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A Short Simplicial *h*-Vector and the Upper Bound Theorem

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Abstract. The Upper Bound Conjecture is verified for a class of odd-dimensional simplicial complexes that in particular includes all Eulerian simplicial complexes with isolated singularities. The proof relies on a new invariant of simplicial complexes—a short simplicial h-vector.

1. Introduction

The goal of this note is to prove an extension of the Upper Bound Theorem for (simplicial) polytopes. The main tool in the proof is a certain new invariant of simplicial complexes, which is a simplicial analog of a short cubical h-vector introduced by Adin [1].

We start by recalling several definitions. A (finite) simplicial complex Δ is *pure* if each maximal face of Δ has the same dimension. A pure simplicial complex Δ is *Eulerian* if for every face F of Δ (including the empty face) the Euler characteristic of its link is equal to the Euler characteristic of the sphere of the same dimension, that is,

$$\chi(\operatorname{lk} F) = 1 + (-1)^{\dim(\operatorname{lk} F)}.$$

In particular, by Poincaré duality, every odd-dimensional homology manifold is Eulerian. (Recall that a simplicial complex Δ is a *homology manifold* if its geometric realization X possesses the following property: for every $p \in X$ and every $i < \dim X$, $H_i(X, X-p) = 0$, while $H_{\dim X}(X, X-p) \cong \mathbb{Z}$. Here $H_i(X, X-p)$ denotes the ith relative singular homology with coefficients $\mathbb{Z}$.)

The Upper Bound Conjecture (abbreviated UBC) proposed by Motzkin in 1957 (see [6]) asserts that among all d-dimensional (simplicial) polytopes with n vertices, the number of i-dimensional faces (for every $i = 1, \dots, d-1$) is maximized by the cyclic polytope $C_d(n)$. Over the last 40 years this conjecture has been treated extensively by many mathematicians: in 1970 McMullen [5] proved the UBC for polytopes; McMullen's result was preceded in 1964 by a surprising work of Klee, where he verified that the UBC holds for all Eulerian complexes with a sufficiently large number of vertices, and conjectured that it holds for all Eulerian complexes [4]; in 1975 Stanley proved the UBC for arbitrary triangulations of spheres [9], [11], and in 1998 Novik verified the UBC for triangulations of odd-dimensional manifolds and several classes of even-dimensional manifolds [7].

In this note we prove the UBC for a class of odd-dimensional simplicial complexes that in particular includes all odd-dimensional Eulerian complexes whose geometric realization has isolated singularities. More precisely, we obtain the following theorem in which $f_i(\Delta)$ denotes the number of i-dimensional faces of a complex Δ, the values $\beta_i(\Delta) = \dim(\tilde{H}_i(\Delta))$ denote the reduced Betti numbers of Δ over a field of characteristic 0, and $C_d(n)$ is a d-dimensional cyclic polytope on n vertices.

Theorem 1. *Let Δ be a pure $(2k+1)$-dimensional simplicial complex on n vertices, such that for every vertex v of Δ, the link of v is either a homology manifold whose Euler characteristic is 2, or an oriented homology manifold satisfying the following condition*:

$$\beta_k(\mathrm{lk}\, v) \le 2\beta_{k-1}(\mathrm{lk}\, v) + 2\sum_{i=0}^{k-3} \beta_i(\mathrm{lk}\, v).$$

Then $f_i(\Delta) \le f_i(C_{2k+2}(n))$ for $i = 1, 2, \dots, 2k+1$.

The main ingredient in the proof is a new invariant of simplicial complexes, $\tilde{h}(\Delta) = (\tilde{h}_0, \tilde{h}_1, \dots, \tilde{h}_{\dim(\Delta)})$, which is a simplicial analog of the short cubical h-vector introduced by Adin (see [1]). We give its definition and list some of its properties in the next section. Section 3 is devoted to a proof of Theorem 1. Section 4 contains several remarks and additional results on the UBC and the $\tilde{h}$-vector.

2. The $\tilde{h}$-vector

In this section we introduce the notion of the $\tilde{h}$-vector for pure simplicial complexes and list some of its properties. We begin by recalling definitions of f-vectors and h-vectors. For a $(d-1)$-dimensional simplicial complex Δ, its f-vector, denoted $f(\Delta)$, is the vector $(f_{-1}, f_0, f_1, \dots, f_{d-1})$ where f_i counts the number of i-dimensional faces. In particular, $f_{-1} = 1$, f_0 is the number of vertices of Δ, and f_1 is the number of edges. The h-vector of Δ, denoted $h(\Delta)$, is the vector $(h_0, h_1, \dots, h_d)$ where

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(\Delta), \qquad i = 0, 1, \dots, d. \tag{1}$$

Equivalently,

$$f_{j-1}(\Delta) = \sum_{i=0}^{j} \binom{d-i}{d-j} h_i(\Delta), \qquad j = 0, 1, \ldots, d. \tag{2}$$

Adin [1, eqns. (1), (11)] defined for any cubical complex C its *short cubical* h-vector, denoted $h^{(\mathrm{sc})}(C) = (h_0^{(\mathrm{sc})}, h_1^{(\mathrm{sc})}, \ldots, h_{\dim(C)}^{(\mathrm{sc})})$. It was later observed by Hetyei that if C is pure, then $h^{(\mathrm{sc})}(C) = \sum_{v \in V} h(\mathrm{lk}\, v)$, where V is the set of vertices of C. (Note that the links of the vertices in a cubical complex are simplicial complexes, and hence the h-vector $h(\mathrm{lk}\, v)$ is well-defined.)

Similarly to the short cubical h-vector, we define a short simplicial h-vector, denoted $\tilde{h}$, as follows.

Definition 1. Let Δ be a pure $(d-1)$-dimensional simplicial complex on the vertex set V. Define

$$\tilde{h}(\Delta) = (\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_{d-1}) := \sum_{v \in V} h(\mathrm{lk}\, v),$$

so in particular $\tilde{h}_i(\Delta) := \sum_{v \in V} h_i(\mathrm{lk}\, v)$.

The next lemma gives several properties of $\tilde{h}$.

Lemma 1.

(i) *Let Δ be a pure $(d-1)$-dimensional simplicial complex. Then*

$$\tilde{h}_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} (j+1) \binom{d-1-j}{d-1-i} f_j(\Delta) \qquad (0 \le i \le d-1)$$

and

$$f_j(\Delta) = (j+1)^{-1} \sum_{i=0}^{j} \binom{d-1-i}{d-1-j} \tilde{h}_i(\Delta) \qquad (0 \le j \le d-1).$$

In particular, the f-numbers of a simplicial complex are non-negative linear combinations of its $\tilde{h}$-numbers.

(ii) *If Δ is a pure $(2k+1)$-dimensional simplicial complex such that the link of every vertex is a homology manifold, then the f-numbers of Δ are non-negative linear combinations of $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_{k+1}$. In other words,*

$$f_j(\Delta) = \sum_{i=0}^{k+1} b_i^j \tilde{h}_i(\Delta), \qquad 0 \le j \le 2k+1,$$

where the coefficients b_i^j are independent of Δ and are non-negative.

Proof. Since every j-dimensional simplex has $j+1$ vertices, it follows that

$$\sum_{v \in V} f_{j-1}(\mathrm{lk}\, v) = (j+1) f_j(\Delta), \tag{3}$$

where V is the set of vertices of Δ. This equation together with relations (1) and (2) (applied to the links of vertices) implies part (i).

Part (ii) is a consequence of (3) and [7, Lemma 6.1], which asserts that the f-numbers of a $2k$-dimensional homology manifold are non-negative linear combinations of its h-numbers $h_0, h_1, \ldots, h_{k+1}$. □

3. The Proof of the Upper Bound Theorem

In this section we prove Theorem 1. This will require the following facts and definitions.

Definition 2. A simplicial complex Δ is *l-neighborly* if each set of l of its vertices forms a face in Δ.

It is well known that all d-dimensional cyclic polytopes are $\lfloor d/2 \rfloor$-neighborly, and that all $\lfloor d/2 \rfloor$-neighborly d-dimensional polytopes with r vertices have the same h-vector:

$$h_i = h_{d-i} = \binom{r-d+i-1}{i} \qquad \text{for} \quad 0 \leq i \leq \lfloor d/2 \rfloor.$$

In the proof of Theorem 1 we will also use the following version of the Upper Bound Theorem for even-dimensional homology manifolds.

Lemma 2. *Let K be a $2k$-dimensional homology manifold on r vertices. Furthermore, assume that either $\chi(K) = 2$, or K is an oriented homology manifold such that*

$$\beta_k(K) \leq 2\beta_{k-1}(K) + 2\sum_{i=0}^{k-3} \beta_i(K). \tag{4}$$

Then

$$h_i(K) \leq h_i(C_{2k+1}(r)) \qquad \textit{for} \quad 0 \leq i \leq k+1.$$

Proof. In the case of $\chi(K) = 2$, the lemma follows from Theorem 6.6 of [7] and the Dehn–Sommerville relations for Eulerian complexes [3]. In the second case the result is a part of the proof of Theorem 6.7 of [7]. □

We are now ready to verify Theorem 1. The argument is very similar to the proof of a special case of the cubical UBC (see Theorem 4.3 of [2]). The only difference is that we use the $\tilde{h}$-vector instead of the short cubical h-vector.

Proof of Theorem 1. Let Δ be a simplicial complex satisfying the conditions of the theorem. By Lemma 1(ii), it suffices to check that $\tilde{h}_i(\Delta) \leq \tilde{h}_i(C_{2k+2}(n))$ for $0 \leq$

$i \leq k+1$. To this end, note that for every vertex v of Δ, $\operatorname{lk} v$ is a $2k$-dimensional simplicial complex on at most $n-1$ vertices that is either a homology manifold with Euler characteristic 2, or an oriented homology manifold satisfying condition (4). Thus, by Lemma 2,

$$h_i(\operatorname{lk} v) \leq h_i(C_{2k+1}(n-1)) \quad \text{for} \quad 0 \leq i \leq k+1.$$

Since $C_{2k+2}(n)$ is a $(k+1)$-neighborly polytope, it follows that the link of every vertex of $C_{2k+2}(n)$ is a k-neighborly $(2k+1)$-dimensional polytope on $n-1$ vertices. Hence,

$$\tilde{h}_i(\Delta) = \sum_v h_i(\operatorname{lk} v) \leq \sum_v h_i(C_{2k+1}(n-1)) = \tilde{h}_i(C_{2k+2}(n)) \qquad \text{for} \quad 0 \leq i \leq k+1,$$

implying the theorem. □

Corollary 1. *Let Δ be a $(2k+1)$-dimensional oriented pseudomanifold on n vertices such that the link of every vertex is either a $2k$-dimensional homology manifold with vanishing middle homology, or it is a $2k$-dimensional homology manifold whose Euler characteristic χ satisfies $(-1)^k(\chi-2) \leq 0$. Then*

$$f_i(\Delta) \leq f_i(C_{2k+2}(n)) \qquad \textit{for} \quad 1 \leq i \leq 2k+1.$$

Proof. Any such complex Δ satisfies the assumptions of Theorem 1. □

4. Additional Remarks and Results

1. Theorem 1 proves a special case of Kalai's conjecture [7, Section 7] that the UBC holds for all simplicial complexes having the property that every link (of a face) of dimension $2k$ $(k = 1, 2, \ldots)$ satisfies condition (4).

2. In his proof of the UBC for spheres [9], [11], Stanley showed that if K is a $(d-1)$-dimensional homology sphere on n vertices, then

$$h_i(K) \leq h_i(C_d(n)) \qquad \text{for} \quad 0 \leq i \leq d-1. \tag{5}$$

Since the f-numbers of any simplicial complex Δ are non-negative combinations of its $\tilde{h}$-numbers (by Lemma 1(i)), arguing exactly as in the proof of Theorem 1, but using (5) instead of Lemma 2, we obtain a new proof of the UBC for odd-dimensional homology manifolds. This proof is shorter and more elementary than the one presented in Theorem 1.4 of [7]. (It does not use any facts about Buchsbaum complexes!)

3. It would be interesting to clarify whether for a $(2k+1)$-dimensional complex Δ satisfying the assumptions of Theorem 1, the inequality $h_i(\Delta) \leq h_i(C_{2k+2}(n))$ $(0 \leq i \leq k+1)$ necessarily holds. We have the expression

$$\begin{aligned} h_r(\Delta) &= \sum_{j=0}^{r} (-1)^{r-j} \binom{2k+2-j}{2k+2-r} f_{j-1}(\Delta) \\ &= (-1)^r \binom{2k+2}{r} + \sum_{i=0}^{r-1} \widetilde{h}_i(\Delta) \binom{2k+1-i}{2k+2-r} \sum_{j=i+1}^{r} \frac{1}{j} (-1)^{r-j} \binom{r-i-1}{r-j} \end{aligned}$$

$$= (-1)^r \binom{2k+2}{r} + \sum_{i=0}^{r-1} \widetilde{h}_i(\Delta) \binom{2k+1-i}{2k+2-r} \int_0^1 x^i (x-1)^{r-i-1}\, dx$$

$$= (-1)^r \binom{2k+2}{r} + \sum_{i=0}^{r-1} (-1)^{r-i-1} \frac{(2k+1-i)!\, i!}{(2k+2-r)!\, r!} \widetilde{h}_i(\Delta).$$

Hence the coefficients of $\tilde{h}$-numbers in the expression for h_r alternate in sign so that short simplicial h-vectors are not sufficient to resolve this question.

4. Lower bounds. Let Δ be a simplicial complex, let $\mathrm{Skel}_i(\Delta)$ denote its i-dimensional skeleton, and let $\chi_i(\Delta) := \chi(\mathrm{Skel}_i(\Delta)) = \sum_{j=0}^{i}(-1)^j f_j(\Delta)$ denote the Euler characteristic of $\mathrm{Skel}_i(\Delta)$. It was shown in [8] that if Δ is a $(2k-1)$-dimensonal manifold, then $(-1)^i \chi_i(\Delta) \geq 0$ for $0 \leq i \leq 2k-1$. The proof relied on several facts about Buchsbaum complexes. Using $\tilde{h}$-numbers we provide a short proof of the following related result.

Proposition 1. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex* (*i.e. a pure simplicial complex such that for every vertex $v \in \Delta$ the link of v is Cohen–Macaulay*). *Then $(-1)^i \chi_i(\Delta) \geq 0$ for $0 \leq i \leq \lfloor (d-1)/2 \rfloor$.*

Proof. Since lk v is Cohen–Macaulay for every vertex $v \in \Delta$, it follows that $h_i(\mathrm{lk}\, v) \geq 0$ for $i = 0, 1, \ldots, d-1$, and, hence, $\tilde{h}_i(\Delta) \geq 0$ for $i = 0, 1, \ldots, d-1$. Expressing the f-numbers of Δ in terms of its $\tilde{h}$-numbers (Lemma 1(i)), we obtain

$$(-1)^i \chi_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} f_j = \sum_{l=0}^{i} \left(\sum_{j=l}^{i} (-1)^{i-j} \frac{1}{j+1} \binom{d-1-l}{d-1-j} \right) \tilde{h}_l. \qquad (6)$$

It is straightforward to show that if $0 \leq i \leq \lfloor (d-1)/2 \rfloor$ and $0 \leq l \leq i$, then

$$\frac{1}{i+1}\binom{d-1-l}{d-1-i} \geq \frac{1}{i}\binom{d-1-l}{d-i} \geq \cdots \geq \frac{1}{l+1}\binom{d-1-l}{d-1-l}.$$

Hence for any $0 \leq i \leq \lfloor (d-1)/2 \rfloor$, all coefficients of $\tilde{h}$-numbers in (6) are non-negative, implying the proposition. □

5. Semi-Eulerian complexes. One may also use short simplicial h-vectors and the Dehn–Sommerville relations to give a new proof of the fact that all odd-dimensional semi-Eulerian simplicial (or regular cell) complexes are Eulerian. This result was proven more generally for posets in Exercise 3.69(c) of [10] by a very different approach.

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