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A SHORTEST SINGLE AXIOM FOR THE CLASSICAL EQUIVALENTIAL CALCULUS

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Ten shortest single axioms for the classical equivalential calculus **E** are known. Of these three are due to Łukasiewicz [2] and the remaining seven to Meredith [5], p. 185. Meredith gave proofs for two of his axioms, and the other five were proved by Peterson [8], who found his proofs with the aid of the theorem-proving computer program described in [9]. Meredith also stated (*cf.* [5], p. 185 and [10], p. 307) that

(6) *EpEEqErpEqr*

(we follow the numbering of [8]) is a single axiom for **E**, but this is incorrect, as noted by Peterson [8], p. 270.

The main object of this paper is to show that

(6′) *EpEEqErpErq*

is a single axiom for **E**; (6′) evidently corrects a misprint in (6) in both [5] and [10]. The proof that (6′) is a single axiom for **E** was found with the aid of a computer program similar to Peterson's, and an account of the way in which the computer obtained this proof is included below. Further discussions of the use of computers to prove theorems in Hilbert-type sentential calculi may be found in [1] and [6].

The following deduction shows that

(10) *EEEpEqrrEqp*

is derivable from (6′) by condensed detachment; for the proof that (10) is a single axiom for **E**, see [8], p. 270.

1.	*EpEEqErpErq*	
2.	*EEpEqErEEsEtrEtsEqp*	= **D**1.1
3.	*EEEEpEqrEqpErss*	= **D**2.1
4.	*Epp*	= **D**3.3
5.	*EEpEqErrEqp*	= **D**1.4
6.	*EEpEpqq*	= **D**5.1

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7. *EpEqEqEpErr* = D5.6
8. *EpEqEqp* = D2.7
9. *EEpEqErEsEsrEqp* = D1.8
10. *EEEpEpqEqrr* = D9.1
11. *EpEEqEqrErEpEss* = D5.10
12. *EEpEpqErErq* = D2.11
13. *EEpEqEErErsEtEtsEqp* = D1.12
14. *EpEqEqEErEspEsr* = D2.12
15. *EpEpEEqEqrr* = D12.12
16. *EEpEpqErErEsEsq* = D13.12
17. *EEpEqErErEEsEsttEqp* = D1.15
18. *EEpEqEErEsEttEsrEqp* = D1.5
19. *EpEpEqq* = D5.4
20. *EEpEEqEqprr* = D17.14
21. *EEEpqEEqEpErrss* = D18.1
22. *EEEpqEEqEprEssr* = D18.14
23. *EEEpqEpErrEsEsq* = D18.16
24. *EEpEqEErErssEqp* = D1.6
25. *EEpEpqEErEsqEsr* = D20.14
26. *EEpEpqEErqErEss* = D24.23
27. *EEpEEqEqprEsEsr* = D24.25
28. *EEEEpEpqrEqEssr* = D21.27
29. *EEEpqEpEqErrEss* = D28.22
30. *EEpEqErrEEpqEss* = D21.29
31. *EEEEpEpqqrEsEsr* = D17.26
32. *EpEqErErEqp* = D3.31
33. *EEpEqErErEqpEss* = D19.32
34. *EEpEqEqEprErEss* = D21.33
35. *EEEpEqEqEprrEss* = D30.34
36. *EpEqEpq* = D3.35
37. *EpEEqErEqrp* = D36.36
38. *EEpEqErEEsEtEstrEqp* = D1.37
39. *EEpqEqp* = D38.14
40. *EEpEqEErsEsrEqp* = D1.39
41. *EEEpqEEqprr* = D40.1
42. *EpEEEqErsErqEsp* = D39.3
43. *EpEEqrErEqp* = D41.42
44. *EEEpEqrrEqp* = D2.43

Four essentially different computer runs, $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$, and $\mathbf{R}_4$, were used in obtaining the above proof. In $\mathbf{R}_1$, an "ordered-list search" (*cf.* [1], p. 184) was used, i.e., each time a theorem derived from (6′) was selected for "development" (i.e., for use in deriving further theorems, *cf.* [1], p. 183) it was chosen from among the shortest of the theorems undeveloped at that time; also the "length-parameter", i.e., the maximum length of theorems stored, was set initially at 27, was reduced to 23 after 300 theorems had been stored, and was further reduced to 15 after 600 theorems had been stored. With these heuristics, Theorem 11 of the above proof was derived as Theorem 673 of the run $\mathbf{R}_1$, and appeared to play an important part in the production of further theorems. This theorem was therefore selected as a "lemma", *cf.* [7], i.e., for $\mathbf{R}_2$ this theorem was used as an additional axiom. In $\mathbf{R}_2$ a "breadth-first search" was used until 100 theorems had been stored, i.e., until then each theorem selected for development was the first-computed of the theorems undeveloped at the time of selection; thereafter the ordered-list search was used. The length-parameter was set initially at 19, and was reduced to 15 after 50 theorems had been stored. When this was done, Theorem 36 of the above proof was derived as Theorem 757 of the run $\mathbf{R}_2$, and because of its shortness was selected as a further lemma. In $\mathbf{R}_3$ the axioms were Theorems 1, 11, and 36 of the above proof, together with Theorem 899 of the run $\mathbf{R}_2$, namely *EEpEqpq*; the change from breadth-first to ordered-list search was made after 300 theorems had been stored; and the length-parameter was reduced from 19 to 15 after 50 theorems had been stored. Theorem 39 of the above proof was then derived as Theorem 167 of the run $\mathbf{R}_3$, i.e., before the change to ordered-list search, and was selected as a further lemma. In $\mathbf{R}_4$ the axioms were Theorems 1 and 39 of the above proof together with the previous lemmas, and the change from breadth-first to ordered-list search was again made after 300 theorems had been stored; the length-parameter was reduced from 19 to 15 after 50 theorems had been stored, and was further reduced to 11 after 1000 theorems had been stored. Certain other devices were also introduced in order to stave off exhaustion of the available core storage and processing time. Theorem 44 of the above proof was then derived as Theorem 1015 of the run $\mathbf{R}_4$.

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