# A Short Proof of the Transcendence of Thue-Morse Continued Fractions

Boris ADAMCZEWSKI and Yann BUGEAUD

The Thue-Morse sequence $\mathbf{t} = (t_n)_{n\geq 0}$ on the alphabet $\{a, b\}$ is defined as follows: $t_n = a$ (respectively, $t_n = b$) if the sum of binary digits of $n$ is even (respectively, odd). This famous binary sequence was first introduced by A. Thue [12] in 1912. It was considered nine years later by M. Morse [7] in a totally different context. These pioneering papers have led to a number of investigations and a broad literature devoted to $\mathbf{t}$. There are many other ways to define the Thue-Morse sequence. Each of them gives rise to specific interests, problems, and most of the time solutions. Such ubiquity is well described in the survey [1], where the occurrence of $\mathbf{t}$ in combinatorics, number theory, differential geometry, theoretical computer science, physics, and even chess is documented. For $a$ and $b$ distinct integers K. Mahler [6] (see also [2]) established that the sum of the series $\sum_{n\geq 0} t_n 2^{-n}$ is transcendental. The present note adresses another Diophantine result related to the Thue-Morse sequence.

It is widely believed that the continued fraction expansion of every irrational algebraic number $\alpha$ either is eventually periodic (and we know from Lagange's theorem that this is the case if and only if $\alpha$ is a quadratic irrational) or contains arbitrarily large partial quotients. Apparently, this challenging question was first considered by A. Ya. Khintchin in [4] (see also [5], [11], or [13] for surveys or books including discussions of this subject). A preliminary step towards its resolution consists in providing explicit examples of transcendental continued fractions with bounded partial quotients. In this direction, M. Queffélec [8] showed in 1998 that the Thue-Morse continued fractions are transcendental.

**Theorem 1 (Queffélec).** *If $a$ and $b$ are distinct positive integers and $\mathbf{t} = (t_n)_{n\geq 0}$ is the Thue-Morse sequence on the alphabet $\{a, b\}$, then the number*

$$\xi = [t_0, t_1, t_2, \ldots] = t_0 + \cfrac{1}{t_1 + \cfrac{1}{t_2 + \cfrac{1}{t_3 + \cdots}}}$$

*is transcendental.*

Choosing $a = 1$ and $b = -1$, we infer from the definition of $\mathbf{t}$ that $t_0 = 1$, $t_1 = -1$, $t_{2n} = t_n$, and $t_{2n+1} = -t_n$ for each positive integer $n$. The generating function $F(z) = \sum_{n\geq 0} t_n z^n$ of $\mathbf{t}$ thus satisfies the equation $F(z) = (1 - z)F(z^2)$. Iterating this identity we arrive at

$$F(z) = \left( \prod_{i=0}^{k-1} \left(1 - z^{2^i}\right) \right) F\left(z^{2^k}\right)$$

for each positive integer $k$. We deduce that $F$ is not a rational function, for otherwise it would have either infinitely many roots or infinitely many poles in the complex plane. Consequently, the sequence $\mathbf{t}$ is not eventually periodic. Thanks to Lagrange's theorem, we can assert that the associated number $\xi$ is not a quadratic irrational. To prove that $\xi$ cannot be algebraic of larger degree requires much more work and the use of a deep result from Diophantine approximation. The purpose of our note is to give a new, simpler proof of Theorem 1 that illustrates the fruitful interplay between combinatorics on words and Diophantine approximation.

We first briefly sketch Queffélec's proof of Theorem 1. To this end we recall another useful description of $\mathbf{t}$. An easy induction shows that the infinite word

$$\mathbf{t} = t_0 t_1 t_2 \ldots = abbabaabbaababbabaababba \ldots$$

is the fixed point beginning with $a$ of the morphism $\mu$ defined by

$$\mu(a) = ab, \qquad \mu(b) = ba,$$

that is,

$$\mathbf{t} = \lim_{n \to +\infty} \mu^n(a). \tag{1}$$

Set $U = abb$ and $V = ab$. Observe that $\mathbf{t}$ begins with $abbab = UV$. Equation (1) shows that for each positive integer $k$ the word $\mathbf{t}$ begins with $\mu^k(U)\mu^k(V)$. Moreover, it is easily checked that $\mu^k(U)$ begins with $\mu^k(V)$ and that the length of $\mu^k(V)$ is two-thirds that of $\mu^k(U)$. Consequently, $\xi$ is *very close* to the quadratic irrational $\xi_k$ whose sequence of partial quotients is given by the periodic sequence $\mu^k(U)\mu^k(U)\mu^k(U)\ldots$.

Queffélec quantified precisely the meaning of "very close" and concluded that $\xi$ admits infinitely many *very good* quadratic approximants. The fact that $\xi$ must be transcendental is then derived from a deep theorem of W. M. Schmidt [10] (see Theorem 2). Here, we denote by $H(\alpha)$ the *height* of an algebraic number $\alpha$ (i.e., $H(\alpha)$ is the maximum of the moduli of the coefficients of its minimal polynomial).

**Theorem 2 (Schmidt).** *Let $\zeta$ be a real number that is neither rational nor quadratic irrational. If there exist a real number $w$ larger than 3 and infinitely many quadratic irrationals $\alpha$ such that*

$$|\zeta - \alpha| < H(\alpha)^{-w},$$

*then $\zeta$ is transcendental.*

In order to apply Theorem 2, Queffélec's proof requires rather precise estimates of the heights of the quadratic approximants $\xi_k$ described earlier. In particular, it is necessary to estimate the growth of the denominators of the convergents to $\xi$. This strongly depends on the values of the positive integers $a$ and $b$. As Queffélec remarked [9], there is a way to overcome this difficulty and to obtain quickly the estimates that are needed. However, there is a price to pay for this, namely, the use of deep tools from ergodic theory via consideration of the Thue-Morse symbolic dynamical system. Thus, one difficulty is in some sense just replaced with another.

Now we show how the proof of Theorem 1 can be simplified considerably by taking a different point of view. The only nonelementary argument we use is an equivalent formulation of Theorem 2 recalled in Theorem 3, which is also found in [10]. In particular, there is absolutely no need here to estimate the growth of the denominators of the convergents to $\xi$. The main novelty is that, instead of using the quasi-periodicity of the Thue-Morse sequence, we focus on a symmetry property of $\mathbf{t}$: it begins with arbitrarily large palindromes. In this respect, the proof we give strongly differs from the original one.

**Theorem 3 (Schmidt).** *Let $\zeta$ be a real number that is neither rational nor quadratic irrational. If there exist a real number $w$ larger than $3/2$ and infinitely many triples $(p, q, r)$ of nonzero integers such that*

$$\max\left\{\left|\zeta - \frac{p}{q}\right|, \left|\zeta^2 - \frac{r}{q}\right|\right\} < \frac{1}{|q|^w},$$

*then $\zeta$ is transcendental.*

We demonstrate how Theorem 3 implies Theorem 1. Let $\zeta = [a_0, a_1, \ldots]$ be a positive real irrational number, and let $n$ be a nonnegative integer. Denote by $p_n/q_n$ the $n$th convergent to $\zeta$, that is, $p_n/q_n = [a_0, a_1, \ldots, a_n]$. By the theory of continued fractions (see, for instance, [3]), we have

$$M_n := \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \quad (n \geq 1).$$

Assume that the word $a_0 a_1 \ldots a_n$ is a *palindrome* (i.e., $a_j = a_{n-j}$ for any integer $j$ with $0 \leq j \leq n$). Then the transpose ${}^tM_n$ of the matrix $M_n$ satisfies

$$\begin{aligned} {}^tM_n &= {}^t\left(\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\right) \\ &= {}^t\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} {}^t\begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots {}^t\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} = M_n. \end{aligned}$$

Since $M_n$ is symmetric, $q_n = p_{n-1}$. Recalling that

$$\left|\zeta - \frac{p_{n-1}}{q_{n-1}}\right| < \frac{1}{q_{n-1}^2},$$

we infer from the fact that $a_0 < \zeta < a_0 + 1$, $a_0 = a_n$, and $|p_n q_{n-1} - p_{n-1} q_n| = 1$ that

$$\begin{aligned} \left|\zeta^2 - \frac{p_n}{q_{n-1}}\right| &= \left|\zeta^2 - \frac{p_{n-1}}{q_{n-1}} \cdot \frac{p_n}{q_n}\right| \leq \left|\zeta + \frac{p_n}{q_n}\right| \cdot \left|\zeta - \frac{p_{n-1}}{q_{n-1}}\right| + \frac{a_0 + 1}{q_n q_{n-1}} \\ &\leq 2(a_0 + 1)\left|\zeta - \frac{p_{n-1}}{q_{n-1}}\right| + \frac{a_0 + 1}{q_n q_{n-1}} < \frac{3(a_0 + 1)}{q_{n-1}^2}. \end{aligned}$$

In other words, if the sequence of partial quotients of $\zeta$ begins with arbitrarily large palindromes, then $\zeta$ and $\zeta^2$ are simultaneously very well approximable by rational numbers having the same denominator. In particular, Theorem 3 implies that $\zeta$ is either quadratic irrational or transcendental.

We next show how this observation applies to the real number $\xi$. First, we remark that the Thue-Morse word $\mathbf{t}$ begins with the palindrome $abba$. Second, notice that $\mu^2(a) = abba$ and $\mu^2(b) = baab$ are palindromes. Consequently, for each positive integer $k$ the prefix of length $4^k$ of $\mathbf{t}$ is a palindrome. Denoting by $p_n/q_n$ the $n$th convergent to $\xi$, we have $p_n/q_n = [t_0, t_1, \ldots, t_n]$ and, in view of the forgoing discussion, we learn that

$$\max\left\{\left|\xi - \frac{p_{4^k-2}}{q_{4^k-2}}\right|, \left|\xi^2 - \frac{p_{4^k-1}}{q_{4^k-2}}\right|\right\} < \frac{3(a+1)}{q_{4^k-2}^2} \tag{2}$$

holds for each positive integer $k$. Recall that we have already established that $\xi$ is not quadratic irrational. Thus, it follows from Theorem 3 and (2) that $\xi$ is transcendental. This ends the proof of Theorem 1.

**ACKNOWLEDGMENTS.** We would like to sincerely thank the referee for a number of valuable suggestions concerning the presentation of this paper.

## References

[1] J.-P. Allouche and J. Shallit, The ubiquitous Prouhet-Thue-Morse sequence, in *Sequences and Their Applications (Singapore, 1998)*, Springer Ser. Discrete Math. Theor. Comput. Sci., Springer-Verlag, London, 1999, pp. 1–16.

[2] F. M. Dekking, Transcendance du nombre de Thue-Morse, *C. R. Acad. Sci. Paris* **285** (1977) 157–160.

[3] J. S. Frame, Continued fractions and matrices, this MONTHLY **56** (1949) 98–103.

[4] A. Ya. Khintchin, *Continued Fractions*, 2nd ed., Gosudarstv. Izdat. Tehn.-Theor. Lit. Moscow-Leningrad, 1949 (Russian); English translation, University of Chicago Press, Chicago-London, 1964.

[5] S. Lang, *Introduction to Diophantine Approximations*, 2nd ed., Springer-Verlag, New-York, 1995.

[6] K. Mahler, Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen, *Math. Ann.* **101** (1929) 342–366; corrigendum **103** (1930) 532.

[7] M. Morse, Recurrent geodesics on a surface of negative curvature, *Trans. Amer. Math. Soc.* **22** (1921) 84–100.

[8] M. Queffélec, Transcendance des fractions continues de Thue-Morse, *J. Number Theory* **73** (1998) 201–211.

[9] —, Irrational numbers with automaton-generated continued fraction expansion, in *Dynamical systems (Luminy-Marseille, 1998)*, World Scientific, River Edge, NJ, 2000, pp. 190–198.

[10] W. M. Schmidt, On simultaneous approximations of two algebraic numbers by rationals, *Acta Math.* **119** (1967) 27–50.

[11] J. O. Shallit, Real numbers with bounded partial quotients, *Enseign. Math.* **38** (1992) 151–187.

[12] A. Thue, Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen, *Norske vid. Selsk. Skr. Mat. Nat. Kl.* **1** (1912) 1–67; reprinted in *Selected Mathematical Papers of Axel Thue*, T. Nagell, ed., Universitetsforlaget, Oslo, 1977, pp. 413–478.

[13] M. Waldschmidt, Un demi-siècle de transcendance, in *Development of Mathematics 1950–2000*, Birkhäuser, Basel, 2000, pp. 1121–1186.

*Boris Adamczewski, CNRS, Institut Camille Jordan, Université Claude Bernard Lyon 1, Bât. Braconnier, 21 avenue Claude Bernard, 69622 Villeurbanne Cedex, France.*
*Boris.Adamczewski@math.univ-lyon1.fr*

*Yann Bugeaud, Université Louis Pasteur, U. F. R. de mathématiques, 7, rue René Descartes, 67084 Strasbourg Cedex, France*
*bugeaud@math.u-strasbg.fr*