

A SIEVE M-THEOREM FOR BUNDLED PARAMETERS IN SEMIPARAMETRIC MODELS, WITH APPLICATION TO THE EFFICIENT ESTIMATION IN A LINEAR MODEL FOR CENSORED DATA

BY YING DING¹ AND BIN NAN^{1,2}

University of Michigan

In many semiparametric models that are parameterized by two types of parameters—a Euclidean parameter of interest and an infinite-dimensional nuisance parameter—the two parameters are bundled together, that is, the nuisance parameter is an unknown function that contains the parameter of interest as part of its argument. For example, in a linear regression model for censored survival data, the unspecified error distribution function involves the regression coefficients. Motivated by developing an efficient estimating method for the regression parameters, we propose a general sieve M-theorem for bundled parameters and apply the theorem to deriving the asymptotic theory for the sieve maximum likelihood estimation in the linear regression model for censored survival data. The numerical implementation of the proposed estimating method can be achieved through the conventional gradient-based search algorithms such as the Newton–Raphson algorithm. We show that the proposed estimator is consistent and asymptotically normal and achieves the semiparametric efficiency bound. Simulation studies demonstrate that the proposed method performs well in practical settings and yields more efficient estimates than existing estimating equation based methods. Illustration with a real data example is also provided.

1. Introduction. In a semiparametric model that is parameterized by two types of parameters—a finite-dimensional Euclidean parameter and an infinite-dimensional parameter—oftentimes the infinite-dimensional parameter is considered as a nuisance parameter, and the two parameters are *separated*. In many interesting statistical models, however, the parameter of interest and the nuisance parameter are *bundled together*, a terminology used by Huang and Wellner [13] when they reviewed the linear models under interval censoring, which means that the infinite-dimensional parameter is an unknown function of the parameter of interest. For example, in a linear regression model for censored survival data, the unspecified error distribution function, often treated as a nuisance parameter, is a

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function of the regression coefficients. Other examples include the single index model and the Cox regression model with an unspecified link function.

There is a rich literature of asymptotic distributional theories for M-estimation in a variety of semiparametric models with well-separated parameters; see, for example, [10–12, 24, 30, 33], among many others. Though many methodologies of M-estimation for bundled parameters have been proposed in the literature, general asymptotic distributional theories for such problems are still lacking. The only estimation theories for bundled parameters we are aware of are the sieve generalized method of moment of [1] and the estimating equation approach of [5, 19].

In this article, we consider an extension of existing asymptotic distributional theories to accommodate situations where the estimation criteria are parameterized with bundled parameters. The proposed theory has similar flavor of Theorem 2 in [5], but they are different because the latter requires an existing uniform consistent estimator of the infinite-dimensional nuisance parameter with a convergence rate faster than $n^{-1/4}$, which is then treated as a fixed function of the parameter of interest in their estimating procedure, while we need to simultaneously estimate both parameters through a sieve parameter space; furthermore, their existing nuisance parameter estimator needs to satisfy their condition (2.6), which is usually hard to verify when its convergence rate is slower than $n^{-1/2}$. Our proposed theory is general enough to cover a wide range of problems for bundled parameters including the aforementioned single index model, the Cox model with unknown link function and a linear model under different censoring mechanisms. Rigorous proofs for each of the models, however, will take lengthy derivations. We only use the efficient estimation in the semiparametric linear regression model with right censored data as an illustrative example that motivates such a theoretical development and will present results for other models elsewhere. Note that the considered example cannot be directly put into the framework of restricted moments due to right censoring, thus cannot be handled by the method of [1].

Suppose that the failure time transformed by a known monotone transformation is linearly related to a set of covariates, where the failure time is subject to right censoring. Let T_i denote the transformed failure time and C_i denote the transformed censoring time by the same transformation for subject i , $i = 1, \dots, n$. Let $Y_i = \min(T_i, C_i)$ and $\Delta_i = I(T_i \leq C_i)$. Then the semiparametric linear model we consider here can be written as

$$(1.1) \quad T_i = X_i' \beta_0 + e_{0,i}, \quad i = 1, \dots, n,$$

where the errors $e_{0,i}$ are independent and identically distributed (i.i.d.) with an unspecified distribution. When the failure time is log-transformed, this model corresponds to the well-known accelerated failure time model [16]. Here we assume that (X_i, C_i) , $i = 1, \dots, n$, are i.i.d. and independent of $e_{0,i}$. This is a common assumption for linear models with censored survival data, which is particularly needed in [22] to derive the efficient score function for β_0 . Such an assumption,

however, is stronger than necessary in the usual linear regression without censoring, for which the error is only required to be uncorrelated with covariates; see, for example, [3]. We also avoid trivial transformations such as $\log(0)$ so that we always have Y_i 's bounded from below.

The semiparametric linear regression model relates the failure time to the covariates directly. It provides a straightforward interpretation of the data and serves as an attractive alternative to the Cox model [6] in many applications. Several estimators of the regression parameters have been proposed in the literature since late 1970s, including the rank-based estimators (see, e.g., [14, 15, 20, 26, 29, 31]) and the Buckley–James estimator (see, e.g., [2, 17, 21]). There are two major challenges in the estimation for such a linear model: (1) the estimating functions in the aforementioned methods are discrete, leading to potential multiple solutions as well as numerical difficulties; (2) none of the aforementioned methods is efficient. Recently, Zeng and Lin [32] developed a kernel-smoothed profile likelihood estimating procedure for the accelerated failure time model. In this article, we consider a sieve maximum likelihood approach for model (1.1) for censored data. The proposed approach is much intuitive, easy to implement numerically and asymptotically efficient.

It is easy to see that T and C are independent conditional on X under the assumption $e_0 \perp (C, X)$. Hence the joint density function of $Z = (Y, \Delta, X)$ can be written as

$$(1.2) \quad f_{Y, \Delta, X}(y, \delta, x) = \lambda_0(y - x' \beta_0)^\delta \exp\{-\Lambda_0(y - x' \beta_0)\} H(y, \delta, x),$$

where $\Lambda_0(\cdot)$ is the true cumulative hazard function for the error term e_0 and $\lambda_0(\cdot)$ is its derivative. $H(y, \delta, x)$ only depends on the conditional distribution of C given X and the marginal distribution of X , and is free of β_0 and λ_0 . To simplify the notation, we will ignore the factor H from the likelihood function. Then for i.i.d. observations (Y_i, Δ_i, X_i) , $i = 1, \dots, n$, from (1.2) we obtain the log likelihood function for β and λ as

$$(1.3) \quad l_n(\beta, \lambda) = n^{-1} \sum_{i=1}^n \left\{ \Delta_i \log\{\lambda(Y_i - X_i' \beta)\} - \int I(Y_i \geq t) \lambda(t - X_i' \beta) dt \right\}.$$

The log likelihood given in (1.3) apparently is a semiparametric model, where the argument of the nuisance parameter λ involves β ; thus β and λ are bundled parameters. To keep the positivity of λ , let $g(\cdot) = \log \lambda(\cdot)$. Then the log likelihood function for β and g , using the counting process notation, can be written as

$$(1.4) \quad l_n(\beta, g) = n^{-1} \sum_{i=1}^n \left\{ \int g(t - X_i' \beta) dN_i(t) - \int I(Y_i \geq t) e^{g(t - X_i' \beta)} dt \right\},$$

where $N_i(t) = \Delta_i I(Y_i \leq t)$ is the counting process for subject i .

We propose a new approach by directly maximizing the log likelihood function in a sieve space in which function $g(\cdot)$ is approximated by B-splines. Numerically, the estimator can be easily obtained by the Newton–Raphson algorithm or

any gradient-based search algorithms. We show that the proposed estimator is consistent and asymptotically normal, and the limiting covariance matrix reaches the semiparametric efficiency bound, which can be estimated either by inverting the information matrix based on the efficient score function of the regression parameters derived by [22], or by inverting the observed information matrix of all parameters, taking into account that we are also estimating the nuisance parameters in the sieve space for the log hazard function.

2. The sieve M-theorem on the asymptotic normality of semiparametric estimation for bundled parameters. In this section, we extend the general theorem introduced by [30], which deals with the asymptotic normality of semiparametric M-estimators of regression parameters when the convergence rate of the estimator for nuisance parameters can be slower than $n^{-1/2}$. In their theorem, the parameters of interest and the nuisance parameters are assumed to be separated. We consider a more general setting where the nuisance parameter can be a function of the parameters of interest. The theorem is crucial in the proof of asymptotic normality given in Theorem 4.2 for our proposed estimators.

Some empirical process notation will be used from now on. We denote $Pf = \int f(z) dP(z)$ and $\mathbb{P}_n f = n^{-1} \sum_{i=1}^n f(Z_i)$, where P is a probability measure, and \mathbb{P}_n is an empirical probability measure, and denote $\mathbb{G}_n f = n^{1/2}(\mathbb{P}_n - P)f$. Given i.i.d. observations $Z_1, Z_2, \dots, Z_n \in \mathcal{Z}$, we estimate the unknown parameters $(\beta, \zeta(\cdot, \beta))$ by maximizing an objective function for $(\beta, \zeta(\cdot, \beta))$, $n^{-1} \sum_{i=1}^n m(\beta, \zeta(\cdot, \beta); Z_i) = \mathbb{P}_n m(\beta, \zeta(\cdot, \beta); Z)$, where β is the parameter of interest, and $\zeta(\cdot, \beta)$ is the nuisance parameter that can be a function of β . Here “.” denotes the other arguments of ζ besides β , which can be some components of $Z \in \mathcal{Z}$. If the objective function m is the log-likelihood function of a single observation, then the estimator becomes the semiparametric maximum likelihood estimator. Here we adopt similar notation in [30].

Let $\theta = (\beta, \zeta(\cdot, \beta))$, $\beta \in \mathcal{B} \subset \mathbb{R}^d$ and $\zeta \in \mathcal{H}$, where \mathcal{B} is the parameter space of β , and \mathcal{H} is a class of functions mapping from $\mathcal{Z} \times \mathcal{B}$ to \mathbb{R} . Let $\Theta = \mathcal{B} \times \mathcal{H}$ be the parameter space of θ . Define a distance between $\theta_1, \theta_2 \in \Theta$ by

$$d(\theta_1, \theta_2) = \{|\beta_2 - \beta_1|^2 + \|\zeta_2(\cdot, \beta_2) - \zeta_1(\cdot, \beta_1)\|^2\}^{1/2},$$

where $|\cdot|$ is the Euclidean distance, and $\|\cdot\|$ is some norm. Let Θ_n be the sieve parameter space, a sequence of increasing subsets of the parameter space Θ growing dense in Θ as $n \rightarrow \infty$. We aim to find $\hat{\theta}_n \in \Theta_n$ such that $d(\hat{\theta}_n, \theta_0) = o_p(1)$ and $\hat{\beta}_n$ is asymptotically normal.

For any fixed $\zeta(\cdot, \beta) \in \mathcal{H}$, let $\{\zeta_\eta(\cdot, \beta) : \eta \text{ in a neighborhood of } 0 \in \mathbb{R}\}$ be a smooth curve in \mathcal{H} running through $\zeta(\cdot, \beta)$ at $\eta = 0$, that is, $\zeta_\eta(\cdot, \beta)|_{\eta=0} = \zeta(\cdot, \beta)$. Assume all $\zeta(\cdot, \beta) \in \mathcal{H}$ are at least twice-differentiable with respect to β , and denote

$$\mathbb{H} = \left\{ h : h(\cdot, \beta) = \left. \frac{\partial \zeta_\eta(\cdot, \beta)}{\partial \eta} \right|_{\eta=0}, \zeta_\eta \in \mathcal{H}, \beta \in \mathcal{B} \right\}.$$

Assume the objective function m is twice Frechet differentiable. Since for a small δ , we have $\zeta(\cdot, \beta + \delta) - \zeta(\cdot, \beta) = \dot{\zeta}_\beta(\cdot, \beta)\delta + o(\delta)$, here $\dot{\zeta}_\beta(\cdot, \beta) = \partial\zeta(\cdot, \beta)/\partial\beta$; then by the definition of functional derivatives it follows that

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \frac{1}{\delta} \{m(\beta, \zeta(\cdot, \beta + \delta); z) - m(\beta, \zeta(\cdot, \beta); z)\} \\ &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \{m(\beta, \zeta(\cdot, \beta) + \dot{\zeta}_\beta(\cdot, \beta)\delta + o(\delta); z) \\ &\quad - m(\beta, \zeta(\cdot, \beta) + \dot{\zeta}_\beta(\cdot, \beta)\delta; z)\} \\ &\quad + \lim_{\delta \rightarrow 0} \frac{1}{\delta} \{m(\beta, \zeta(\cdot, \beta) + \dot{\zeta}_\beta(\cdot, \beta)\delta; z) - m(\beta, \zeta(\cdot, \beta); z)\} \\ &= \lim_{\delta \rightarrow 0} \dot{m}_2(\beta, \zeta(\cdot, \beta) + \dot{\zeta}_\beta(\cdot, \beta)\delta; z)[o(\delta)/\delta] \\ &\quad + \dot{m}_2(\beta, \zeta(\cdot, \beta); z)[\dot{\zeta}_\beta(\cdot, \beta)] \\ &= \dot{m}_2(\beta, \zeta(\cdot, \beta); z)[\dot{\zeta}_\beta(\cdot, \beta)], \end{aligned}$$

where the subscript 2 indicates that the derivatives are taken with respect to the second argument of the function. The last equality holds because

$$\lim_{\delta \rightarrow 0} \dot{m}_2(\beta, \zeta(\cdot, \beta) + \dot{\zeta}_\beta(\cdot, \beta)\delta; z)[o(\delta)/\delta] = 0.$$

Similarly we have

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \frac{1}{\delta} \{\dot{m}_2(\beta, \zeta(\cdot, \beta + \delta); z)[h(\cdot, \beta)] - \dot{m}_2(\beta, \zeta(\cdot, \beta); z)[h(\cdot, \beta)]\} \\ &= \ddot{m}_{22}(\beta, \zeta(\cdot, \beta); z)[h(\cdot, \beta), \dot{\zeta}_\beta(\cdot, \beta)] \end{aligned}$$

and

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \frac{1}{\delta} \{\dot{m}_2(\beta, \zeta(\cdot, \beta); z)[h(\cdot, \beta + \delta)] - \dot{m}_2(\beta, \zeta(\cdot, \beta); z)[h(\cdot, \beta)]\} \\ &= \dot{m}_2(\beta, \zeta(\cdot, \beta); z)[\dot{h}_\beta(\cdot, \beta)]. \end{aligned}$$

Thus according to the chain rule of the functional derivatives, we have

$$\begin{aligned} \dot{m}_\beta(\beta, \zeta(\cdot, \beta); z) &= \frac{\partial m(\beta, \zeta(\cdot, \beta); z)}{\partial \beta} \\ &= \dot{m}_1(\beta, \zeta(\cdot, \beta); z) + \dot{m}_2(\beta, \zeta(\cdot, \beta); z)[\dot{\zeta}_\beta(\cdot, \beta)], \\ \dot{m}_\zeta(\beta, \zeta(\cdot, \beta); z)[h] &= \left. \frac{\partial m(\beta, (\zeta + \eta h)(\cdot, \beta); z)}{\partial \eta} \right|_{\eta=0} \\ &= \dot{m}_2(\beta, \zeta(\cdot, \beta); z)[h(\cdot, \beta)], \end{aligned}$$

$$\begin{aligned}
 \ddot{m}_{\beta\beta}(\beta, \zeta(\cdot, \beta); z) &= \frac{\partial^2 m(\beta, \zeta(\cdot, \beta); z)}{\partial\beta \partial\beta'} = \frac{\partial \dot{m}_\beta(\beta, \zeta(\cdot, \beta); z)}{\partial\beta'} \\
 &= \ddot{m}_{11}(\beta, \zeta(\cdot, \beta); z) + \ddot{m}_{12}(\beta; \zeta(\cdot, \beta); z)[\dot{\zeta}_\beta(\cdot, \beta)] \\
 &\quad + \ddot{m}_{21}(\beta, \zeta(\cdot, \beta); z)[\dot{\zeta}_\beta(\cdot, \beta)] \\
 &\quad + \ddot{m}_{22}(\beta, \zeta(\cdot, \beta); z)[\dot{\zeta}_\beta(\cdot, \beta), \dot{\zeta}_\beta(\cdot, \beta)] \\
 &\quad + \dot{m}_2(\beta, \zeta(\cdot, \beta); z)[\ddot{\zeta}_{\beta\beta}(\cdot, \beta)], \\
 \ddot{m}_{\beta\zeta}(\beta, \zeta(\cdot, \beta); z)[h] &= \left. \frac{\partial \dot{m}_\beta(\beta, (\zeta + \eta h)(\cdot, \beta); z)}{\partial\eta} \right|_{\eta=0} \\
 &= \ddot{m}_{12}(\beta, \zeta(\cdot, \beta); z)[h(\cdot, \beta)] \\
 &\quad + \ddot{m}_{22}(\beta, \zeta(\cdot, \beta); z)[\dot{\zeta}_\beta(\cdot, \beta), h(\cdot, \beta)] \\
 &\quad + \dot{m}_2(\beta, \zeta(\cdot, \beta); z)[\dot{h}_\beta(\cdot, \beta)], \\
 \ddot{m}_{\zeta\beta}(\beta, \zeta(\cdot, \beta); z)[h] &= \frac{\partial \dot{m}_2(\beta, \zeta(\cdot, \beta); z)[h(\cdot, \beta)]}{\partial\beta} \\
 &= \ddot{m}_{21}(\beta, \zeta(\cdot, \beta); z)[h(\cdot, \beta)] \\
 &\quad + \ddot{m}_{22}(\beta, \zeta(\cdot, \beta); z)[h(\cdot, \beta), \dot{\zeta}_\beta(\cdot, \beta)] \\
 &\quad + \dot{m}_2(\beta, \zeta(\cdot, \beta); z)[\dot{h}_\beta(\cdot, \beta)], \\
 \ddot{m}_{\zeta\zeta}(\beta, \zeta(\cdot, \beta); z)[h_1, h_2] &= \ddot{m}_{22}(\beta, \zeta(\cdot, \beta); z)[h_1(\cdot, \beta), h_2(\cdot, \beta)].
 \end{aligned}$$

As noted before, the subscript 1 or 2 in the derivatives indicates that the derivatives are taken with respect to the first or the second argument of the function, and h inside the square brackets is a function denoting the direction of the functional derivative with respect to ζ . Note that for the second derivatives $\ddot{m}_{\beta\zeta}$ and $\ddot{m}_{\zeta\beta}$, we implicitly require the direction h to be a differentiable function with respect to β . It is easily seen that when ζ is free of β , all the above derivatives reduce to that in [30]. Following [30], we also define

$$\begin{aligned}
 \dot{S}_\beta(\beta, \zeta(\cdot, \beta)) &= P\dot{m}_\beta(\beta, \zeta(\cdot, \beta); Z), \\
 \dot{S}_\zeta(\beta, \zeta(\cdot, \beta))[h] &= P\dot{m}_\zeta(\beta, \zeta(\cdot, \beta); Z)[h], \\
 \dot{S}_{\beta,n}(\beta, \zeta(\cdot, \beta)) &= \mathbb{P}_n\dot{m}_\beta(\beta, \zeta(\cdot, \beta); Z), \\
 \dot{S}_{\zeta,n}(\beta, \zeta(\cdot, \beta))[h] &= \mathbb{P}_n\dot{m}_\zeta(\beta, \zeta(\cdot, \beta); Z)[h], \\
 \ddot{S}_{\beta\beta}(\beta, \zeta(\cdot, \beta)) &= P\ddot{m}_{\beta\beta}(\beta, \zeta(\cdot, \beta); Z), \\
 \ddot{S}_{\zeta\zeta}(\beta, \zeta(\cdot, \beta))[h, h] &= P\ddot{m}_{\zeta\zeta}(\beta, \zeta(\cdot, \beta); Z)[h, h]
 \end{aligned}$$

and

$$\ddot{S}_{\beta\zeta}(\beta, \zeta(\cdot, \beta))[h] = \ddot{S}'_{\zeta\beta}(\beta, \zeta(\cdot, \beta))[h] = P\ddot{m}_{\beta\zeta}(\beta, \zeta(\cdot, \beta); Z)[h].$$

Furthermore, for $\mathbf{h} = (h_1, h_2, \dots, h_d)' \in \mathbb{H}^d$, we denote

$$\begin{aligned} \dot{m}_\zeta(\beta, \zeta(\cdot, \beta); z)[\mathbf{h}] &= (\dot{m}_\zeta(\beta, \zeta(\cdot, \beta); z)[h_1], \dots, \dot{m}_\zeta(\beta, \zeta(\cdot, \beta); z)[h_d])', \\ \ddot{m}_{\beta\zeta}(\beta, \zeta(\cdot, \beta); z)[\mathbf{h}] &= (\ddot{m}_{\beta\zeta}(\beta, \zeta(\cdot, \beta); z)[h_1], \dots, \ddot{m}_{\beta\zeta}(\beta, \zeta(\cdot, \beta); z)[h_d]), \\ \ddot{m}_{\zeta\beta}(\beta, \zeta(\cdot, \beta); z)[\mathbf{h}] &= (\ddot{m}_{\zeta\beta}(\beta, \zeta(\cdot, \beta); z)[h_1], \dots, \ddot{m}_{\zeta\beta}(\beta, \zeta(\cdot, \beta); z)[h_d])', \\ \ddot{m}_{\zeta\zeta}(\beta, \zeta(\cdot, \beta); z)[\mathbf{h}, h] &= (\ddot{m}_{\zeta\zeta}(\beta, \zeta(\cdot, \beta); z)[h_1, h], \dots, \\ &\quad \ddot{m}_{\zeta\zeta}(\beta, \zeta(\cdot, \beta); z)[h_d, h])' \end{aligned}$$

and define correspondingly

$$\begin{aligned} \dot{S}_\zeta(\beta, \zeta(\cdot, \beta))[\mathbf{h}] &= P\dot{m}_\zeta(\beta, \zeta(\cdot, \beta); Z)[\mathbf{h}], \\ \dot{S}_{\zeta,n}(\beta, \zeta(\cdot, \beta))[\mathbf{h}] &= \mathbb{P}_n\dot{m}_\zeta(\beta, \zeta(\cdot, \beta); Z)[\mathbf{h}], \\ \ddot{S}_{\beta\zeta}(\beta, \zeta(\cdot, \beta))[\mathbf{h}] &= P\ddot{m}_{\beta\zeta}(\beta, \zeta(\cdot, \beta); Z)[\mathbf{h}], \\ \ddot{S}_{\zeta\beta}(\beta, \zeta(\cdot, \beta))[\mathbf{h}] &= P\ddot{m}_{\zeta\beta}(\beta, \zeta(\cdot, \beta); Z)[\mathbf{h}], \\ \ddot{S}_{\zeta\zeta}(\beta, \zeta(\cdot, \beta))[\mathbf{h}, h] &= P\ddot{m}_{\zeta\zeta}(\beta, \zeta(\cdot, \beta); Z)[\mathbf{h}, h]. \end{aligned}$$

To obtain the asymptotic normality result for the sieve M-estimator $\hat{\beta}_n$, the assumptions we will make in the following look similar to those in [30], but all the derivatives with respect to β involve the chain rule and hence are more complicated, which is the key difference to [30]. Additionally, we focus on sieve estimators in the sieve parameter space. We list the following assumptions:

(A1) (Rate of convergence) For an estimator $\hat{\theta}_n = (\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n)) \in \Theta_n$ and the true parameter $\theta_0 = (\beta_0, \zeta_0(\cdot, \beta_0)) \in \Theta$, $d(\hat{\theta}_n, \theta_0) = O_p(n^{-\xi})$ for some $\xi > 0$.

(A2) $\dot{S}_\beta(\beta_0, \zeta_0(\cdot, \beta_0)) = 0$ and $\dot{S}_\zeta(\beta_0, \zeta_0(\cdot, \beta_0))[h] = 0$ for all $h \in \mathbb{H}$.

(A3) (Positive information) There exists an $\mathbf{h}^* = (h_1^*, \dots, h_d^*)'$, where $h_j^* \in \mathbb{H}$ for $j = 1, \dots, d$, such that

$$\ddot{S}_{\beta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0))[h] - \ddot{S}_{\zeta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*, h] = 0$$

for all $h \in \mathbb{H}$. Furthermore, the matrix

$$\begin{aligned} A &= -\ddot{S}_{\beta\beta}(\beta_0, \zeta_0(\cdot, \beta_0)) + \ddot{S}_{\zeta\beta}(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*] \\ &= -P\{\ddot{m}_{\beta\beta}(\beta_0, \zeta_0(\cdot, \beta_0); Z) - \ddot{m}_{\zeta\beta}(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*]\} \end{aligned}$$

is nonsingular.

(A4) The estimator $(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n))$ satisfies

$$\dot{S}_{\beta,n}(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n)) = o_p(n^{-1/2}) \quad \text{and} \quad \dot{S}_{\zeta,n}(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n))[\mathbf{h}^*] = o_p(n^{-1/2}).$$

(A5) (Stochastic equicontinuity) For some $C > 0$,

$$\begin{aligned} \sup_{d(\theta, \theta_0) \leq Cn^{-\xi}, \theta \in \Theta_n} |\sqrt{n}(\dot{S}_{\beta,n} - \dot{S}_\beta)(\beta, \zeta(\cdot, \beta)) \\ - \sqrt{n}(\dot{S}_{\beta,n} - \dot{S}_\beta)(\beta_0, \zeta_0(\cdot, \beta_0))| = o_p(1) \end{aligned}$$

and

$$\sup_{d(\theta, \theta_0) \leq Cn^{-\xi}, \theta \in \Theta_n} |\sqrt{n}(\dot{S}_{\zeta, n} - \dot{S}_{\zeta})(\beta, \zeta(\cdot, \beta))[\mathbf{h}^*(\cdot, \beta)] - \sqrt{n}(\dot{S}_{\zeta, n} - \dot{S}_{\zeta})(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*(\cdot, \beta_0)]| = o_p(1).$$

(A6) (Smoothness of the model) For some $\alpha > 1$ satisfying $\alpha\xi > 1/2$, and for θ in a neighborhood of $\theta_0 : \{\theta : d(\theta, \theta_0) \leq Cn^{-\xi}, \theta \in \Theta_n\}$,

$$\begin{aligned} &|\dot{S}_{\beta}(\beta, \zeta(\cdot, \beta)) - \dot{S}_{\beta}(\beta_0, \zeta_0(\cdot, \beta_0)) - \ddot{S}_{\beta\beta}(\beta_0, \zeta_0(\cdot, \beta_0))(\beta - \beta_0) \\ &\quad - \ddot{S}_{\beta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0))[\zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)]| \\ &= O(d^\alpha(\theta, \theta_0)) \end{aligned}$$

and

$$\begin{aligned} &|\dot{S}_{\zeta}(\beta, \zeta(\cdot, \beta))[\mathbf{h}^*(\cdot, \beta)] - \dot{S}_{\zeta}(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*(\cdot, \beta_0)] \\ &\quad - \ddot{S}_{\zeta\beta}(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*(\cdot, \beta_0)](\beta - \beta_0) \\ &\quad - \ddot{S}_{\zeta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*(\cdot, \beta_0), \zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)]| \\ &= O(d^\alpha(\theta, \theta_0)). \end{aligned}$$

Note that ξ in (A1) depends on the entropy of the sieve parameter space for ζ and cannot be arbitrarily small; it is controlled by the smoothness of the model in (A6). The convergence rate in (A1) needs to be achieved prior to obtaining asymptotic normality. Assumption (A2) is a common assumption for the maximum likelihood estimation and usually holds. The direction \mathbf{h}^* in (A3) may be found through the equation in (A3). It is the least favorable direction when m is the likelihood function. Assumptions (A4) and (A5) are usually verified either by the Donsker property or the maximal inequality of [28]. Assumption (A6) can be obtained by a Taylor expansion. The following theorem is an extension to Theorem 6.1 in [30] when the infinite-dimensional parameter ζ is a function of the finite-dimensional parameter β .

THEOREM 2.1. *Suppose that assumptions (A1)–(A6) hold. Then*

$$\begin{aligned} \sqrt{n}(\hat{\beta}_n - \beta_0) &= A^{-1}\sqrt{n}\mathbb{P}_n m^*(\beta_0, \zeta_0(\cdot, \beta_0); Z) + o_p(1) \\ &\rightarrow_d N(0, A^{-1}B(A^{-1})'), \end{aligned}$$

where

$$\begin{aligned} m^*(\beta_0, \zeta_0(\cdot, \beta_0); z) &= \dot{m}_{\beta}(\beta_0, \zeta_0(\cdot, \beta_0); z) - \dot{m}_{\zeta}(\beta_0, \zeta_0(\cdot, \beta_0); z)[\mathbf{h}^*], \\ B &= P\{m^*(\beta_0, \zeta_0(\cdot, \beta_0); Z)^{\otimes 2}\}, \end{aligned}$$

and A is given in assumption (A3). Here $a^{\otimes 2} = aa'$.

PROOF. The proof follows similarly along the proof of Theorem 6.1 in [30]. Assumptions (A1) and (A5) yield

$$\sqrt{n}(\dot{S}_{\beta,n} - \dot{S}_{\beta})(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n)) - \sqrt{n}(\dot{S}_{\beta,n} - \dot{S}_{\beta})(\beta_0, \zeta_0(\cdot, \beta_0)) = o_p(1).$$

Since $\dot{S}_{\beta,n}(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n)) = o_p(n^{-1/2})$ by (A4) and $\dot{S}_{\beta}(\beta_0, \zeta_0(\cdot, \beta_0)) = 0$ by (A2), we have

$$\sqrt{n}\dot{S}_{\beta}(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n)) + \sqrt{n}\dot{S}_{\beta,n}(\beta_0, \zeta_0(\cdot, \beta_0)) = o_p(1).$$

Similarly,

$$\sqrt{n}\dot{S}_{\zeta}(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n))[\mathbf{h}^*(\cdot, \hat{\beta}_n)] + \sqrt{n}\dot{S}_{\zeta,n}(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*(\cdot, \beta_0)] = o_p(1).$$

Combining these equalities and assumption (A6) yields

$$\begin{aligned} & \ddot{S}_{\beta\beta}(\beta_0, \zeta_0(\cdot, \beta_0))(\hat{\beta}_n - \beta_0) + \ddot{S}_{\beta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0))[\hat{\zeta}_n(\cdot, \hat{\beta}_n) - \zeta_0(\cdot, \beta_0)] \\ (2.1) \quad & + \dot{S}_{\beta,n}(\beta_0, \zeta_0(\cdot, \beta_0)) + O(d^\alpha(\hat{\theta}_n, \theta_0)) \\ & = o_p(n^{-1/2}) \end{aligned}$$

and

$$\begin{aligned} & \ddot{S}_{\zeta\beta}(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*(\cdot, \beta_0)](\hat{\beta}_n - \beta_0) \\ (2.2) \quad & + \ddot{S}_{\zeta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*(\cdot, \beta_0), \hat{\zeta}_n(\cdot, \hat{\beta}_n) - \zeta_0(\cdot, \beta_0)] \\ & + \dot{S}_{\zeta,n}(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*(\cdot, \beta_0)] + O(d^\alpha(\hat{\theta}_n, \theta_0)) \\ & = o_p(n^{-1/2}). \end{aligned}$$

Since $\alpha > 1$ with $\alpha\xi > 1/2$, the rate of convergence assumption (A1) implies $\sqrt{n}O(d^\alpha(\hat{\theta}_n, \theta_0)) = o_p(1)$, then (2.1) and (2.2) together with (A3) yields

$$\begin{aligned} & (\ddot{S}_{\beta\beta}(\beta_0, \zeta_0(\cdot, \beta_0)) - \ddot{S}_{\zeta\beta}(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*(\cdot, \beta_0)])(\hat{\beta}_n - \beta_0) \\ & = -(\dot{S}_{\beta,n}(\beta_0, \zeta_0(\cdot, \beta_0)) - \dot{S}_{\zeta,n}(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*(\cdot, \beta_0)]) + o_p(n^{-1/2}), \end{aligned}$$

that is,

$$-A(\hat{\beta}_n - \beta_0) = -\mathbb{P}_n m^*(\beta_0, \zeta_0(\cdot, \beta_0); Z) + o_p(n^{-1/2}).$$

This yields

$$\begin{aligned} \sqrt{n}(\hat{\beta}_n - \beta_0) &= A^{-1}\sqrt{n}\mathbb{P}_n m^*(\beta_0, \zeta_0(\cdot, \beta_0); Z) + o_p(1) \\ &\rightarrow_d N(0, A^{-1}B(A^{-1})'). \end{aligned}$$

□

3. Back to the linear model: The sieve maximum likelihood estimation.

By taking logarithm to the positive function $\lambda(\cdot)$ in (1.3), the function $g(\cdot)$ in (1.4) is no longer restricted to be positive, which eases the estimation. We now describe the spline-based sieve maximum likelihood estimation for model (1.1). Under the regularity conditions (C.1)–(C.3) stated in Section 4, we know that the observed residual times $\{Y_i - X_i'\beta : \beta \in \mathcal{B}, i = 1, \dots, n\}$ are confined in some finite interval. Let $[a, b]$ be an interval of interest, where $-\infty < a < b < \infty$. Let $T_{K_n} = \{t_1, \dots, t_{K_n}\}$ be a set of partition points of $[a, b]$ with $K_n = O(n^\nu)$ and $\max_{1 \leq j \leq K_n+1} |t_j - t_{j-1}| = O(n^{-\nu})$ for some constant $\nu \in (0, 1/2)$. Let $\mathcal{S}_n(T_{K_n}, K_n, p)$ be the space of polynomial splines of order $p \geq 1$ defined in [23], Definition 4.1. According to Schumaker ([23], Corollary 4.10), there exists a set of B-spline basis functions $\{B_j, 1 \leq j \leq q_n\}$ with $q_n = K_n + p$ such that for any $s \in \mathcal{S}_n(T_{K_n}, K_n, p)$, we can write

$$(3.1) \quad s(t) = \sum_{j=1}^{q_n} \gamma_j B_j(t),$$

where we follow [25] by requiring $\max_{j=1, \dots, q_n} |\gamma_j| \leq c_n$ that is allowed to grow with n slowly enough.

Let $\gamma = (\gamma_1, \dots, \gamma_{q_n})'$. Under suitable smoothness assumptions, $g_0(\cdot) = \log \lambda_0(\cdot)$ can be well approximated by some function in $\mathcal{S}_n(T_{K_n}, K_n, p)$. Therefore, we seek a member of $\mathcal{S}_n(T_{K_n}, K_n, p)$ together with a value of $\beta \in \mathcal{B}$ that maximizes the log likelihood function. Specifically, let $\hat{\theta}_n = (\hat{\beta}_n, \hat{\gamma}_n)$ be the value that maximizes

$$(3.2) \quad l_n(\beta, \gamma) = n^{-1} \sum_{i=1}^n \left[\int \sum_{j=1}^{q_n} \gamma_j B_j(t - X_i'\beta) dN_i(t) - \int I(Y_i \geq t) \exp \left\{ \sum_{j=1}^{q_n} \gamma_j B_j(t - X_i'\beta) \right\} dt \right].$$

Taking the first order derivatives of $l_n(\beta, \gamma)$ with respect to β and γ and setting them to zero, we can obtain the score equations. Since the integrals here are univariate integrals, their numerical implementation can be easily done by the one-dimensional Gaussian-quadrature method. A Newton–Raphson algorithm or any other gradient-based search algorithm can be applied to solve the score equations for all parameters $\theta = (\beta, \gamma)$, for example,

$$\theta^{(m+1)} = \theta^{(m)} - H(\theta^{(m)})^{-1} \cdot S(\theta^{(m)}),$$

where $\theta^{(m)} = (\beta^{(m)}, \gamma^{(m)})$ is the parameter estimate from the m th iteration, and

$$S(\theta) = \begin{pmatrix} \frac{\partial l_n(\beta, \gamma)}{\partial \beta} \\ \frac{\partial l_n(\beta, \gamma)}{\partial \gamma} \end{pmatrix}, \quad H(\theta) = \begin{pmatrix} \frac{\partial^2 l_n(\beta, \gamma)}{\partial \beta \partial \beta'} & \frac{\partial^2 l_n(\beta, \gamma)}{\partial \beta \partial \gamma'} \\ \frac{\partial^2 l_n(\beta, \gamma)}{\partial \gamma \partial \beta'} & \frac{\partial^2 l_n(\beta, \gamma)}{\partial \gamma \partial \gamma'} \end{pmatrix}$$

are the score function and Hessian matrix of parameter θ . For any fixed β and n , it is clearly seen that $l_n(\beta, \gamma)$ in (3.2) is concave with respect to γ and goes to $-\infty$ if any γ_j approaches either ∞ or $-\infty$; hence $\hat{\gamma}_n$ must be bounded which yields an estimator of s in $\mathcal{S}_n(T_{K_n}, K_n, p)$.

As stated in the next section, the distribution of $\hat{\beta}_n$ can be approximated by a normal distribution. One way to estimate the variance matrix of $\hat{\beta}_n$ is to approximate the (inverse of the) information matrix based on the efficient score function for β_0 by plugging in the estimated parameters $(\hat{\beta}_n, \hat{\lambda}_n(\cdot))$. The consistency of such a variance estimator is given in Theorem 4.3. Another way is to invert the observed information matrix from the last Newton–Raphson iteration, taking into account that we are also estimating the nuisance parameter γ . The consistency of the latter approach may be proved in a similar way as Example 4 in [24] or via Theorem 2.2 in [9], and we leave detailed derivation to interested readers. Simulations indicate that both estimators work reasonably well.

4. Asymptotic results. Denote $\epsilon_\beta = Y - X'\beta$ and $\epsilon_0 = Y - X'\beta_0$. We assume the following regularity conditions:

(C.1) The true parameter β_0 belongs to the interior of a compact set $\mathcal{B} \subseteq \mathbb{R}^d$.

(C.2) (a) The covariate X takes values in a bounded subset $\mathcal{X} \subseteq \mathbb{R}^d$;

(b) $E(XX')$ is nonsingular.

(C.3) There is a truncation time $\tau < \infty$ such that, for some constant δ , $P(\epsilon_0 > \tau|X) \geq \delta > 0$ almost surely with respect to the probability measure of X . This implies that $\Lambda_0(\tau) \leq -\log \delta < \infty$.

(C.4) The error e_0 's density f and its derivative \dot{f} are bounded and

$$\int (\dot{f}(t)/f(t))^2 f(t) dt < \infty.$$

(C.5) The conditional density of C given X and its derivative $\dot{g}_{C|X}$ are uniformly bounded for all possible values of X , that is,

$$\sup_{x \in \mathcal{X}} g_{C|X}(t|X = x) \leq K_1, \quad \sup_{x \in \mathcal{X}} |\dot{g}_{C|X}(t|X = x)| \leq K_2$$

for all $t \leq \tau$ with some constants $K_1, K_2 > 0$, where τ is the truncation time defined in condition (C.3).

(C.6) Let \mathcal{G}^p denote the collection of bounded functions g on $[a, b]$ with bounded derivatives $g^{(j)}$, $j = 1, \dots, k$, and the k th derivative $g^{(k)}$ satisfies the following Lipschitz continuity condition:

$$|g^{(k)}(s) - g^{(k)}(t)| \leq L|s - t|^m \quad \text{for } s, t \in [a, b],$$

where k is a positive integer and $m \in (0, 1]$ such that $p = k + m \geq 3$, and $L < \infty$ is an unknown constant. The true log hazard function $g_0(\cdot) = \log \lambda_0(\cdot)$ belongs to \mathcal{G}^p , where $[a, b]$ is a bounded interval.

(C.7) For some $\eta \in (0, 1)$, $u' \text{Var}(X|\epsilon_0)u \geq \eta u' E(XX'|\epsilon_0)u$ almost surely for all $u \in \mathbb{R}^d$.

Condition (C.1) is a common regularity assumption that has been imposed in the literature; see, for example, [17]. Conditions (C.2)(a), (C.3) and (C.4) were also assumed in [26]. Condition (C.5) implies Condition *B* in [26]. In condition (C.6), we require $p \geq 3$ to provide desirable controls of the spline approximation error rates of the first and second derivatives of g_0 (see Corollary 6.21 of [23]), which are needed in verifying assumptions (A4)–(A6). Condition (C.7) was also proposed for the panel count data model in [30]. As noted in their Remark 3.4, this condition (C.7) can be justified in many applications when condition (C.2)(b) is satisfied. The bounded interval $[a, b]$ in (C.6) may be chosen as $a = \inf_{y,x}(y - x'\beta_0) > -\infty$ and $b = \tau < \infty$ under (C.1)–(C.3), which is what we use in the following.

Now define the collection of functions \mathcal{H}^p as follows:

$$\mathcal{H}^p = \{\zeta(\cdot, \beta) : \zeta(t, x, \beta) = g(\psi(t, x, \beta)), g \in \mathcal{G}^p, t \in [a, b], x \in \mathcal{X}, \beta \in \mathcal{B}\},$$

where

$$\psi(t, x, \beta) = t - x'(\beta - \beta_0)$$

and \mathcal{G}^p is defined in (C.6). Here ζ is a composite function of g composed with ψ . Note that $\zeta(t, x, \beta_0) = g(t)$. Then for $\zeta(\cdot, \beta) \in \mathcal{H}^p$ we define the following norm:

$$(4.1) \quad \|\zeta(\cdot, \beta)\|_2 = \left\{ \int_{\mathcal{X}} \int_a^b \{g(t - x'(\beta - \beta_0))\}^2 d\Lambda_0(t) dF_X(x) \right\}^{1/2}.$$

We also have the following collection of scores:

$$\mathbb{H} = \left\{ h : h(\cdot, \beta) = \frac{\partial \zeta_\eta(\cdot, \beta)}{\partial \eta} \Big|_{\eta=0} = w(\psi(\cdot, \beta)), \zeta_\eta \in \mathcal{H}^p \right\},$$

in which $h(t, x, \beta) = w(\psi(t, x, \beta)) = w(t - x'(\beta - \beta_0))$.

For any $\theta_1 = (\beta_1, \zeta_1(\cdot, \beta_1))$ and $\theta_2 = (\beta_2, \zeta_2(\cdot, \beta_2))$ in the space of $\Theta^p = \mathcal{B} \times \mathcal{H}^p$, define the following distance:

$$(4.2) \quad d(\theta_1, \theta_2) = \{|\beta_1 - \beta_2|^2 + \|\zeta_1(\cdot, \beta_1) - \zeta_2(\cdot, \beta_2)\|_2^2\}^{1/2}.$$

Let $\mathcal{G}_n^p = \mathcal{S}_n(T_{K_n}, K_n, p)$. Denote

$$\mathcal{H}_n^p = \{\zeta(\cdot, \beta) : \zeta(t, x, \beta) = g(\psi(t, x, \beta)), g \in \mathcal{G}_n^p, t \in [a, b], x \in \mathcal{X}, \beta \in \mathcal{B}\}$$

and $\Theta_n^p = \mathcal{B} \times \mathcal{H}_n^p$. Clearly $\mathcal{H}_n^p \subseteq \mathcal{H}_{n+1}^p \subseteq \dots \subseteq \mathcal{H}^p$ for all $n \geq 1$. The sieve estimator $\hat{\theta}_n = (\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n))$, where $\hat{\zeta}_n(t, x, \hat{\beta}_n) = \hat{g}_n(t - x'(\hat{\beta}_n - \beta_0))$, is the maximizer of the empirical log-likelihood $n^{-1}l_n(\theta; Z)$ over the sieve space Θ_n^p . The following theorem gives the convergence rate of the proposed estimator $\hat{\theta}_n$ to the true parameter $\theta_0 = (\beta_0, \zeta_0(\cdot, \beta_0)) = (\beta_0, g_0)$.

THEOREM 4.1. *Let $K_n = O(n^\nu)$, where ν satisfies the restriction $\frac{1}{2(1+p)} < \nu < \frac{1}{2p}$ with p being the smoothness parameter defined in condition (C.6). Suppose conditions (C.1)–(C.7) hold, and the failure time T follows model (1.1). Then*

$$d(\hat{\theta}_n, \theta_0) = O_p\{n^{-\min(p\nu, (1-\nu)/2)}\},$$

where $d(\cdot, \cdot)$ is defined in (4.2).

REMARK. It is worth pointing out that the sieve space \mathcal{G}_n^p does not have to be restricted to the B-spline space; it can be any sieve space as long as the estimator $\hat{\theta}_n \in \mathcal{B} \times \mathcal{H}_n^p$ satisfies the conditions of Theorem 1 in [25]. We refer to [4] for a comprehensive discussion of the sieve estimation for semiparametric models in general sieve spaces. Our choice of the B-spline space is primarily motivated by its simplicity of numerical implementation, which is a tremendous advantage of the proposed approach over exiting numerical methods for the accelerated failure time models, in particular, the linear programming approach.

We provide a proof of Theorem 4.1 in the supplementary material [8] by checking the conditions of Theorem 1 in [25]. Theorem 4.1 implies that if $\nu = 1/(1 + 2p)$, $d(\hat{\theta}_n, \theta_0) = O_p(n^{-p/(1+2p)})$ which is the optimal convergence rate in the nonparametric regression setting. Although the overall convergence rate is slower than $n^{-1/2}$, the next theorem states that the proposed estimator of the regression parameter is still asymptotically normal and semiparametrically efficient.

THEOREM 4.2. *Given the following efficient score function for the censored linear model derived by [22]:*

$$l_{\beta_0}^*(Y, \Delta, X) = \int \{X - P(X|Y - X'\beta_0 \geq t)\} \left\{ -\frac{\dot{\lambda}_0}{\lambda_0}(t) \right\} dM(t),$$

where

$$M(t) = \Delta I(Y - X'\beta_0 \leq t) - \int_{-\infty}^t I(Y - X'\beta_0 \geq s) \lambda_0(s) ds$$

is the failure counting process martingale, and

$$P(X|Y - X'\beta_0 \geq t) = \frac{P\{XI(Y - X'\beta_0 \geq t)\}}{P\{I(Y - X'\beta_0 \geq t)\}}$$

was shown by [21]. Suppose that the conditions in Theorem 4.1 hold, and $I(\beta_0) = P\{l_{\beta_0}^*(Y, \Delta, X)^{\otimes 2}\}$ is nonsingular, then

$$n^{1/2}(\hat{\beta}_n - \beta_0) = n^{-1/2}I^{-1}(\beta_0) \sum_{i=1}^n l_{\beta_0}^*(Y_i, \Delta_i, X_i) + o_p(1) \rightarrow N(0, I^{-1}(\beta_0))$$

in distribution.

The proof of Theorem 4.2 is where we need to apply our general sieve M-theorem proposed in Section 2. We prove by checking assumptions (A1)–(A6). Details are provided in Section 7. The following theorem gives consistency of the variance estimator based on the above efficient score.

THEOREM 4.3. *Suppose the conditions in Theorem 4.2 hold. Denote*

$$l_{\hat{\beta}_n}^*(Y, \Delta, X) = \int \{X - \bar{X}(t; \hat{\beta}_n)\} \{-\hat{g}_n(t)\} d\hat{M}(t),$$

where

$$\bar{X}(t; \hat{\beta}_n) = \frac{\mathbb{P}_n\{XI(Y - X'\hat{\beta}_n \geq t)\}}{\mathbb{P}_n\{I(Y - X'\hat{\beta}_n \geq t)\}}$$

and

$$\hat{M}(t) = \Delta I(Y - X'\hat{\beta}_n \leq t) - \int_{-\infty}^t I(Y - X'\hat{\beta}_n \geq s) \exp\{\hat{g}_n(s)\} ds.$$

Then $\mathbb{P}_n\{l_{\hat{\beta}_n}^*(Y, \Delta, X)^{\otimes 2}\} \rightarrow P\{l_{\beta_0}^*(Y, \Delta, X)^{\otimes 2}\} = I(\beta_0)$ in probability.

It is clearly seen that $\bar{X}(t, \hat{\beta}_n)$ in Theorem 4.3 estimates $P(X|Y - X'\beta_0 \geq t)$ in Theorem 4.2. The proof of Theorem 4.3 is provided in the supplementary material [8].

5. Numerical examples.

5.1. *Simulations.* Extensive simulations are carried out to evaluate the finite sample performance of the proposed method. In the simulation studies, failure times are generated from the model

$$\log T = 2 + X_1 + X_2 + e_0,$$

where X_1 is Bernoulli with success probability 0.5, X_2 is independent normal with mean 0 and standard deviation 0.5 truncated at ± 2 . This is the same model used by [15] and [32]. We consider six error distributions: standard normal; standard extreme-value; mixtures of $N(0, 1)$ and $N(0, 3^2)$ with mixing probabilities (0.5, 0.5) and (0.95, 0.05), denoted by $0.5N(0, 1) + 0.5N(0, 3^2)$ and $0.95N(0, 1) + 0.05N(0, 3^2)$, respectively; Gumbel($-0.5\mu, 0.5$) with μ being the Euler constant and $0.5N(0, 1) + 0.5N(-1, 0.5^2)$. The first four distributions were also considered by [32]. Similar to [32], the censoring times are generated from uniform $[0, c]$ distribution, where c is chosen to produce a 25% censoring rate. We set the sample size n to 200, 400 and 600.

We choose cubic B-splines with one interior knot for $n = 200$ and 400, and two interior knots for $n = 600$. We perform the sieve maximum likelihood analysis and obtain the estimates of the slope parameters using the Newton–Raphson algorithm

that updates (β, γ) iteratively. We stop iteration when the change of parameter estimates or the gradient value is less than a pre-specified tolerance value that is set to be 10^{-5} in our simulations. Log-rank and Gehan-weighted estimators are included for efficiency comparisons. We calculate the theoretical semiparametric efficiency bound $I^{-1}(\beta_0)$, and scale it by the sample size, that is, $\sigma^* = \sqrt{I^{-1}(\beta_0)/n}$, which serves as the reference standard error under the fully efficient situation. Table 1 summarizes the results of these studies based on 1,000 simulated datasets. The bias of the proposed estimators of β_1 and β_2 are negligible. Both variance estimation procedures, denoted as ${}^1\text{SEE}$ (the standard error estimates by inverting the information matrix based on the efficient score function) and ${}^2\text{SEE}$ (the standard error estimates by inverting the observed information matrix of all parameters including nuisance parameters), yield nice standard error estimates for the parameter estimators comparing to the empirical standard error SE, and the 95% confidence intervals have proper coverage probabilities, especially when the sample size is large. For the $N(0, 1)$ error and the two mixtures of normal errors that are also considered in [32], the proposed estimators are more efficient than the log-rank estimators and have similar variances to the Gehan-weighted estimators. For the standard extreme-value error, the proposed estimators are more efficient than the Gehan-weighted estimator and similar to the log-rank estimator that is known to be the most efficient estimator under this particular error distribution. For the Gumbel $(-0.5\mu, 0.5)$ and $0.5N(0, 1) + 0.5N(-1, 0.5^2)$ errors, the proposed estimators are more efficient than the other two estimators. Under all six error distributions, the standard errors of the proposed estimators are close to the efficient theoretical standard errors. The sample averages of the estimates for λ_0 under different simulation settings are reasonably close to corresponding true curves (results not shown here; see [7] for details).

5.2. A real data example. We use the Stanford heart transplant data [18] as an illustrative example. This dataset was also analyzed by [15] using their proposed least squares estimators. Following their analysis, we consider the same two models: the first one regresses the base-10 logarithm of the survival time on age at transplant and T5 mismatch score for the 157 patients with complete records on T5 measure, and the second one regresses the base-10 logarithm of the survival time on age and age². There were 55 censored patients. We fit these two models using the proposed method with five cubic B-spline basis functions.

We report the parameter estimates and the standard error estimates in Table 2 and compare them with the Gehan-weighted estimators reported by [15] and the Buckley–James estimators reported by [18]. For the first model, the parameter estimates for the age effect are fairly similar among all estimators, and the standard error estimate from the proposed method tends to be smaller, while the parameter estimates for the T5 mismatch score vary across different estimators with none of them being significant at the 0.05 level. The disparity of the T5 effect may be due

TABLE 1

Summary statistics for the simulation studies. The true slope parameters are $\beta_1 = 1$ and $\beta_2 = 1$. (a): $N(0, 1)$; (b): standard extreme-value; (c): $0.5N(0, 1) + 0.5N(0, 3^2)$; (d): $0.95N(0, 1) + 0.05N(0, 3^2)$; (e): $Gumbel(-0.5\mu, 0.5)$; (f): $0.5N(0, 1) + 0.5N(-1, 0.5^2)$

Err. dist	n	B-spline MLE				Log-rank		Gehan		σ^*	
		Bias	SE	¹ SEE (CP)	² SEE (CP)	Bias	SE	Bias	SE		
(a)	200	β_1	0.003	0.168	0.149 (0.912)	0.155 (0.924)	0.000	0.170	0.002	0.159	0.155
		β_2	0.003	0.167	0.153 (0.928)	0.156 (0.928)	0.004	0.171	0.002	0.160	0.156
	400	β_1	0.006	0.110	0.108 (0.948)	0.110 (0.950)	0.005	0.115	0.008	0.108	0.110
		β_2	0.001	0.110	0.109 (0.944)	0.110 (0.945)	0.002	0.116	0.001	0.109	0.110
	600	β_1	0.001	0.092	0.088 (0.939)	0.090 (0.943)	0.001	0.096	0.002	0.093	0.090
		β_2	0.005	0.091	0.089 (0.945)	0.090 (0.944)	0.005	0.097	0.003	0.092	0.090
(b)	200	β_1	-0.009	0.180	0.154 (0.894)	0.161 (0.903)	-0.008	0.168	-0.007	0.190	0.165
		β_2	0.004	0.182	0.162 (0.903)	0.163 (0.915)	0.005	0.170	0.005	0.195	0.169
	400	β_1	0.000	0.126	0.113 (0.914)	0.115 (0.923)	-0.001	0.124	0.000	0.143	0.117
		β_2	0.008	0.118	0.116 (0.934)	0.116 (0.938)	0.010	0.116	0.012	0.135	0.120
	600	β_1	0.001	0.102	0.093 (0.919)	0.094 (0.923)	0.001	0.100	0.000	0.114	0.095
		β_2	0.011	0.098	0.095 (0.944)	0.095 (0.945)	0.011	0.097	0.007	0.114	0.098
(c)	200	β_1	0.014	0.300	0.281 (0.930)	0.279 (0.924)	-0.020	0.315	-0.019	0.292	0.259
		β_2	0.000	0.306	0.285 (0.916)	0.282 (0.918)	0.002	0.317	0.002	0.288	0.260
	400	β_1	0.034	0.199	0.206 (0.955)	0.200 (0.949)	0.002	0.218	0.002	0.197	0.183
		β_2	-0.003	0.207	0.208 (0.949)	0.202 (0.942)	-0.001	0.222	-0.002	0.200	0.184
	600	β_1	0.035	0.168	0.171 (0.957)	0.165 (0.949)	0.003	0.185	0.001	0.163	0.150
		β_2	-0.007	0.169	0.172 (0.956)	0.166 (0.956)	-0.004	0.190	-0.002	0.168	0.150
(d)	200	β_1	-0.013	0.172	0.157 (0.926)	0.164 (0.927)	-0.010	0.181	-0.007	0.166	0.167
		β_2	-0.004	0.180	0.160 (0.908)	0.164 (0.913)	-0.005	0.184	-0.005	0.173	0.166
	400	β_1	0.003	0.119	0.113 (0.944)	0.116 (0.948)	0.004	0.126	0.006	0.117	0.118
		β_2	0.003	0.117	0.114 (0.942)	0.116 (0.953)	0.004	0.126	0.003	0.115	0.118
	600	β_1	-0.003	0.097	0.093 (0.948)	0.095 (0.952)	-0.002	0.105	0.002	0.097	0.096
		β_2	0.001	0.096	0.094 (0.942)	0.095 (0.944)	0.002	0.105	0.003	0.094	0.096
(e)	200	β_1	0.004	0.080	0.077 (0.944)	0.078 (0.946)	-0.001	0.111	0.004	0.088	0.079
		β_2	-0.001	0.083	0.080 (0.929)	0.078 (0.934)	0.000	0.114	0.000	0.091	0.080
	400	β_1	-0.005	0.055	0.055 (0.946)	0.055 (0.951)	-0.003	0.079	-0.004	0.061	0.056
		β_2	0.003	0.055	0.056 (0.954)	0.056 (0.950)	0.003	0.081	0.003	0.063	0.056
	600	β_1	-0.003	0.047	0.045 (0.940)	0.045 (0.938)	0.000	0.067	-0.001	0.052	0.045
		β_2	-0.001	0.047	0.046 (0.944)	0.045 (0.943)	-0.002	0.066	-0.001	0.051	0.046
(f)	200	β_1	-0.002	0.126	0.117 (0.918)	0.120 (0.929)	-0.002	0.159	-0.001	0.128	0.119
		β_2	0.000	0.133	0.120 (0.917)	0.121 (0.926)	0.002	0.164	0.001	0.134	0.116
	400	β_1	-0.002	0.087	0.084 (0.949)	0.085 (0.950)	0.003	0.114	0.000	0.091	0.084
		β_2	0.004	0.086	0.086 (0.951)	0.086 (0.953)	0.003	0.111	0.004	0.090	0.082
	600	β_1	0.003	0.074	0.070 (0.929)	0.070 (0.931)	0.005	0.101	0.001	0.074	0.069
		β_2	0.003	0.074	0.070 (0.936)	0.070 (0.936)	0.009	0.104	0.004	0.075	0.067

TABLE 2

Regression parameter estimates and standard error estimates for the Stanford heart transplant data. The proposed estimators are compared with Gehan-weighted estimators reported in [14] and Buckley–James estimators reported in [18]

	Covariate	B-spline MLE		Gehan-weighted		Buckley–James	
		Est.	SE	Est.	SE	Est.	SE
M.1	Age	−0.0237	0.0068	−0.0211	0.0106	−0.015	0.008
	T5	−0.2118	0.1271	−0.0265	0.1507	−0.003	0.134
M.2	Age	0.1022	0.0245	0.1046	0.0474	0.107	0.037
	Age ²	−0.0016	0.0004	−0.0017	0.0006	−0.0017	0.0005

to what was pointed out by [18]: the accelerated failure time model with age and T5 as covariates does not fit the data ideally. For the second model with age and age² being the covariates, the point estimates are very similar across all methods and the standard error estimates from the proposed method are the smallest.

6. Discussion. By applying the proposed general sieve M-estimation theory for semiparametric models with bundled parameters, we are able to derive the asymptotic distribution for the sieve maximum likelihood estimator in a linear regression model where the response variable is subject to right censoring. By providing a both statistically and computationally efficient estimating procedure, this work makes the linear model a more viable alternative to the Cox proportional hazards model. Comparing to the existing methods for estimating β in a linear model, the proposed method has three advantages. First, the estimating functions are smooth functions in contrast to the discrete estimating functions in the existing estimation methods; thus the root search is easier and can be done quickly by conventional iterative methods such as the Newton–Raphson algorithm. Second, the standard error estimates are obtained directly by inverting either the efficient information matrix for the regression parameters or the observed information matrix of all parameters; either method is more computationally tractable compared to the re-sampling techniques. Third, the proposed estimator achieves the semiparametric efficiency bound.

The proposed general sieve M-estimation theory can also be applied to other statistical models, for example, the single index model, the Cox model with an unknown link function and the linear model under different censoring mechanisms. Such research is undergoing and will be presented elsewhere.

7. Proof of Theorem 4.2. Empirical process theory developed in [27, 28] will be heavily involved in the proof. We use the symbol \lesssim to denote that the left-hand side is bounded above by a constant times the right-hand side and \gtrsim to denote that the left-hand side is bounded below by a constant times the right-hand side. For

notational simplicity, we drop the superscript $*$ in the outer probability measure P^* whenever an outer probability applies.

7.1. *Technical lemmas.* We first introduce several lemmas that will be used for the proofs of Theorems 4.1, 4.2 and 4.3. Proofs of these lemmas are provided in the supplementary material [8].

LEMMA 7.1. *Under conditions (C.1)–(C.3) and (C.6), the log-likelihood*

$$l(\beta, \zeta(\cdot, \beta); Z) = \Delta g(\epsilon_0 - X'(\beta - \beta_0)) - \int_a^b 1(\epsilon_0 \geq t) \exp\{g(t - X'(\beta - \beta_0))\} dt,$$

where $\epsilon_0 = Y - X'\beta_0$, has bounded and continuous first and second derivatives with respect to $\beta \in \mathcal{B}$ and $\zeta(\cdot, \beta) \in \mathcal{H}^p$.

LEMMA 7.2. *For $g_0 \in \mathcal{G}^p$, there exists a function $g_{0,n} \in \mathcal{G}_n^p$ such that*

$$\|g_{0,n} - g_0\|_\infty = O(n^{-p\nu}).$$

LEMMA 7.3. *Let $\theta_{0,n} = (\beta_0, \zeta_{0,n}(\cdot, \beta_0))$ with $\zeta_{0,n}(\cdot, \beta_0) \equiv g_{0,n}$ defined in Lemma 7.2. Denote $\mathcal{F}_n = \{l(\theta; z) - l(\theta_{0,n}; z) : \theta \in \Theta_n^p\}$. Assume that conditions (C.1)–(C.3) and (C.6) hold, then the ε -bracketing number associated with $\|\cdot\|_\infty$ norm for \mathcal{F}_n is bounded by $(1/\varepsilon)^{c q_n + d}$, that is, $N_{[\cdot]}(\varepsilon, \mathcal{F}_n, \|\cdot\|_\infty) \lesssim (1/\varepsilon)^{c q_n + d}$ for some constant $c > 0$.*

LEMMA 7.4. *Let $h_j^*(t, x, \beta) = w_j^*(\psi(t, x, \beta))$, where $h_j^*(t, x, \beta_0) = w_j^*(t) = -\dot{g}_0(t)P(X_j|\epsilon_0 \geq t)$, $j = 1, \dots, d$. Assume conditions (C.1)–(C.6) hold, then there exists $h_{j,n}^*(t, x, \beta) = w_{j,n}^*(\psi(t, x, \beta)) \in \mathcal{H}_n^2$ such that $\|h_{j,n}^* - h_j^*\|_\infty = O(n^{-2\nu})$, or equivalently, $\|w_{j,n}^* - w_j^*\|_\infty = O(n^{-2\nu})$ where $w_{j,n}^* \in \mathcal{G}_n^2$.*

LEMMA 7.5. *For h_j^* defined in Lemma 7.4, denote the class of functions*

$$\mathcal{F}_n^j(\eta) = \{\dot{l}_\zeta(\theta; z)[h_j^* - h_j] : \theta \in \Theta_n^p, h_j \in \mathcal{H}_n^2, d(\theta, \theta_0) \leq \eta, \|h_j - h_j^*\|_\infty \leq \eta\}.$$

Assume conditions (C.1)–(C.6) hold, then $N_{[\cdot]}(\varepsilon, \mathcal{F}_n^j(\eta), \|\cdot\|_\infty) \lesssim (\eta/\varepsilon)^{c q_n + d}$ for some constant $c > 0$.

LEMMA 7.6. *For $j = 1, \dots, d$, define the following two classes of functions:*

$$\mathcal{F}_{n,j}^\beta(\eta) = \{\dot{l}_{\beta_j}(\theta; z) - \dot{l}_{\beta_j}(\theta_0; z) : \theta \in \Theta_n^p, d(\theta, \theta_0) \leq \eta, \|\dot{g}(\psi(\cdot, \beta)) - \dot{g}_0(\psi(\cdot, \beta_0))\|_2 \leq \eta\}$$

and

$$\mathcal{F}_{n,j}^\zeta(\eta) = \{\dot{l}_\zeta(\theta; z)[h_j^*(\cdot, \beta)] - \dot{l}_\zeta(\theta_0; z)[h_j^*(\cdot, \beta_0)] : \theta \in \Theta_n^p, d(\theta, \theta_0) \leq \eta\},$$

where $\dot{l}_{\beta_j}(\theta; Z)$ is the j th element of $\dot{l}_\beta(\theta; Z)$, $\dot{g}(\cdot)$ denotes the derivative of $g(\cdot)$ and h_j^* is defined in Lemma 7.5. Assume conditions (C.1)–(C.6) hold, then $N_{[\cdot]}(\varepsilon, \mathcal{F}_{n,j}^\beta(\eta), \|\cdot\|_\infty) \lesssim (\eta/\varepsilon)^{c_1 q_n + d}$ and $N_{[\cdot]}(\varepsilon, \mathcal{F}_{n,j}^\zeta(\eta), \|\cdot\|_\infty) \lesssim (\eta/\varepsilon)^{c_2 q_n + d}$ for some constants $c_1, c_2 > 0$.

7.2. *Proof of Theorem 4.2.* We prove the theorem by checking assumptions (A1)–(A6) in Section 2. Here the criterion function of a single observation is the log-likelihood function $l(\beta, \zeta(\cdot, \beta); Z)$. So instead of m , we use l to denote the criterion function. By Theorem 4.1 we know that assumption (A1) holds with $\xi = \min(p\nu, (1 - \nu)/2)$ and the norm $\|\cdot\|_2$ defined in (4.1). Assumption (A2) automatically holds for the scores. For (A3), we need to find an $\mathbf{h}^* = (h_1^*, \dots, h_d^*)'$ with $\mathbf{h}^*(t, x, \beta_0) = \mathbf{w}^*(t)$ such that

$$\begin{aligned} & \ddot{S}_{\beta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0))[h] - \ddot{S}_{\zeta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0))[\mathbf{h}^*, h] \\ & = P\{\ddot{l}_{\beta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0); Z)[h] - \ddot{l}_{\zeta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*, h]\} = 0 \end{aligned}$$

for all $h \in \mathbb{H}$ with $h(t, x, \beta) = w(t - x'(\beta - \beta_0))$. Note that

$$\begin{aligned} & P\{\ddot{l}_{\beta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0); Z)[h] - \ddot{l}_{\zeta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*, h]\} \\ & = P\left\{-X\left[\Delta\dot{w}(\epsilon_0) - \int_a^b 1(\epsilon_0 \geq t) \exp\{g_0(t)\}\dot{w}(t) dt\right] \right. \\ & \quad \left. + \int_a^b 1(\epsilon_0 \geq t) \exp\{g_0(t)\}w(t)[X\dot{g}_0(t) + \mathbf{w}^*(t)] dt\right\}. \end{aligned}$$

Since $P\{\dot{l}_\zeta(\beta_0, \zeta_0(\cdot, \beta_0); Z)[h]|X\} = 0$ for all $h \in \mathbb{H}$, replacing $h(\cdot, \beta_0)$ by \dot{w} we have

$$\begin{aligned} & P\left\{-X\left[\Delta\dot{w}(\epsilon_0) - \int_a^b 1(\epsilon_0 \geq t) \exp\{g_0(t)\}\dot{w}(t) dt\right]\right\} \\ & = P\left\{-X \cdot P\left[\Delta\dot{w}(\epsilon_0) - \int_a^b 1(\epsilon_0 \geq t) \exp\{g_0(t)\}\dot{w}(t) dt \mid X\right]\right\} \\ & = P\{-X \cdot 0\} = 0. \end{aligned}$$

Hence we only need to find a \mathbf{w}^* such that

$$\begin{aligned} & P\left\{\int_a^b 1(\epsilon_0 \geq t) \exp\{g_0(t)\}w(t)[X\dot{g}_0(t) + \mathbf{w}^*(t)] dt\right\} \\ & = \int_a^b \exp\{g_0(t)\}w(t)\{\dot{g}_0(t)P[1(\epsilon_0 \geq t)X] + \mathbf{w}^*(t)P[1(\epsilon_0 \geq t)]\} dt = 0. \end{aligned}$$

One obvious choice for \mathbf{w}^* (or \mathbf{h}^*) is

$$(7.1) \quad \mathbf{h}^*(t, x, \beta_0) = \mathbf{w}^*(t) = -\dot{g}_0(t) \frac{P[1(\epsilon_0 \geq t)X]}{P[1(\epsilon_0 \geq t)]} = -\dot{g}_0(t)P(X|\epsilon_0 \geq t).$$

Then it follows

$$\begin{aligned} & \dot{l}_\beta(\beta_0, \zeta_0(\cdot, \beta_0); Z) - \dot{l}_\zeta(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*] \\ &= \Delta\{-\dot{g}_0(Y - X'\beta_0)\}\{X - P(X|\epsilon_0 \geq Y - X'\beta_0)\} \\ &\quad - \int 1(Y - X'\beta_0 \geq t)\{X - P(X|\epsilon_0 \geq t)\}\{-\dot{g}_0(t)\} \exp\{g_0(t)\} dt \\ &= \int \{X - P(X|\epsilon_0 \geq t)\}\{-\dot{g}_0(t)\} dM(t) \\ &= l_{\beta_0}^*(Y, \Delta, X), \end{aligned}$$

which is the efficient score function for β_0 originally derived by [22], where

$$M(t) = \Delta I(Y - X'\beta_0 \leq t) - \int_{-\infty}^t I(Y - X'\beta_0 \geq s) \exp\{g_0(s)\} ds.$$

By the fact of zero-mean for a score function, it is straightforward to verify the following equalities:

$$\begin{aligned} P\ddot{l}_{\beta\zeta}(\beta, \zeta(\cdot, \beta); Z)[h] &= -P\{\dot{l}_\beta(\beta, \zeta(\cdot, \beta); Z)\dot{l}'_\zeta(\beta, \zeta(\cdot, \beta); Z)[h]\}, \\ P\ddot{l}_{\zeta\beta}(\beta, \zeta(\cdot, \beta); Z)[h] &= -P\{\dot{l}_\zeta(\beta, \zeta(\cdot, \beta); Z)[h]\dot{l}'_\beta(\beta, \zeta(\cdot, \beta); Z)\}, \\ P\ddot{l}_{\beta\beta}(\beta, \zeta(\cdot, \beta); Z) &= -P\{\dot{l}_\beta(\beta, \zeta(\cdot, \beta); Z)\dot{l}'_\beta(\beta, \zeta(\cdot, \beta); Z)\}, \\ P\ddot{l}_{\zeta\zeta}(\beta, \zeta(\cdot, \beta); Z)[h_1, h_2] &= -P\{\dot{l}_\zeta(\beta, \zeta(\cdot, \beta); Z)[h_1]\dot{l}'_\zeta(\beta, \zeta(\cdot, \beta); Z)[h_2]\}. \end{aligned}$$

Then together with the fact that

$$P\{\ddot{l}_{\beta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*] - \ddot{l}_{\zeta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*, \mathbf{h}^*]\} = 0,$$

the matrix A in assumption (A3) of Theorem 2.1 is given by

$$\begin{aligned} A &= P\{-\ddot{l}_{\beta\beta}(\beta_0, \zeta_0(\cdot, \beta_0); Z) + \ddot{l}_{\zeta\beta}(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*] \\ &\quad + \ddot{l}_{\beta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*] - \ddot{l}_{\zeta\zeta}(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*, \mathbf{h}^*]\} \\ &= P\{\dot{l}_\beta(\beta_0, \zeta_0(\cdot, \beta_0); Z)\dot{l}'_\beta(\beta_0, \zeta_0(\cdot, \beta_0); Z) \\ &\quad - \dot{l}_\zeta(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*]\dot{l}'_\beta(\beta_0, \zeta_0(\cdot, \beta_0); Z) \\ &\quad - \dot{l}_\beta(\beta_0, \zeta_0(\cdot, \beta_0); Z)\dot{l}'_\zeta(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*] \\ &\quad + \dot{l}_\zeta(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*]\dot{l}'_\zeta(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*]\} \\ &= P\{\dot{l}_\beta(\beta_0, \zeta_0(\cdot, \beta_0); Z) - \dot{l}_\zeta(\beta_0, \zeta_0(\cdot, \beta_0); Z)[\mathbf{h}^*]\}^{\otimes 2} \\ &= Pl_{\beta_0}^*(Y, \Delta, X)^{\otimes 2}, \end{aligned}$$

which is the information matrix for β_0 .

To verify (A4), we note that the first part automatically holds since $\hat{\beta}_n$ satisfies the score equation $\dot{S}_{\beta,n}(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n)) = \mathbb{P}_n \dot{l}_\beta(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n); Z) = 0$. Next we shall show that

$$\begin{aligned} & \dot{S}_{\zeta,n}(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n))[h_j^*] \\ &= \mathbb{P}_n \left\{ \Delta w_j^*(Y - X' \hat{\beta}_n) - \int 1(Y \geq t) \exp\{\hat{\zeta}_n(t, X, \hat{\beta}_n)\} w_j^*(t - X' \hat{\beta}_n) dt \right\} \\ &= o_p(n^{-1/2}), \end{aligned}$$

where $w_j^*(t) = -\dot{g}_0(t)P(X_j | \epsilon_0 \geq t)$, $j = 1, \dots, d$, is the j th component of $\mathbf{w}^*(t)$ given in (7.1). According to Lemma 7.4, there exists $h_{j,n}^* \in \mathcal{H}_n^2$ such that $\|h_j^* - h_{j,n}^*\|_\infty = O(n^{-2\nu})$. Then by the score equation for $\gamma: \dot{S}_{\gamma,n}(\hat{\beta}_n, \hat{\gamma}_n) = \mathbb{P}_n \dot{l}_\gamma(\hat{\beta}_n, \hat{\gamma}_n; Z) = 0$ and the fact that $w_{j,n}^*(t)$ can be written as $w_{j,n}^*(t) = \sum_{k=1}^{q_n} \gamma_{j,k}^* B_k(t)$ for some coefficients $\{\gamma_{j,1}^*, \dots, \gamma_{j,q_n}^*\}$ and the basis functions $B_k(t)$ of the spline space, it follows that

$$\mathbb{P}_n \left\{ \Delta w_{j,n}^*(Y - X' \hat{\beta}_n) - \int 1(Y \geq t) \exp\{\hat{\zeta}_n(t, X, \hat{\beta}_n)\} w_{j,n}^*(t - X' \hat{\beta}_n) dt \right\} = 0.$$

So it suffices to show that for each $1 \leq j \leq d$,

$$I_n = \mathbb{P}_n \dot{l}_\zeta(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n); Z)[h_j^* - h_{j,n}^*] = o_p(n^{-1/2}).$$

Since $P\{\dot{l}_\zeta(\beta_0, \zeta_0(\cdot, \beta_0); Z)[h_j^* - h_{j,n}^*]\} = 0$, we decompose I_n into $I_n = I_{1n} + I_{2n}$, where

$$I_{1n} = (\mathbb{P}_n - P)\dot{l}_\zeta(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n); Z)[h_j^* - h_{j,n}^*]$$

and

$$I_{2n} = P\{\dot{l}_\zeta(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n); Z)[h_j^* - h_{j,n}^*] - \dot{l}_\zeta(\beta_0, \zeta_0(\cdot, \beta_0); Z)[h_j^* - h_{j,n}^*]\}.$$

We will show that I_{1n} and I_{2n} are both $o_p(n^{-1/2})$.

First consider I_{1n} . According to Lemma 7.5, the ε -bracketing number associated with $\|\cdot\|_\infty$ norm for the class $\mathcal{F}_n^j(\eta)$ defined in Lemma 7.5 is bounded by $(\eta/\varepsilon)^{cq_n+d}$. This implies that

$$\log N_{[\cdot]}(\varepsilon, \mathcal{F}_n^j(\eta), L_2(P)) \leq \log N_{[\cdot]}(\varepsilon, \mathcal{F}_n^j(\eta), \|\cdot\|_\infty) \lesssim q_n \log(\eta/\varepsilon),$$

which leads to the bracketing integral

$$\begin{aligned} J_{[\cdot]}(\eta, \mathcal{F}_n^j(\eta), L_2(P)) &= \int_0^\eta \sqrt{1 + \log N_{[\cdot]}(\varepsilon, \mathcal{F}_n^j(\eta), L_2(P))} d\varepsilon \\ &\lesssim q_n^{1/2} \eta. \end{aligned}$$

Now we pick η to be $\eta_n = O\{n^{-\min(2\nu, (1-\nu)/2)}\}$, then

$$\|h_j^* - h_{j,n}^*\|_\infty = O(n^{-2\nu}) \leq O\{n^{-\min(2\nu, (1-\nu)/2)}\} = \eta_n,$$

and since $p \geq 3$,

$$d(\hat{\theta}_n, \theta_0) = O_p\{n^{-\min(p\nu, (1-\nu)/2)}\} \leq O_p\{n^{-\min(2\nu, (1-\nu)/2)}\} = \eta_n.$$

Therefore, $\dot{l}_\zeta(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n); z)[h_j^* - h_{j,n}^*] \in \mathcal{F}_n^j(\eta_n)$. Denote $t_\beta = t - X'(\beta - \beta_0)$ for notational simplicity, for any $\dot{l}_\zeta(\theta; Z)[h_j^* - h] \in \mathcal{F}_n^j(\eta_n)$, it follows that

$$\begin{aligned} & P\{\dot{l}_\zeta(\theta; Z)[h_j^* - h]\}^2 \\ &= P\left\{\Delta(w_j^* - w)(\epsilon_\beta) + \int_a^b 1(\epsilon_0 \geq t) \exp\{g(t_\beta)\}(w_j^* - w)(t_\beta) dt\right\}^2 \\ &\lesssim \|w_j^* - w\|_\infty^2 + P\left\{\int_a^b \exp\{2g(t_\beta)\}(w_j^* - w)^2(t_\beta) dt\right\} \\ &\lesssim \|w_j^* - w\|_\infty^2 + \|w_j^* - w\|_\infty \int_a^b P[\exp\{2g(t_\beta)\}] dt, \end{aligned}$$

where the first inequality holds because of the Cauchy–Schwarz inequality. Since $\|w_j^* - w\|_\infty \leq \eta_n$, by the same argument as ([25], page 591), for slowly growing c_n (their l_n), for example, $c_n = o(\log(\eta_n^{-1}))$, we know that $\|\dot{l}_\zeta(\theta; Z)[h_j^* - h]\|_\infty$ is bounded by some constant $0 < M < \infty$ and $P\{\dot{l}_\zeta(\theta; Z)[h_j^* - h]\}^2 \lesssim \eta_n$ for a slightly enlarged η_n obtained by a fine adjustment of ν . Then by the maximal inequality in Lemma 3.4.2 of [28], it follows that

$$\begin{aligned} E_P \|\mathbb{G}_n\|_{\mathcal{F}_n^j(\eta_n)} &\lesssim J_{[\cdot]}(\eta_n, \mathcal{F}_n^j(\eta_n), L_2(P)) \left(1 + \frac{J_{[\cdot]}(\eta_n, \mathcal{F}_n^j(\eta_n), L_2(P))}{\eta_n^2 \sqrt{n}} M\right) \\ &\lesssim q_n^{1/2} \eta_n + q_n n^{-1/2} \\ &= O\{n^{\nu/2 - \min(2\nu, (1-\nu)/2)}\} + O(n^{\nu-1/2}) \\ &= O\{n^{-\min(3\nu/2, 1/2-\nu)}\} + O(n^{\nu-1/2}) = o(1), \end{aligned}$$

where the last equality holds because $0 < \nu < 1/2$. Thus by Markov’s inequality, $I_{1n} = n^{-1/2} \mathbb{G}_n \dot{l}_\zeta(\hat{\theta}_n; Z)[h_j^* - h_{j,n}^*] = o_p(n^{-1/2})$.

Next for I_{2n} , the Taylor expansion for $\dot{l}_\zeta(\hat{\theta}_n; Z)[h_j^* - h_{j,n}^*]$ at θ_0 yields

$$\begin{aligned} & \dot{l}_\zeta(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n); Z)[h_j^* - h_{j,n}^*] - \dot{l}_\zeta(\beta_0, \zeta_0(\cdot, \beta_0); Z)[h_j^* - h_{j,n}^*] \\ &= (\hat{\beta}_n - \beta_0)' \ddot{l}_{\beta\zeta}(\tilde{\beta}_n, \tilde{\zeta}_n(\cdot, \tilde{\beta}_n); Z)[h_j^* - h_{j,n}^*] \\ &\quad + \ddot{l}_{\zeta\zeta}(\tilde{\beta}_n, \tilde{\zeta}_n(\cdot, \tilde{\beta}_n); Z)[h_j^* - h_{j,n}^*, \hat{\zeta}_n - \zeta_0], \end{aligned}$$

where $(\tilde{\beta}_n, \tilde{\zeta}_n(\cdot, \tilde{\beta}_n))$ is between $(\beta_0, \zeta_0(\cdot, \beta_0))$ and $(\hat{\beta}_n, \hat{\zeta}_n(\cdot, \hat{\beta}_n))$. Then it follows that

$$\begin{aligned} & |\ddot{l}_{\beta\zeta}(\tilde{\beta}_n, \tilde{\zeta}_n(\cdot, \tilde{\beta}_n); Z)[h_j^* - h_{j,n}^*]| \\ &= \left| X \left\{ \Delta(\dot{w}_j^* - \dot{w}_{j,n}^*)(\epsilon_{\tilde{\beta}_n}) \right. \right. \\ &\quad \left. \left. - \int_a^b 1(\epsilon_0 \geq t) \exp\{\tilde{g}_n(t_{\tilde{\beta}_n})\} [(\dot{w}_j^* - \dot{w}_{j,n}^*)(t_{\tilde{\beta}_n}) \right. \right. \\ &\quad \left. \left. + \dot{\tilde{g}}_n(t_{\tilde{\beta}_n})(w_j^* - w_{j,n}^*)(t_{\tilde{\beta}_n})] dt \right\} \right| \\ &\lesssim \|\dot{w}_j^* - \dot{w}_{j,n}^*\|_\infty + \|\dot{w}_j^* - \dot{w}_{j,n}^*\|_\infty \left\{ \int_a^b \exp\{\tilde{g}_n(t_{\tilde{\beta}_n})\} dt \right\} \\ &\quad + \|w_j^* - w_{j,n}^*\|_\infty \left\{ \int_a^b \exp\{\tilde{g}_n(t_{\tilde{\beta}_n})\} \dot{\tilde{g}}_n(t_{\tilde{\beta}_n}) dt \right\} \\ &\lesssim \|\dot{w}_j^* - \dot{w}_{j,n}^*\|_\infty + \|w_j^* - w_{j,n}^*\|_\infty \\ &= O(n^{-\nu}) + O(n^{-2\nu}) \\ &= O(n^{-\nu}), \end{aligned}$$

where the second inequality holds because \tilde{g}_n and its first derivative $\dot{\tilde{g}}_n$ are bounded (or growing with n slowly enough so it can be effectively treated as bounded based on the same argument of [25] on page 591), and the last equality holds due to the Corollary 6.21 of [23] that $\|\dot{w}_j^* - \dot{w}_{j,n}^*\|_\infty = O(n^{-(2-1)\nu}) = O(n^{-\nu})$. Thus,

$$\begin{aligned} & P(|\hat{\beta}_n - \beta_0| \ddot{l}_{\beta\zeta}(\tilde{\beta}_n, \tilde{\zeta}_n(\cdot, \tilde{\beta}_n); Z)[h_j^* - h_{j,n}^*]) \\ &= |\hat{\beta}_n - \beta_0| \cdot O(n^{-\nu}) \\ &= O_p\{n^{-\min(p\nu, (1-\nu)/2)}\} \cdot O(n^{-\nu}) \\ &= O_p\{n^{-\min((p+1)\nu, (1+3\nu)/2)}\}. \end{aligned}$$

Also,

$$\begin{aligned} & |\ddot{l}_{\zeta\zeta}(\tilde{\beta}_n, \tilde{\zeta}_n(\cdot, \tilde{\beta}_n); Z)[h_j^* - h_{j,n}^*, \hat{\zeta}_n - \zeta_0]| \\ &= \left| \int_a^b 1(\epsilon_0 \geq t) \exp\{\tilde{g}_n(t_{\tilde{\beta}_n})\} (w_j^* - w_{j,n}^*)(t_{\tilde{\beta}_n}) (\hat{g}_n - g_0)(t_{\tilde{\beta}_n}) dt \right| \\ &\leq \|w_j^* - w_{j,n}^*\|_\infty \cdot \left\{ \int_a^b \exp\{\tilde{g}_n(t_{\tilde{\beta}_n})\} (\hat{g}_n - g_0)(t_{\tilde{\beta}_n}) dt \right\} \\ &= \|w_j^* - w_{j,n}^*\|_\infty \cdot I_{3n}. \end{aligned}$$

By the Cauchy–Schwarz inequality and the boundedness of \tilde{g}_n , we have

$$\begin{aligned} P\{I_{3n}\}^2 &= P\left\{\int_a^b \exp\{\tilde{g}_n(t_{\tilde{\beta}_n})\}(\hat{g}_n - g_0)(t_{\tilde{\beta}_n}) dt\right\}^2 \\ &\lesssim \int_{\mathcal{X}} \int_a^b (\hat{g}_n - g_0)^2(t_{\tilde{\beta}}) d\Lambda_0(t) dF_X(x) = \|\hat{\zeta}_n(\cdot, \tilde{\beta}_n) - \zeta_0(\cdot, \tilde{\beta}_n)\|_2^2 \\ &\lesssim |\tilde{\beta}_n - \hat{\beta}_n|^2 + \|\hat{\zeta}_n(\cdot, \hat{\beta}_n) - \zeta_0(\cdot, \beta_0)\|_2^2 + |\beta_0 - \tilde{\beta}_n|^2 \\ &\lesssim |\hat{\beta}_n - \beta_0|^2 + \|\hat{\zeta}_n(\cdot, \hat{\beta}_n) - \zeta_0(\cdot, \beta_0)\|_2^2 = d(\hat{\theta}_n, \theta_0)^2. \end{aligned}$$

Hence $P|I_{3n}| \lesssim d(\hat{\theta}_n, \theta_0)$ and

$$\begin{aligned} P|\ddot{l}_{\zeta\zeta}(\tilde{\beta}_n, \tilde{g}_n; Z)[h_j^* - h_{j,n}^*, \hat{\zeta}_n - \zeta_0]| \\ \lesssim \|w_j^* - w_{j,n}^*\|_\infty \cdot d(\hat{\theta}_n, \theta_0) = O(n^{-2\nu}) \cdot O_p\{n^{-\min(p\nu, (1-\nu)/2)}\} \\ = O_p\{n^{-\min((p+2)\nu, (1+3\nu)/2)}\}. \end{aligned}$$

Since $\frac{1}{2(1+p)} < \nu < \frac{1}{1+2p}$, it follows that $I_{2n} = O\{n^{-\min((p+1)\nu, (1+3\nu)/2)}\} = o(n^{-1/2})$. Thus $I_n = I_{1n} + I_{2n} = o_p(n^{-1/2})$, and condition (A4) holds.

Now we verify assumption (A5). First by Lemma 7.6, the ε -bracketing numbers for the classes of functions $\mathcal{F}_{n,j}^\beta(\eta)$ and $\mathcal{F}_{n,j}^\zeta(\eta)$ are both bounded by $(\eta/\varepsilon)^{cq_n+d}$, which implies that the corresponding ε -bracketing integrals are both bounded by $q_n^{1/2}\eta$, that is,

$$J_{[\cdot]}(\eta, \mathcal{F}_{n,j}^\beta(\eta), L_2(P)) \lesssim q_n^{1/2}\eta \quad \text{and} \quad J_{[\cdot]}(\eta, \mathcal{F}_{n,j}^\zeta(\eta), L_2(P)) \lesssim q_n^{1/2}\eta.$$

Then for $\dot{l}_{\beta_j}(\theta; z) - \dot{l}_{\beta_j}(\theta_0; z)$, by applying the Cauchy–Schwarz inequality, together with subtracting and adding the terms $\dot{g}(\epsilon_0)$, $e^{g_0(t_\beta)}\dot{g}(t_\beta)$, $e^{g_0(t)}\dot{g}(t_\beta)$ and $e^{g_0(t)}\dot{g}_0(t_\beta)$, we have

$$\begin{aligned} \{\dot{l}_{\beta_j}(\theta; Z) - \dot{l}_{\beta_j}(\theta_0; Z)\}^2 \\ = \left\{ -\Delta X_j[\dot{g}(\epsilon_\beta) - \dot{g}_0(\epsilon_0)] \right. \\ \left. + X_j \int_a^b 1(\epsilon_0 \geq t)[e^{g(t_\beta)}\dot{g}(t_\beta) - e^{g_0(t)}\dot{g}_0(t)] dt \right\}^2 \\ \lesssim \{\Delta[\dot{g}(\epsilon_\beta) - \dot{g}_0(\epsilon_0)]\}^2 + \left\{ \int_a^b [e^{g(t_\beta)}\dot{g}(t_\beta) - e^{g_0(t)}\dot{g}_0(t)]^2 dt \right\} \\ \lesssim \{\Delta[\dot{g}(\epsilon_\beta) - \dot{g}_0(\epsilon_0)]\}^2 + \{\Delta[\dot{g}(\epsilon_0) - \dot{g}_0(\epsilon_0)]\}^2 \\ + \int_a^b \{[e^{g(t_\beta)} - e^{g_0(t_\beta)}]^2 + [e^{g_0(t_\beta)} - e^{g_0(t)}]^2\} \dot{g}^2(t_\beta) dt \end{aligned}$$

$$\begin{aligned}
 & + \int_a^b e^{2g_0(t)} \{[\dot{g}(t_\beta) - \dot{g}_0(t_\beta)]^2 + e^{2g_0(t)}[\dot{g}_0(t_\beta) - \dot{g}_0(t)]^2\} dt \\
 & = B_1 + B_2 + B_3 + B_4.
 \end{aligned}$$

For B_1 , since \ddot{g} is bounded and the largest eigenvalue of $P(XX')$ satisfies $0 < \lambda_d < \infty$ by condition (C.2)(b), it follows that

$$\begin{aligned}
 PB_1 & \leq P[\ddot{g}(Y - X'\tilde{\beta})X'(\beta - \beta_0)]^2 \lesssim P[X'(\beta - \beta_0)]^2 \\
 & \leq \lambda_d |\beta - \beta_0|^2 \lesssim |\beta - \beta_0|^2 \leq \eta^2.
 \end{aligned}$$

For B_2 , we have

$$\begin{aligned}
 PB_2 & \leq \int_{\mathcal{X}} \left\{ \int_a^b (\dot{g}(t) - \dot{g}_0(t))^2 d\Lambda_0(t) \right\} dF_X(x) \\
 & = \|\dot{g}(\psi(\cdot, \beta_0)) - \dot{g}_0(\psi(\cdot, \beta_0))\|_2^2 \\
 & \lesssim |\beta - \beta_0|^2 + \|\dot{g}(\psi(\cdot, \beta)) - \dot{g}_0(\psi(\cdot, \beta_0))\|_2^2 \lesssim \eta^2.
 \end{aligned}$$

For B_3 , by using the mean value theorem, it follows that

$$\begin{aligned}
 PB_3 & = P \left\{ \int_a^b \{ [e^{\tilde{g}(t_\beta)}(g - g_0)(t_\beta)]^2 + [e^{g_0(t_\beta)}X'(\beta - \beta_0)]^2 \} \dot{g}^2(t_\beta) dt \right\} \\
 & \lesssim \int_{\mathcal{X}} \int_a^b (g - g_0)^2(t_\beta) d\Lambda_0(t) dF_X(x) + P[X'(\beta - \beta_0)]^2 \\
 & \lesssim \|\zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)\|_2^2 + |\beta - \beta_0|^2 \leq \eta^2,
 \end{aligned}$$

where $\tilde{g} = g_0 + \xi(g - g_0)$ for some $0 < \xi < 1$ and thus is bounded. Finally for B_4 , by the mean value theorem, it follows that

$$\begin{aligned}
 PB_4 & = P \left\{ \int_a^b e^{2g_0(t)} \{ [\dot{g}(t_\beta) - \dot{g}_0(t_\beta)]^2 + e^{2g_0(t)}[\dot{g}_0(t_\beta) - \dot{g}_0(t)]^2 \} dt \right\} \\
 & \lesssim \int_{\mathcal{X}} \int_a^b (\dot{g} - \dot{g}_0)^2(t_\beta) d\Lambda_0(t) dF_X(x) + P \int_a^b [\ddot{g}_0(t_{\tilde{\beta}})X'(\beta - \beta_0)]^2 dt \\
 & \lesssim \|\dot{g}(\psi(\cdot, \beta)) - \dot{g}_0(\psi(\cdot, \beta_0))\|_2^2 + P[X'(\beta - \beta_0)]^2 \\
 & \lesssim \|\dot{g}(\psi(\cdot, \beta)) - \dot{g}_0(\psi(\cdot, \beta_0))\|_2^2 + |\beta - \beta_0|^2 \lesssim \eta^2.
 \end{aligned}$$

Therefore we have $P\{\dot{l}_{\beta_j}(\theta; Z) - \dot{l}_{\beta_j}(\theta_0; Z)\}^2 \lesssim \eta^2$. Using the similar argument, we can show that $P\{\dot{l}_\zeta(\theta; Z)[h_j^*] - \dot{l}_\zeta(\theta_0; Z)[h_j^*]\}^2 \lesssim \eta^2$. By Lemma 7.1, we also have $\|\dot{l}_{\beta_j}(\theta; Z) - \dot{l}_{\beta_j}(\theta_0; Z)\|_\infty$ and $\|\dot{l}_\zeta(\theta; Z)[h_j^*] - \dot{l}_\zeta(\theta_0; Z)[h_j^*]\|_\infty$ are both bounded. Now we pick η as $\eta_n = O\{n^{-\min((p-1)\nu, (1-\nu)/2)}\}$, then by the maximal

inequality in Lemma 3.4.2 of [28], it follows that

$$\begin{aligned} E_P \|\mathbb{G}_n\|_{\mathcal{F}_{n,j}^\beta(\eta_n)} &\lesssim q_n^{1/2} \eta_n + q_n n^{-1/2} \\ &= O\{n^{\max((3/2-p)v, v-1/2)}\} + O(n^{v-1/2}) = o(1), \end{aligned}$$

where the last equality holds since $p \geq 3$ and $v < \frac{1}{2}$. Similarly, we have $E_P \|\mathbb{G}_n\|_{\mathcal{F}_{n,j}^\xi(\eta_n)} = o(1)$. Thus for $\xi = \min(pv, (1 - v)/2)$ and

$$Cn^{-\xi} = O\{n^{-\min(pv, (1-v)/2)}\}$$

by Markov’s inequality,

$$\begin{aligned} \sup_{d(\theta, \theta_0) \leq Cn^{-\xi}} \mathbb{G}_n\{\dot{l}_{\beta_j}(\beta, \zeta(\cdot, \beta); Z) - \dot{l}_{\beta_j}(\beta_0, \zeta_0(\cdot, \beta_0); Z)\} &= o_p(1), \\ \sup_{d(\theta, \theta_0) \leq Cn^{-\xi}} \mathbb{G}_n\{\dot{l}_\zeta(\beta, \zeta(\cdot, \beta); Z)[h_j^*] - \dot{l}_\zeta(\beta_0, \zeta_0(\cdot, \beta_0); Z)[h_j^*]\} &= o_p(1). \end{aligned}$$

This completes the verification of assumption (A5).

Finally, assumption (A6) can be verified by using the Taylor expansion. Since the proofs for the two equations in (A6) are essentially identical, we just prove the first equation. In a neighborhood of $\theta_0: \{\theta: d(\theta, \theta_0) \leq Cn^{-\xi}, \theta \in \Theta_n^p\}$ with $\xi = \min(pv, (1 - v)/2)$, the Taylor expansion for $\dot{l}_\beta(\theta; Z)$ yields

$$\begin{aligned} \dot{l}_\beta(\theta; Z) &= \dot{l}_\beta(\theta_0; Z) + \ddot{l}_{\beta\beta}(\tilde{\theta}; Z)(\beta - \beta_0) + \ddot{l}_{\beta\zeta}(\tilde{\theta}; Z)[\zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)] \\ &= \dot{l}_\beta(\theta_0; Z) + \ddot{l}_{\beta\beta}(\theta_0; Z)(\beta - \beta_0) + \ddot{l}_{\beta\zeta}(\theta_0; Z)[\zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)] \\ &\quad + \{\ddot{l}_{\beta\beta}(\tilde{\theta}; Z)(\beta - \beta_0) - \ddot{l}_{\beta\beta}(\theta_0; Z)(\beta - \beta_0)\} \\ &\quad + \{\ddot{l}_{\beta\zeta}(\tilde{\theta}; Z)[\zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)] - \ddot{l}_{\beta\zeta}(\theta_0; Z)[\zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)]\}, \end{aligned}$$

where $\tilde{\theta} = (\tilde{\beta}, \tilde{\zeta}(\cdot, \tilde{\beta}))$ is a midpoint between θ_0 and θ . So

$$\begin{aligned} P\{\dot{l}_\beta(\theta; Z) - \dot{l}_\beta(\theta_0; Z) - \ddot{l}_{\beta\beta}(\theta_0; Z)(\beta - \beta_0) - \ddot{l}_{\beta\zeta}(\theta_0; Z)[\zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)]\} \\ = P\{\ddot{l}_{\beta\beta}(\tilde{\theta}; Z) - \ddot{l}_{\beta\beta}(\theta_0; Z)\}(\beta - \beta_0) \\ + P\{\ddot{l}_{\beta\zeta}(\tilde{\theta}; Z)[\zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)] - \ddot{l}_{\beta\zeta}(\theta_0; Z)[\zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)]\}. \end{aligned}$$

Then by direct calculation we have

$$\begin{aligned} P|\ddot{l}_{\beta\beta}(\tilde{\theta}; Z) - \ddot{l}_{\beta\beta}(\theta_0; Z)| \\ \leq P|XX' \Delta\{\ddot{g}(\epsilon_{\tilde{\beta}}) - \ddot{g}_0(\epsilon_0)\}| \\ + P\left\{XX' \left| \int_a^b 1(\epsilon_0 \geq t) \{\exp\{\tilde{g}(t_{\tilde{\beta}})\} \ddot{g}(t_{\tilde{\beta}}) - \exp\{g_0(t)\} \ddot{g}_0(t)\} dt \right. \right. \\ \left. \left. + \int_a^b 1(\epsilon_0 \geq t) \{\exp\{\tilde{g}(t_{\tilde{\beta}})\} \dot{\tilde{g}}^2(t_{\tilde{\beta}}) - \exp\{g_0(t)\} \dot{g}_0^2(t)\} dt \right| \right\} \end{aligned}$$

$$\begin{aligned} &\lesssim P|\Delta\{\ddot{\tilde{g}}(\epsilon_{\tilde{\beta}}) - \ddot{g}_0(\epsilon_0)\}| \\ &\quad + P\left\{\int_a^b |\exp\{\tilde{g}(t_{\tilde{\beta}})\}\ddot{\tilde{g}}(t_{\tilde{\beta}}) - \exp\{g_0(t)\}\ddot{g}_0(t)| dt\right\} \\ &\quad + P\left\{\int_a^b |\exp\{\tilde{g}(t_{\tilde{\beta}})\}\dot{\tilde{g}}^2(t_{\tilde{\beta}}) - \exp\{g_0(t)\}\dot{g}_0^2(t)| dt\right\} \\ &= C_1 + C_2 + C_3. \end{aligned}$$

By applying a similar argument that we used before for verifying (A5) and condition (C.6), we can show

$$\begin{aligned} C_1 &\lesssim |\beta - \beta_0| + \|\ddot{g}(\psi(\cdot, \beta)) - \ddot{g}_0(\psi(\cdot, \beta_0))\|_2 \\ &= O(n^{-\xi}) + O\{n^{-\min((p-2)v, (1-v)/2)}\}. \end{aligned}$$

Similarly, we can show

$$\begin{aligned} C_2 &\lesssim |\beta - \beta_0| + \|\ddot{g}(\psi(\cdot, \beta)) - \ddot{g}_0(\psi(\cdot, \beta_0))\|_2 \\ &= O(n^{-\xi}) + O\{n^{-\min((p-2)v, (1-v)/2)}\} \end{aligned}$$

and

$$\begin{aligned} C_3 &\lesssim |\beta - \beta_0| + \|\dot{g}(\psi(\cdot, \beta)) - \dot{g}_0(\psi(\cdot, \beta_0))\|_2 \\ &= O(n^{-\xi}) + O\{n^{-\min((p-1)v, (1-v)/2)}\}, \end{aligned}$$

where $\xi = \min(pv, (1 - v)/2)$. Therefore,

$$P|\ddot{l}_{\beta\beta}(\tilde{\theta}; Z) - \ddot{l}_{\beta\beta}(\theta_0; Z)| = O\{n^{-\min((p-2)v, (1-v)/2)}\}$$

and thus

$$\begin{aligned} &P|\ddot{l}_{\beta\beta}(\tilde{\theta}; Z) - \ddot{l}_{\beta\beta}(\theta_0; Z)|(\beta - \beta_0) \\ &= O\{n^{-\min((p-2)v, (1-v)/2)}\} \cdot O\{n^{-\min(pv, (1-v)/2)}\} \\ &= O\{n^{-\min(2(p-1)v, 1/2+(p-5/2)v, 1-v)}\} \\ &= o(n^{-1/2}), \end{aligned}$$

where the last equality holds since $p \geq 3$, so $2(p-1)v > \frac{p-1}{p+1} \geq \frac{1}{2}$, $\frac{1}{2} + (p - \frac{5}{2})v > \frac{1}{2}$ and $1 - v > \frac{1}{2}$. Similarly we can show

$$\begin{aligned} &P|\ddot{l}_{\beta\zeta}(\tilde{\theta}; Z)[\zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)] - \ddot{l}_{\beta\zeta}(\theta_0; Z)[\zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)]| \\ &= O\{n^{-\min(2(p-1)v, 1/2+(p-5/2)v, 1-v)}\} \\ &= o(n^{-1/2}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & |P\{\dot{l}_\beta(\theta; Z) - \dot{l}_\beta(\theta_0; Z) - \ddot{l}_{\beta\beta}(\theta_0; Z)(\beta - \beta_0) - \ddot{l}_{\beta\zeta}(\tilde{\theta}; Z)[\zeta(\cdot, \beta) - \zeta_0(\cdot, \beta_0)]\}| \\ &= O\{n^{-\min(2(p-1)v, 1/2+(p-5/2)v, 1-v)}\} = O(n^{-\alpha\xi}), \end{aligned}$$

where $\alpha = \min(2(p-1)v, \frac{1}{2} + (p-\frac{5}{2})v, 1-v) / \min(pv, \frac{1-v}{2}) > 1$ and $\alpha\xi > 1/2$.

Therefore, we have verified all six assumptions, and thus we have

$$\sqrt{n}(\hat{\beta}_n - \beta_0) = A^{-1}\sqrt{n}\mathbb{P}_n l_{\beta_0}^*(\beta_0, \zeta_0(\cdot, \beta_0); Z) + o_p(1) \rightarrow N(0, A^{-1}B(A^{-1})'),$$

where $l_{\beta_0}^*(\theta_0; Z) = \dot{l}_\beta(\theta_0; Z) - \dot{l}_\zeta(\theta_0; Z)[\mathbf{h}^*]$ is the efficient score function for β_0 and $A = P\{l_{\beta_0}^*(Y, \Delta, X)\}^{\otimes 2} = I(\beta_0)$, which is shown when verifying (A3). Hence $A = B$ and $A^{-1}B(A^{-1})' = A^{-1} = I^{-1}(\beta_0)$, and

$$\sqrt{n}\mathbb{P}_n l_{\beta_0}^*(\theta_0; Z) = n^{-1/2} \sum_{i=1}^n l_{\beta_0}^*(Y_i, \Delta_i, X_i).$$

Thus we complete the proof of Theorem 4.2.

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SUPPLEMENTARY MATERIAL

Additional proofs (DOI: [10.1214/11-AOS934SUPP](https://doi.org/10.1214/11-AOS934SUPP); .pdf). The supplementary document contains proofs of technical lemmas and Theorems 4.1 and 4.3.

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DEPARTMENT OF BIostatISTICS
UNIVERSITY OF MICHIGAN
1420 WASHINGTON HEIGHTS
ANN ARBOR, MICHIGAN 48109-2029
USA
E-MAIL: yingding@umich.edu
bnan@umich.edu