf{C}\,\Xi_n\, H_{n+1}^{g-1}(y_1,\ldots,y_{n+1})\right)\Big|_{y_{n+1}=y_i},\\
T_2' &= \underset{1,1}{\overset{y}{\mathsf{sym}}}\, y_1^2\frac{y_2-1}{y_1-y_2}\,\Delta_1^y\,\mathsf{C}\,\Xi_n\, H_{n-1}^g(y_1,y_3,\ldots,y_n),\\
T_3' &= \sum_{k=3}^n \underset{1,k-1}{\overset{y}{\mathsf{sym}}}\left(\Delta_1^y\,\mathsf{C}\,\Xi_n\, H_k^0(y_1,\ldots,y_k)\right)\left(\Delta_1^y\,\mathsf{C}\,\Xi_n\, H_{n-k+1}^g(y_1,y_{k+1},\ldots,y_n)\right),\\
T_4' &= \tfrac{1}{2}\sum_{\substack{1\leq k\leq n,\\ 1\leq a\leq g-1}} \underset{1,k-1}{\overset{y}{\mathsf{sym}}}\left(\Delta_1^y\,\mathsf{C}\,\Xi_n\, H_k^a(y_1,\ldots,y_k)\right)\left(\Delta_1^y\,\mathsf{C}\,\Xi_n\, H_{n-k+1}^{g-a}(y_1,y_{k+1},\ldots,y_n)\right).
\end{aligned}$$

Now apply M to (17), and use (16). With the notation $\Omega_n^g = \mathsf{M}\,\mathsf{C}\,\Xi_n\, H_n^g$, the only non-zero contributions on the left hand side arise from

$$\left(-\sum_{i=1}^n y_i\frac{\partial}{\partial y_i} + n + 2g - 2\right)\Omega_n^g = \left(-(2g-3+2n) + n + 2g - 2\right)\Omega_n^g = (1-n)\Omega_n^g,$$

since all terms in Ω_n^g have total degree $2g-3+2n$. On the right hand side, all contributions from terms T_1', T_3' and T_4' are zero. For T_2', the only non-zero contributions arise from

$$\underset{1,1}{\overset{y}{\mathsf{sym}}}\,\frac{y_1^4}{y_1-y_2}\frac{\partial}{\partial y_1}\Omega_{n-1}^g(y_1,y_3,\ldots,y_n),$$

from degree considerations alone. However, note that $y_1^4/(y_1-y_2) = y_1^3 + y_1^3 y_2/(y_1-y_2)$, and we conclude that, for *full* terms, the non-zero contributions from T_2' are given by

$$\underset{1,1}{\overset{y}{\mathsf{sym}}}\,\frac{y_1^3 y_2}{y_1-y_2}\frac{\partial}{\partial y_1}\Omega_{n-1}^g(y_1,y_3,\ldots,y_n).$$

Thus, we obtain the partial differential equation

$$(1-n)\Omega_n^g(\mathbf{y}) = \underset{1,1}{\overset{y}{\mathsf{sym}}}\,\frac{y_1^3 y_2}{y_1-y_2}\frac{\partial}{\partial y_1}\Omega_{n-1}^g(y_1,y_3,\ldots,y_n), \tag{18}$$

for $n \geq 2, g \geq 1$.

Now multiply this equation for Ω_n^g by $(-1)^{3g-4+n}t^{2g-4+n}/c_g(2g-4+n)!$, and sum over $g \geq 1$, to obtain the partial differential equation for Ω_n, $n \geq 2$. For $n=1$, we have

$$\Omega_1^g = (-1)^{3g-2}\langle \tau_{2g-2}\lambda_g\rangle_g (2g-2)! y_1^{2g-1},$$

from Theorem 2.1, which gives $\Omega_1(y_1;t) = \sum_{g\geq 1} y_1^{2g-1}t^{2g-2}$, and the result follows. □

The partial differential equation in Theorem 3.2 is simple enough that it can be solved explicitly.

Theorem 3.3. *For $n \geq 1$,*

$$\Omega_n(\mathbf{y};t) = \begin{cases} \mathsf{E} \prod_{i=1}^{n} \dfrac{y_i}{1-y_i t}, & \text{for } n \text{ odd,} \\ \mathsf{O} \prod_{i=1}^{n} \dfrac{y_i}{1-y_i t}, & \text{for } n \text{ even.} \end{cases}$$

Proof. Let $F_n(\mathbf{y};t) = \prod_{i=1}^n \frac{y_i}{1-y_i t}$. Then we have

$$\operatorname*{sym}_{1,1}^{y} \frac{y_1^3 y_2}{y_1-y_2}\frac{\partial}{\partial y_1} F_{n-1}(y_1,y_3,\ldots,y_n;t) \quad = \quad F_n(\mathbf{y};t) \operatorname*{sym}_{1,1}^{y} \frac{y_1^2(1-y_2t)}{(y_1-y_2)(1-y_1t)}.$$

But the symmetrized term on the right hand side of this equation becomes

$$\operatorname*{sym}_{2}^{y} \frac{y_1^2(1-y_2t)^2 - y_2^2(1-y_1t)^2}{(y_1-y_2)(1-y_1t)(1-y_2t)} = \operatorname*{sym}_{2}^{y}\left(\frac{y_1}{1-y_1t}+\frac{y_2}{1-y_2t}\right) = (n-1)\sum_{i=1}^{n}\frac{y_1}{1-y_1t},$$

and we thus have

$$\operatorname*{sym}_{1,1}^{y} \frac{y_1^3 y_2}{y_1-y_2}\frac{\partial}{\partial y_1} F_{n-1}(y_1,y_3,\ldots,y_n;t) = (n-1)\frac{\partial}{\partial t}F_n(\mathbf{y};t).$$

This proves that $F_n(\mathbf{y};t)$ is a solution to the partial differential equation given in Theorem 3.2, and the result follows from the initial conditions and the parity restrictions on the generating series $\Omega_n(\mathbf{y};t)$. □

4. Proof of the λ_g-Conjecture

Now we can prove the λ_g-Conjecture stated as Theorem 1.1.

Proof. We have $\prod_{i=1}^n (1-y_it)^{-1} = \sum_{k\geq 0} h_k(\mathbf{y})t^k$, where $h_k(\mathbf{y})$ is the kth *complete* (or *homogeneous*) symmetric function, given by

$$h_k(\mathbf{y}) = \sum_{\substack{\alpha\vdash_0 k,\\ l(\alpha)=n}} m_\alpha(\mathbf{y}).$$

Then, immediately from Theorem 3.3, we obtain

$$\Omega_n^g(\mathbf{y}) = c_g(-1)^{3g-3+n}(2g-3+n)!\ \ y_1\cdots y_n \sum_{\substack{\alpha\vdash_0 2g-3+n,\\ l(\alpha)=n}} m_\alpha(\mathbf{y}).$$

and the result follows by comparing this result with Theorem 2.1. □

5. The philosophy of the general approach

The approach stands in a more general geometric-combinatorial setting, and although we do not need much of this setting here, we do require it for our proof [GJV3] of Faber's intersection number conjecture (for a small number of points). This more general setting provides a useful perspective for the proof that we have given of the λ_g-Conjecture.

5.1. **A bridge between geometry and combinatorics.** The general approach is based on the observation that localization theory (developed in Gromov-Witten theory by [GP]), when applied to the cases that have been described above, expresses a series in the intersection numbers in terms of a sum over combinatorial structures (such as trees or graphs) that are weighted by Hurwitz numbers H^g_α (or double Hurwitz numbers in the case of Faber's Conjecture). An account of this is given in [V]. In this sense, localization theory provides a bridge from the *geometry* of intersection numbers for the moduli spaces of curves on the one hand, to branched covers on the other. As we have seen, the latter may be regarded as *combinatorial* structures.

Associated with the generating series for transitive ordered factorizations into transpositions is a functional equation that leads to an implicitly defined set of series. These, together with the combinatorial structure (trees, graphs) that are a consequence of the use of localization theory, determine an implicit change of variables. Although the functional equation is transcendental, the derivatives of its solution are, in effect, *rational* in the solution. It is precisely this rationality that leads to the *polynomiality property* and thence to a linear system of equations for the intersection numbers.

The usefulness of this general point of view is reinforced by the following observations. First, it enables us to obtain other Hodge integrals. Secondly, our proof of Faber's intersection number conjecture (for a small number of points) uses localization theory to create a sum over a particular class of trees weighted by genus 0 double Hurwitz numbers, which we subject to a similar but more complex (combinatorial) analysis.

5.2. **Integrable systems, recent developments and closing comments.** The λ_g-Conjecture, a statement about the moduli space of curves, or the factorization of transpositions, should not need to follow from Gromov-Witten theory. This work was motivated by the fact that the other three intersection-number conjectures either follow or might be expected to follow from understanding the algebraic structure of Hurwitz-type numbers. In each case, there is a natural change of variables (motivated by the string and dilaton equations); and in each case, there is a connection to integrable hierarchies. We point out the following recent developments: **i)** Kazarian and Lando's [KaL] and Kim and Liu's [KiL] short proofs of Witten's conjecture (the $\overline{\mathcal{M}}_{g,n}$ case); **ii)** Shadrin and Zvonkine's description and proof of a Witten-type theorem on the conjectural compactified Picard variety (related to one-part double Hurwitz numbers), relating the intersection theory to integrable hierarchies [SZ]; and **iii)** our proof of Faber's intersection number conjecture for up to three points, using "Faber-Hurwitz numbers," [GJV3].

Finally, the Join-cut Equation seems intertwined in some way with integrable hierarchies, but the precise connection is not yet clear. For example, it is a non-trivial task to go from the Join-cut Equation to Witten's Conjecture.

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Appendix A. Intersection numbers k higher than minimum

In principle, the formalism that we have described can be used also to obtain $\langle\tau_{\alpha_1}\cdots\tau_{\alpha_n}\lambda_{g-k}\rangle_g$ for $k>0$. This is a useful property of our formalism and one that is not presently shared by approaches to this question through algebraic geometry. In demonstrating this property, we confine ourselves to stating the necessary results and to giving explicit generating series for the case $k=1$ (next to minimum) and for a few values of (g,n).

A.1. **The general case.** By extending Theorem 2.1 to obtain full terms of total degree one higher than the "minimum," we obtain the following result that identifies $\langle\tau_{\alpha_1}\cdots\tau_{\alpha_n}\lambda_{g-1}\rangle_g$ as a coefficient in the generating series $\Lambda^g_{n,1}$, where we use the notation $\Lambda^g_{n,k} = \mathsf{F}_{2g-3+2n+k}\,\mathsf{C}\,\Xi_n\,H^g_n$ for the terms that are k higher than minimum total degree, $k \geq 0$.

Theorem A.1. *For $n, g \geq 1$ and $n \geq 3, g = 0$,*

$$\begin{aligned}\Lambda^g_{n,1}(\mathbf{y}) &= y_1 \cdots y_n (-1)^{3g-3+n} \sum_{\substack{\beta \vdash_0 2g-2+n,\\ l(\beta)=n}} \langle \tau_{\beta_1} \cdots \tau_{\beta_n} \lambda_{g-1} \rangle_g \beta_1! \cdots \beta_n! m_\beta(\mathbf{y}) \\ &\quad + (-1)^{3g-2+n} c_g (2g-3+n)!\, y_1 \cdots y_n \sum_{k=2}^{2g-2+n} \left(\sum_{j=1}^{k-1} \frac{1}{j} \right) p_k(\mathbf{y}) h_{2g-2+n-k}(\mathbf{y}).\end{aligned}$$

where $\beta = (\beta_1, \ldots, \beta_n)$.

By extending Theorem 3.2, we obtain a partial differential equation that is satisfied by the generating series $\Lambda^g_{n,1}$. This is stated in the following theorem. (Recall that $\Omega^g_n = \Lambda^g_{n,0}$, where Ω^g_n is used in the proof of Theorem 3.2.)

Theorem A.2. *For $g, n \geq 1$,*

$$\Lambda^g_{n,1}(\mathbf{y}) + \frac{1}{n} \underset{1,1}{\overset{y}{\mathsf{sym}}} \frac{y_1^3 y_2}{y_1 - y_2} \frac{\partial}{\partial y_1} \Lambda^g_{n-1,1}(y_1, y_3, \ldots, y_n) = \frac{1}{n} \left(T_1^{''} + \cdots + T_4^{''} \right)$$

where

$$\begin{aligned} T_1^{''} &= \left(\sum_{i=1}^n \frac{y_i^2 \partial}{\partial y_i} \right) \Omega^g_{n-1}(\mathbf{y}), \\ T_2^{''} &= 2 \underset{1,1}{\overset{y}{\mathsf{sym}}} \frac{y_1^4 y_2}{y_1 - y_2} \frac{\partial}{\partial y_1} \Omega^g_{n-1}(y_1, y_3, \ldots, y_n), \\ T_3^{''} &= -\sum_{k=3}^n \underset{1,k-1}{\overset{y}{\mathsf{sym}}} \left(\frac{y_1^2 \partial}{\partial y_1} \Omega^0_k(y_1, \ldots, y_k) \right) \left(\frac{y_1^2 \partial}{\partial y_1} \Omega^g_{n-k+1}(y_1, y_{k+1}, \ldots, y_n) \right), \\ T_4^{''} &= -\tfrac{1}{2} \sum_{\substack{1 \leq k \leq n,\\ 1 \leq a \leq g-1}} \underset{1,k-1}{\overset{y}{\mathsf{sym}}} \left(\frac{y_1^2 \partial}{\partial y_1} \Omega^a_k(y_1, \ldots, y_k) \right) \left(\frac{y_1^2 \partial}{\partial y_1} \Omega^{g-a}_{n-k+1}(y_1, y_{k+1}, \ldots, y_n) \right) \end{aligned}$$

where $\Lambda^g_{0,1} = \Omega^g_0 = 0$ for all g.

Note that the right hand side of the partial differential equation for $\Lambda^g_{n,1}$ in Theorem A.2 involves only the previously determined series $\Omega^g_n = \Lambda^g_{n,0}$. A similar partial differential equation can be derived for the generating series $\Lambda^g_{n,k}$, given in general form in the following result.

Theorem A.3. *For $k \geq 0$,*

$$\Lambda^g_{n,k}(\mathbf{y}) + \frac{1}{n+k-1} \underset{1,1}{\overset{y}{\mathsf{sym}}} \frac{y_1^3 y_2}{y_1 - y_2} \frac{\partial}{\partial y_1} \Lambda^g_{n-1,k}(y_1, y_3, \ldots, y_n)$$

depends only upon $\Lambda^l_{j,i}$ for $0 \leq i < k$, $0 \leq l \leq g$, $1 \leq j \leq n$.

We observe that Theorem A.2 agrees with the case $k = 1$, and that equation (18) agrees with the case $k = 0$, in which the right hand side is identically zero. We do not know how to exploit the fact that the partial differential operator applied to $\Lambda^g_{n-1,k}$ in Theorem A.3 is independent of k.

A.2. **Explicit results for $k = 1$.**

A.2.1. *The genus $g = 1$ case.* For the genus $g = 1$ case we have the following corollary of Theorem A.2 that, together with Theorem A.1, gives an explicit expression for the generating series $\Lambda^g_{n,1}$ for the intersection numbers $\langle \tau_{\beta_1} \cdots \tau_{\beta_n} \lambda_{g-1} \rangle_g$.

Corollary A.4.

$$\begin{aligned}\Lambda^1_{n,1}(\mathbf{y}) \quad = \quad & \frac{(-1)^{n+1}}{24}(n-1)!y_1 \cdots y_n \sum_{k=2}^{n} \left(\sum_{j=1}^{k-1} \frac{1}{j} \right) p_k(\mathbf{y}) h_{n-k}(\mathbf{y}) + \frac{(-1)^n}{24} n! y_1 \cdots y_n h_n(\mathbf{y}) \\ & + \frac{(-1)^{n-1}}{24} y_1 \cdots y_n \sum_{i=2}^{n} \sum_{m=i}^{n} \sum_{k=0}^{n-m} (i-2)!(n-i)!(-1)^{m-i} \binom{m}{i} e_m(\mathbf{y}) h_k(\mathbf{y}) h_{n-k-m}(\mathbf{y}).\end{aligned}$$

The resolutions of the generating series $24\Lambda^1_{n,1}$, in which $g = 1$, with respect to the monomial symmetric functions m_θ, where θ is a partition, are listed below for $1 \le n \le 6$. They are obtained directly from Corollary A.4. (Note that $24 = c_1^{-1}$.)

g	n	$24\Lambda^1_{n,1}$
1	1	$-m_2$
1	2	$m_{3\,1} + m_{2^2}$
1	3	$-m_{4\,1^2} - 2m_{3\,2\,1} - 2m_{2^2}$
1	4	$-2m_{5\,1^3} + 3m_{4\,2\,1^2} + 4m_{3^2\,1^2} + 6m_{3\,2^2\,1} + 6m_{2^4}$
1	5	$34m_{6\,1^4} + 8m_{5\,2\,1^3} - 12m_{4\,2^2\,1^2} - 16m_{3^2\,2\,1^2} - 24m_{3\,2^3\,1} - 24m_{2^5}$
1	6	$-324m_{7\,1^5} - 170m_{6\,2\,1^4} - 112m_{5\,3\,1^4} - 40m_{5\,2^2\,1^3} - 96m_{4^2\,1^4}$
		$+60m_{4\,2^3\,1^2} + 24m_{3^3\,1^3} + 80m_{3^2\,2^2\,1^2} + 120m_{3\,2^4\,1} + 120m_{2^6}$.

The intersection numbers $\langle \tau_{\alpha_1} \cdots \tau_{\alpha_n} \lambda_{g-1} \rangle_g$ for $g = 1$ are then given by Theorem A.1.

A.2.2. *The arbitrary genus case.* The next table gives generating series $c_g^{-1}\Lambda^g_{n,1}$ for $g = 2, \ldots, 5$ and for a few values of n. The series are obtained from Theorem A.2.

g	n	$c_g^{-1}\Lambda^g_{n,1}$
2	1	$37m_4$
2	2	$-106m_{5\,1} - 111m_{4\,2} - 116m_{3^2}$
2	3	$362m_{6\,1^2} + 424m_{5\,2\,1} + 444m_{4\,3\,2} + 444m_{4\,2^2} + 464m_{3^2\,2}$
3	1	$-3426m_6$
3	2	$16836m_{7\,1} + 17130m_{6\,2} + 17424m_{5\,3} + 17424m_{4^2}$
4	1	$61164m_8$
4	2	$-4249232m_{9\,1} - 4278148m_{8\,2} - 4307064m_{7\,3} - 4311180m_{6\,4} - 4315296m_{5^2}$
5	1	$-180519696m_{10}$
5	2	$1619765280m_{11\,1} + 1624677264m_{10\,2} + 1629589248m_{9\,3}$
		$+1630276704m_{8\,4} + 1630964160m_{7\,5} + 1630964160m_{6^2}$.

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