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A SIGN-CHANGING SOLUTION FOR A SUPERLINEAR DIRICHLET PROBLEM

ALFONSO CASTRO, JORGE COSSIO AND JOHN M. NEUBERGER

ABSTRACT. We show that a superlinear boundary value problem has at least three nontrivial solutions. A pair are of one sign (positive and negative, respectively), and the third solution changes sign exactly once. The *critical level* of the sign-changing solution is bounded below by the sum of the two lesser levels of the one-sign solutions. If nondegenerate, the one sign solutions are of *Morse index* 1 and the sign-changing solution has Morse index 2. Our results extend and complement those of Z.Q. Wang [**12**].

1. Introduction. Let Ω be a smooth bounded region in $\mathbf{R}^N$, Δ the Laplacian operator, and $f \in C^1(\mathbf{R}, \mathbf{R})$ such that $f(0) = 0$. We assume that there exist constants $A > 0$ and $p \in (1, (N+2)/(N-2))$ such that $|f'(u)| \leq A(|u|^{p-1} + 1)$ for all $u \in \mathbf{R}$. Hence, f is *subcritical*, i.e., there exists $B > 0$ such that $|f(u)| \leq B(|u|^p + 1)$. Also, we assume that there exists $m \in (0,1)$ such that

$$f(u)u - 2F(u) \geq muf(u), \tag{1}$$

where $F(u) = \int_0^u f(s)\,ds$, for all $u \in \mathbf{R}$. Finally, we make the assumption that f satisfies

$$f'(u) > \frac{f(u)}{u} \quad \text{for } u \neq 0 \quad \text{and} \quad \lim_{|u|\to\infty} \frac{f(u)}{u} = \infty, \tag{2}$$

i.e., f is *superlinear*. Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω. In this paper we study the boundary value problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{3}$$

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Let H be the Sobolev space $H_0^{1,2}(\Omega)$ with inner product $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx$ (see [**1**] or [**5**]). We define $J : H \to \mathbf{R}$ by

$$J(u) = \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 - F(u) \right\} dx.$$

By regularity theory for elliptic boundary value problems (see [**5**]), u is a solution to (3) if and only if u is a critical point of the action functional J. We prove the following result:

Theorem 1.1. *If $f'(0) < \lambda_1$, then* (3) *has at least three nontrivial solutions*: $\omega_1 > 0$ *in* Ω, $\omega_2 < 0$ *in* Ω *and* ω_3. *The function* ω_3 *changes sign exactly once in* Ω, *i.e.*, $(\omega_3)^{-1}(\mathbf{R} - \{0\})$ *has exactly two connected components. If nondegenerate, the one-sign solutions are Morse index* 1 *critical points of* J, *and the sign-changing solution has Morse index* 2. *Furthermore,*

$$J(\omega_3) \geq J(\omega_1) + J(\omega_2).$$

Remark. One can relax condition (1) to hold only for $|u|$ sufficiently large. This is necessary in order to consider cases where $f'(0) > 0$.

To the best of our knowledge, these results are the first to establish the existence of a *sign-changing* solution to (3). Developments such as those in [**12**] do not imply that any solution changes sign, much less that it changes sign exactly once. Also, our proofs improve on the information about the action level and Morse index of solutions provided in [**12**]. For the sake of completeness, we have included independent and direct proofs of the existence of ω_1 and ω_2 (see [**9**] and [**12**]). Another proof of the existence of a third nontrivial solution to (3) can be found in [**11**], but the method does not show the solution to change sign. Additionally, one can use odd nonlinearities and sources such as [**3**] and [**7**] to prove some of our one-sign results. However, our methods seem more direct and informative.

We note that if $f'(0) > \lambda_1$, then by multiplying (3) by an eigenfunction corresponding to λ_1 and integrating by parts, it is easily seen that (3) does not have one-signed solutions.

2. Preliminary lemmas. Our assumptions on f imply that $J \in C^2(H, \mathbf{R})$ (see [**9**]), and that

$$
\begin{aligned}
J'(u)(v) &= \langle \nabla J(u), v \rangle \\
&= \int_\Omega \{(\nabla u \cdot \nabla v - f(u)v\} \, dx, \quad \text{for all } v \in H.
\end{aligned} \tag{4}
$$

Define $\gamma : H \to \mathbf{R}$ by $\gamma(u) = \langle \nabla J(u), u \rangle = \int_\Omega \{|\nabla u|^2 - uf(u)\} \, dx$, and compute

$$
\begin{aligned}
\gamma'(u)(v) &= \langle \nabla \gamma(u), v \rangle \\
&= 2 \int_\Omega \nabla u \cdot \nabla v \, dx \\
&\quad - \int_\Omega f(u) v \, dx - \int_\Omega f'(u) uv \, dx.
\end{aligned} \tag{5}
$$

Definition 2.1. For $u \in L^1(\Omega)$, we define $u_+(x) = \max\{u(x), 0\} \in L^1(\Omega)$ and $u_-(x) = \min\{u(x), 0\} \in L^1(\Omega)$. If $u \in H$, then $u_+, u_- \in H$ (see [**8**]). We say that $u \in L^1(\Omega)$ *changes sign* if $u_+ \neq 0$ and $u_- \neq 0$. For $u \neq 0$ we say that u is *positive* (and write $u > 0$) if $u_- = 0$, and similarly, u is *negative* ($u < 0$), if $u_+ = 0$.

We define $S \subset H$ and various subsets of S:

$$
\begin{aligned}
S &= \{u \in H - \{0\} : \gamma(u) = 0\}, & \hat{S} &= \{u \in S : u_+ \neq 0, u_- \neq 0\}, \\
S_1 &= \{u \in \hat{S} : \gamma(u_+) = 0\}, & G^+ &= \{u \in S : u > 0\}, \\
\hat{S}^+ &= \{u \in S : \gamma(u_+) < 0\}, & W^+ &= G^+ \cup \hat{S}^+, \\
G^- &= \{u \in S : u < 0\}, & \hat{S}^- &= \{u \in S : \gamma(u_+) > 0\},
\end{aligned}
$$
$$
W^- = G^- \cup \hat{S}^-.
$$

We have the disjoint unions $S = G^+ \cup \hat{S} \cup G^-$ and $\hat{S} = \hat{S}^+ \cup S_1 \cup \hat{S}^-$. We note that nontrivial solutions to (3) are in S, one-sign solutions are in $G^+ \cup G^-$, and sign-changing solutions are in S_1. We define $S^\infty = \{u \in H : \|u\| = 1\}$.

Lemma 2.2. *Under the above assumptions, we have*

(a) 0 *is a local minimum of* J. *If* $u \in H - \{0\}$, *then there exists a unique* $\bar\lambda = \bar\lambda(u) \in (0,\infty)$ *such that* $\bar\lambda u \in S$. *Moreover,* $J(\bar\lambda u) = \max_{\lambda>0} J(\lambda u) > 0$. *If* $\gamma(u) < 0$, *then* $\bar\lambda < 1$, *and if* $\gamma(u) > 0$, *then* $\bar\lambda > 1$ *and* $J(u) > 0$.

(b) *The function* $\bar\lambda \in C^1(S^\infty, (0,\infty))$. *The set* S *is closed, unbounded, and a connected* C^1-*submanifold of* H *diffeomorphic to* S^∞.

(c) $u \in S$ *is a critical point of* J *on* H *if and only if* u *is a critical point of* $J|_S$.

(d) $J|_S$ *is coercive, i.e.,* $J(u) \to \infty$ *as* $\|u\| \to \infty$ *in* S. *Also,* $0 \notin S$ *and* $\inf_S J > 0$.

Proof. (a) Let $u \in H - \{0\}$ be a fixed function, and define $\Phi : \mathbf{R} \to \mathbf{R}$ by

$$\Phi(\lambda) = J(\lambda u) = \frac{1}{2}\lambda^2\|u\|^2 - \int_\Omega F(\lambda u)\,dx.$$

Differentiating Φ yields

$$\begin{aligned}\Phi'(\lambda) &= \lambda\|u\|^2 - \int_\Omega f(\lambda u)u\,dx = \gamma(\lambda u)/\lambda,\\ \Phi''(\lambda) &= \|u\|^2 - \int_\Omega f'(\lambda u)u^2\,dx.\end{aligned} \tag{6}$$

If $\lambda > 0$ is a critical point of Φ, then $\|u\|^2 = \int_\Omega ((f(\lambda u))/\lambda)u\,dx$, and by (2) we have

$$\begin{aligned}\Phi''(\lambda) &= \int_\Omega \frac{f(\lambda u)}{\lambda} u\,dx - \int_\Omega f'(\lambda u)u^2\,dx\\ &= \int_\Omega \left\{u^2\left(\frac{f(\lambda u)}{\lambda u}\right) - f'(\lambda u)\right\}dx < 0.\end{aligned}$$

Thus, every critical point of Φ in $(0,\infty)$ is a strict local maximum, and hence, Φ has at most one critical point in $(0,\infty)$. Using Poincare's inequality, $\lambda_1 \int_\Omega u^2 \le \|u\|^2$ (see [**1**] or [**5**]), we have

$$\begin{aligned}J''(0)(u,u) = \Phi''(0) &= \|u\|^2 - \int_\Omega f'(0)u^2\,dx\\ &\ge \left(1 - \frac{f'(0)}{\lambda_1}\right)\|u\|^2 > 0,\end{aligned} \tag{7}$$

where we have used $f'(0) < \lambda_1$. Since $\Phi'(0) = 0$, by (7) we see that $\Phi'(\lambda) > 0$ for $\lambda > 0$ small. Since we also have $\Phi(0) = 0$, the above comments imply that J has a local minimum of 0 at $0 \in H$. On the other hand, because f is superlinear and because of (2), it follows that $\lim_{\lambda\to\infty} \Phi'(\lambda) = -\infty$ (see also (6)). Therefore, Φ' has a unique zero $\bar\lambda \in (0,\infty)$ and $\bar\lambda u \in S$. Thus $\Phi'(\lambda) = \gamma(\lambda u)/\lambda > 0$ for $\lambda < \bar\lambda$, and similarly, $\Phi'(\lambda) = \gamma(\lambda u)/\lambda < 0$ for $\lambda > \bar\lambda$. In particular, this shows that, given $u \in H$ such that $\gamma(u) < 0$, $\gamma(u) > 0$, there exists $\alpha < 1$, $\alpha > 1$, such that $\gamma(\alpha u) = 0$, i.e., $\alpha u \in S$.

(b) If $u \in S$, then by (2) we have

$$\begin{aligned}\langle \nabla\gamma(u), u\rangle &= \int_\Omega \{|\nabla u|^2 - f'(u)u^2\}\, dx \\ &< \int_\Omega \{|\nabla u|^2 - f(u)u\}\, dx = 0.\end{aligned} \tag{8}$$

Thus, by the implicit function theorem, S is a C^1-submanifold of H, see [**6**]. By the continuity of γ, the submanifold S is a closed subset of H. The remaining facts can be discerned from [**3**] and [**4**], but for completeness we include arguments. let $\chi : (0,\infty)\times S^\infty \to \mathbf{R}$ be defined by $\chi(a,u) = \gamma(au)$. For $u \in S^\infty$ and $a > 0$ we see that $\chi(a,u) = 0$ if and only if $au \in S$. Hence, as in (8),

$$\frac{\partial\chi}{\partial a}(\bar\lambda(u), u) = \langle \nabla\gamma(\bar\lambda(u)u), u\rangle < 0,$$

and thus the implicit function theorem further implies that $\bar\lambda \in C^1(S^\infty, (0,\infty))$. The map $u \to \bar\lambda(u)u$ is a bijection and also of class C^1, and hence a diffeomorphism between S and S^∞. From this it also follows that S is path-wise connected.

Let us see that S is unbounded. Without loss of generality we may assume that $0 \in \Omega$. Let $d > 0$ be sufficiently small so that $D = (0,d) \times \cdots \times (0,d) \subset \Omega$. For $k \in \mathbf{N}$, we define the functions $\psi_k : \Omega \to \mathbf{R}$ by

$$\psi_k(x_1, \dots, x_n) = \begin{cases} \sin(k\pi x_1/d)\cdots\sin(k\pi x_n/d) & \text{for } x \in D \\ 0 & \text{for } x \in \Omega - D.\end{cases}$$

Let $M_1 = n(\pi/d)^2(d/2)^n$ and $M_2 = d^n \max\{f(1), -f(-1)\}$. Then $\|\psi_k\|^2 = M_1 k^2$. Also, since $|\psi_k| \le 1$ on Ω, we have $\int_\Omega \psi_k f(\psi_k)\, dx \le$

M_2. Thus,

$$\begin{aligned}\gamma(\psi_k) &= \|\psi_k\|^2 - \int_\Omega \psi_k f(\psi_k)\,dx \\ &\geq M_1 k^2 - M_2 \longrightarrow \infty \quad \text{as } k \longrightarrow \infty.\end{aligned}$$

Let k be sufficiently large so that $\gamma(\psi_k) > 0$. Using Lemma 2.2(a), we see that there exist $\alpha_k > 1$ so that $\alpha_k \psi_k \in S$. We conclude S is unbounded, since

$$\|\alpha_k \psi_k\| = \alpha_k \|\psi_k\| \geq \|\psi_k\| \longrightarrow \infty \quad \text{as } k \longrightarrow \infty.$$

(c) The forward implication is obvious. Let $u \in S$ be a critical point of $J|_S$ and $T_u S = \{v : v \perp \nabla\gamma(u)\}$ be the tangent space to S at u. It follows that $J'(u)(v) = 0$ for all $v \in T_u S$ and $J'(u)(u) = \gamma(u) = 0$. Since the submanifold S has codimension 1 in H, we need only show $u \notin T_u S$ to conclude $\nabla J(u) = 0$. In fact, $v \in T_u S$ implies $\langle \nabla\gamma(u), v\rangle = 0$ while $u \in S$ implies $\langle \nabla\gamma(u), u\rangle < 0$, see (8).

(d) For $u \in S$, we recall that $\|u\|^2 = \int_\Omega |\nabla u|^2\,dx = \int_\Omega u f(u)\,dx$, whence it follows that

$$\begin{aligned}J(u) &= \int_\Omega \left\{\frac{1}{2}|\nabla u|^2 - F(u)\right\} dx \\ &= \frac{1}{2}\int_\Omega \{f(u)u - 2F(u)\}\,dx \qquad (9)\\ &\geq \frac{1}{2}\int_\Omega m u f(u)\,dx = \frac{m}{2}\|u\|^2 \longrightarrow \infty, \quad \text{as } \|u\| \longrightarrow \infty,\end{aligned}$$

where we have used (1). Since $S \subset H$ is closed and $0 \notin S$, there exists $\delta > 0$ such that if $u \in S$ then $\|u\| \geq \delta$. This and (9) imply $\inf_S J \geq m\delta^2/2 > 0$. □

Our next lemma, in particular, shows that $\hat{S}$ is an open subset of S, whereas S_1 is closed.

Lemma 2.3. *The function $h : H \to H$ defined by $h(u) = u_+$ is continuous. Also, h defines a continuous function from $L^{p+1}(\Omega)$ into itself.*

Proof. Let $u \in H$. By Corollary A.5 of [**8**], $u_+ \in H$ and $u_- \in H$. By Lemmas A.3 and A.4 of [**8**], $\nabla u = 0$ almost everywhere on $\{x \in \Omega : u(x) = 0\}$. Hence $\nabla h(u) = j(u)\nabla u$ almost everywhere in Ω, where $j(u)(x) = 1$ if $u(x) \geq 0$ and $j(u)(x) = 0$ if $u(x) < 0$. Let $u_n \to u$ in H, where without loss of generality we can assume $u_n \to u$ almost everywhere on Ω. Note that

$$\begin{aligned}
\|h(u_n) - h(u)\|^2 &= \int_\Omega |\nabla h(u_n) - \nabla h(u)|^2 \, dx \\
&= \int_\Omega |j(u_n)(\nabla u_n - \nabla u) \, dx \\
&\quad + (j(u_n) - j(u))\nabla u|^2 \\
&\leq 2 \int_\Omega |\nabla u_n - \nabla u|^2 \, dx \\
&\quad + 2 \int_\Omega (j(u_n) - j(u))^2 |\nabla u|^2 \, dx.
\end{aligned}$$

The first term in the last expression converges to 0 since $u_n \to u$ in H. For almost all $x \in \Omega$ we have $j(u_n) \to j(u) = 0$, $j(u_n) \to j(u) = 1$, or $\nabla u = 0$. Hence, the last integrand converges almost everywhere to 0. This integrand is bounded by $|\nabla u|^2$, thus by the dominated convergence theorem its integral converges to 0. Hence $\|h(u_n) - h(u)\|^2 \to 0$ and $h \in C(H, H)$. Also, by the dominated convergence theorem, h is continuous on $L^{p+1}(\Omega)$. $\square$

An important consequence of Lemmas A.3 and A.4 of [**8**] is the fact that $J(u) = J(u_+) + J(u_-)$ and $\gamma(u) = \gamma(u_+) + \gamma(u_-)$ for all $u \in H$.

Lemma 2.4. *Given $w \in \hat{S}$, there exists a path $r_w \equiv r \in C^1([0,1], S)$ such that*

(a) $r(0) = aw_+ \in G^+$ *for some* $a > 0$, $r(1) = bw_- \in G^-$ *for some* $b > 0$, $r(a/(a+b)) = w$.

(b) $a > 1$ *if and only if* $b < 1$ *if and only if* $w \in \hat{S}^+$, $\gamma(r(t)_+) < 0$ *if and only if* $t \in (0, 1/2)$ *if and only if* $r(t) \in \hat{S}^+$.

(c) $r(1/2) = aw_+ + bw_- \in S_1$, $r([0,1]) \cap S_1 = \{r(1/2)\}$.

(d) $J(r(0)) < J(r(t)) < J(r(1/2))$ *for* $t \in (0, 1/2)$.

Proof. For justification of the steps in this proof, the reader is referred to Lemma 2.2 (a) and (b). Since w changes sign, $w_+, w_- \neq 0$, and hence, there exists $a, b > 0$ such that $aw_+ \in G^+$ and $bw_- \in G^-$. Observe that, for all $t \in [0,1]$, the element $(1-t)aw_+ + tbw_- \neq 0$, and thus there exists $\alpha \in C^1([0,1], \mathbf{R})$ such that

$$r(t) = \alpha(t)[(1-t)aw_+ + tbw_-] \in S.$$

We see also that $r \in C^1([0,1], S)$, $r(0) = aw_+ \in G^+$, $r(1/2) = aw_+ + bw_- \in S_1$, and $r(1) = bw_- \in G^-$. Furthermore, we note that $r(a/(a+b)) = \alpha(ab)/(a+b)w \in S$ implies $\alpha = (a+b)/(ab)$ and $r(a/(a+b)) = w$. We also have $w \in S_1$ if and only if $a = b = 1$ and $a/(a+b) = 1/2$. We observe that $w \in \hat{S}^+$ if and only if $a < 1$, $b > 1$, and $0 < a/(a+b) < 1/2$. We recall from Lemma 2.2 (a) that in this case $J(w_-) > 0$, and hence $J(aw_-) > 0$. Thus, for $w \in \hat{S}^+$, we have

$$\begin{aligned} J(r(0)) &< J(aw_+) + J(aw_-) = J(aw) < J(w) \\ &= J\left(r\left(\frac{a}{a+b}\right)\right) < J(aw_+) + J(bw_-) = J\left(r\left(\frac{1}{2}\right)\right). \end{aligned}$$

Similar inequalities hold for $w \in \hat{S}^-$ and $r(t)$ for $t \in [1/2, 1]$. $\square$

Lemma 2.5. *The sets G^+, $\hat{S}^+$, W^+, S_1, W^-, $\hat{S}^-$ and G^- satisfy*:

(a) *G^+, S_1 and G^- are closed, and G^+, G^- are connected.*

(b) *$\hat{S}$ is open, and the subsets $\hat{S}^+$ and $\hat{S}^-$ are open and separated by S_1.*

(c) *W^+ and W^- are the only two components of $S - S_1$. In particular, G^+ and G^- are separated by S_1.*

(d) *If $w \in G^+$ and $J(w) = \min_{G^+} J$, then $J(w) = \min_{W^+} J$ and w is a critical point of J.*

Proof. By the continuity of $\gamma \circ h$, see Lemma 2.3, G^+, G^- and S_1 are closed. Likewise, $\hat{S}$, $\hat{S}^+$, $\hat{S}^-$ and $W^+ \cup W^- = S - S_1$ are open. Given $u, v \in G^+$, the convex linear combination $\{z(t) = tu + (1-t)v : t \in [0,1]\}$ projects onto the path $\{\alpha(t)z(t) \in G^+ : t \in [0,1]\}$, where $\alpha \in C^1([0,1], \mathbf{R})$, see Lemmas 2.2 (a) and (2.4). Hence, G^+, analogously G^-, is connected.

Let $\sigma : [0,1] \to \hat{S}$ be a path connecting $\sigma(0) \in \hat{S}^+$ and $\sigma(1) \in \hat{S}^-$. By applying the intermediate value theorem to the continuous function $\gamma \circ h \circ \sigma$, we see that $\sigma([0,1]) \cap S_1 \neq \varnothing$. Hence, S_1 separates $\hat{S}^+$ and $\hat{S}^-$.

By Lemma 2.4 (b), for each $w \in \hat{S}^+$, the path $r_2([0, a/(a+b)]) \subset W^+$ connects w with $aw_+ \in G^+$. Since, by (a), G^+ is connected, the set W^+ is connected. Similarly, W^- is connected. Since W^+ and W^- cannot lie in the same component of $S - S_1$ and $S = W^+ \cup S_1 \cup W^-$, it follows that they are exactly the two distinct open connected components of $S - S_1$.

Lastly, by Lemma 2.4, we see that, given $u \in \hat{S}^+$, there exists $v = r_u(0) \in G^+$ such that $J(v) < J(u)$. This implies that $\inf_{W^+} J = \inf_{G^+} J$. Since W^+ is open in S, by Lemma 2.2 if $J(w) = \min_{G^+} J$, then w is a critical point of J. $\square$

Lemma 2.6. *If $w \in S_1$ and $J(w) = \min_{S_1} J$, then w is a critical point of J.*

Proof. In order to conclude that w is a critical point of J, it is sufficient to show that w is a critical point of $J|_S$, see Lemma 2.2 (c). For $u \in S$, let P_u denote the orthogonal projection of H onto the tangent space $T_u S$. Suppose now that $\nabla J(w) \neq 0$. By part c) of Lemma 2.2, we see that $P_w \nabla J(w) \neq 0$. Let $\delta \in (0, (1/2) \min\{\|w_+\|, \|w_-\|\})$, $C = \{w\}$, and $B = \{u \in S : \|u - w\| \geq \delta\}$. Since B is closed and C is compact, by a version of the so-called deformation lemma, see [**4**], there exists a continuous function $\Lambda : [0,1] \times S \to S$ and $t_0 > 0$ such that for all $t \in [0, t_0)$ the following hold:

(a) $\Lambda(t,x) = x$ for all $x \in B$,

(b) $J(\Lambda(t,x)) \leq J(x)$ for all $x \in S$,

(c) $J(\Lambda(t,w)) \leq J(w) - (t/4)\|P_w \nabla J(w)\|$.

Since $w \in S_1$, we can now construct the path $r(t)$ of Lemma 2.4, whereupon we define the deformed path $r_1(t) = \Lambda(t_0/2, r(t))$. For

$t \neq 1/2$, we have $J(r_1(t)) \leq J(r(t)) < J(r(1/2)) = J(w)$. Additionally,

$$\begin{aligned} J(r_1(1/2)) &= J(\Lambda(t_0/2, r(1/2))) = J(\Lambda(t_0/2, w)) \\ &< J(w) - (t_0/8)\|P_w \nabla J(w)\| < J(w), \end{aligned}$$

and, as a result,

$$\max\{J(r_1(t)) : t \in [0,1]\} < J(w) = \min_{S_1} J. \tag{10}$$

By the definition of δ, we see that $w_+ \in B$ and $w_- \in B$, whence $r_1(0) = \Lambda(t_0/2, r(0)) = r(0) = w_+ \in G^+$. Similarly, $r_1(1) = w_- \in G^-$, so Lemma 2.5 (c) shows $r_1[0,1] \cap S_1$ is nonempty. This contradiction of (10) proves the lemma. □

3. Existence of a sign-changing solution. Now we show that there exists a solution ω_3 to (3) which changes sign exactly once. If the solution is a nondegenerate critical point, then it has Morse index 2.

Let $c_3 = \inf_{S_1} J$ and $\{u_n\} \subset S_1$ be such that $J(u_n) \to c_3$. Using $\gamma((u_n)_+) = 0$, we see that $\{(u_n)_+\} \subset G^+$ and $\{(u_n)_-\} \subset G^-$. Since J is coercive, see (9), $\{u_n\}$ is bounded. By the Sobolev imbedding theorem, without loss of generality, we can assume that there exist $u, v, w \in H$ such that

$$\begin{aligned} &u_n \rightharpoonup u, \qquad (u_n)_+ \rightharpoonup v, \qquad (u_n)_- \rightharpoonup w \quad \text{in } H, \\ &u_n \longrightarrow u, \qquad (u_n)_+ \longrightarrow v, \qquad (u_n)_- \longrightarrow w \quad \text{in } L^{p+1}(\Omega). \end{aligned}$$

By Lemma 2.3 we know that $h : L^{p+1} \to L^{p+1}$, $u \to u_+$, is a continuous transformation, so we see that $u_+ = v \geq 0$ and $u_- = w \leq 0$. Let us see that $u \in S_1$. Since $(u_n)_+ \to u_+$ in L^{p+1} and f is subcritical,

$$\int_\Omega F((u_n)_+)\, dx \longrightarrow \int_\Omega F(u_+)\, dx$$

and

$$\int_\Omega f((u_n)_+)(u_n)_+\, dx \longrightarrow \int_\Omega f(u_+)\, dx.$$

By $(u_n)_+ \in S$ and Lemma 2.2 (d), we see that $\int_\Omega u_+ f(u_+)\, dx = \lim \int_\Omega (u_n)_+ f((u_n)_+)\, dx = \lim \|(u_n)_+\|^2 > 0$, consequentially $u_+, u_- \neq$

0 and $u = u_+ + u_-$ is sign changing. Let us see that $(u_n)_+ \to u_+$ in H. If we suppose not, then without loss of generality we may assume that $\|u_+\|^2 < \liminf_{n\to\infty} \|(u_n)_+\|^2$, whence

$$\begin{aligned}\gamma(u_+) &= \|u_+\|^2 - \int_\Omega u_+ f(u_+)\,dx \\ &< \liminf_{n\to\infty} \|(u_n)_+\|^2 - \int_\Omega (u_n)_+ f((u_n)_+)\,dx = 0.\end{aligned}$$

From Lemma 2.2 (a) we see that there exists $0 < \alpha < 1$ such that $\alpha u_+ \in G^+$, and similarly that there exists $0 < \beta \le 1$ such that $\beta u_- \in G^-$. We conclude that $\alpha u_+ + \beta u_- \in S_1$. This provides a contradiction, since

$$\begin{aligned}J(\alpha u_+ + \beta u_-) &< \liminf_{n\to\infty} J(\alpha(u_n)_+ + \beta(u_n)_-) \\ &= \liminf_{n\to\infty}\{J(\alpha(u_n)_+) + J(\beta(u_n)_-)\} \\ &\le \liminf_{n\to\infty}\{J((u_n)_+) + J(u_n)_-)\} \\ &= \liminf_{n\to\infty} J(u_n) = \inf_{S_1} J = c_3.\end{aligned}$$

Hence $(u_n)_+ \to u_+$ in H and $\alpha = 1$. Similarly, we conclude that $(u_n)_- \to u_-$ in H and $\beta = 1$, which proves that $u \in S_1$, $u_n \to u$ in H, and $J(u) = c_3$. Letting $\omega_3 = u$, we see that $J|_{S_1}$ attains its minimum at ω_3. We find that ω_3 is a critical point of J, see Lemma 2.6, and hence a solution to (3).

Let us see that ω_3 changes sign exactly once. Since ω_3 is of class C^2, hence continuous, $E = \{x \in \Omega : u(x) \ne 0\}$ is open. Suppose E has more than two components. Since ω_3 changes sign, without loss of generality we can assume that there exist connected components A, B and C of E such that $u > 0$ in A and $u < 0$ in B. Let u_A, u_B and u_C be the zero extensions of $\omega_3|_A$, $\omega_3|_B$ and $\omega_3|_C$ to all of Ω. Since $\Delta\omega_3 + f(\omega_3) = 0$ on Ω, it follows that $\gamma(u_A) = \gamma(u_B) = \gamma(u_C) = 0$. Hence, $J > 0$ on S implies $J(u_A + u_B) < J(u_A + u_B + u_C) \le J(\omega_3) = c_3$, a contradiction since $u_A + u_B \in S_1$. We conclude that E has exactly two components.

If ω_3 is a nondegenerate critical point, we see that ω_3 has Morse index 2 in H by observing that $J''(\omega_3)(v,v) < 0$ for $v = (\omega_3)_+$ and $v = (\omega_3)_-$, $\langle(\omega_3)_+, (\omega_3)_-\rangle = 0$ and $J|_{S_1}$ has a minimum at ω_3.

4. Existence of one-sign solutions. For the sake of completeness, we establish the existence of the solution $\omega_1 > 0$ (one finds ω_2 similarly). Furthermore, we show that $J|_S$ has local minima at ω_1 and ω_2, and hence these two critical points are of Mores index 1, if nondegenerate.

We define $c_1 = \inf_{G^+} J$ and take $\{u_n\} \subset G^+$ with $\lim_{n\to\infty} J(u_n) = c_1$. As in Section 3, the coercivity of J and the subcritical condition on f allow us to apply the Sobolev imbedding theorem, whence we find $\bar{u} \in H \subset L^{p+1}$ such that, without loss of generality,

$$u_n \rightharpoonup \bar{u} \quad \text{in } H, \qquad u_n \longrightarrow \bar{u} \quad \text{in } L^{p+1},$$

$$\int_\Omega u_n f(u_n)\, dx \longrightarrow \int_\Omega \bar{u} f(\bar{u})\, dx,$$

$$\int_\Omega F(u_n)\, dx \longrightarrow \int_\Omega F(\bar{u})\, dx.$$

That $\bar{u} \neq 0$ is evident, as $\int_\Omega \bar{u} f(\bar{u})\, dx = \lim_{n\to\infty} \int_\Omega u_n f(u_n)\, dx = \lim_{n\to\infty} \|u_n\|^2 > 0$, see Lemma 2.2 (d). By the continuity of $h : L^{p+1} \to L^{p+1}$, we see that $\bar{u} > 0$. We wish to show that $u_n \to \bar{u}$ in H. If we assume to the contrary that $u_n \not\to \bar{u}$ in H, then without loss of generality we may assume that $\|\bar{u}\|^2 < \liminf \|u_n\|^2$. It follows that $\gamma(\bar{u}) < \liminf \gamma(u_n) = 0$, so by Lemma 2.1 (a) there exists $0 < \alpha < 1$ such that $\alpha\bar{u} \in G^+$. Consequently, we get the following contradiction,

$$J(\alpha\bar{u}) < \liminf_{n\to\infty} J(\alpha u_n) \le \liminf_{n\to\infty} J(u_n) = \inf_{G^+} J = c_1.$$

We conclude that $u_n \to \bar{u}$ in H, $\alpha = 1$, $\bar{u} \in G^+$ and $J(\bar{u}) = \min_{G^+} J = c_1$. By Lemma 2.5 (d) we see that $\omega_1 = \bar{u} \in G^+$ is a critical point of J and hence a positive solution to (3).

We obtain the solution $\omega_2 \in G^- \subset W^-$ in the same fashion, whence we can define $c_2 = \inf_{G^-} J = \inf_{W^-} J = J(\omega_2)$. Finally, to conclude the proof of Theorem 1.1, we observe that

$$c_3 = J(\omega_3) = J((\omega_3)_+) + J((\omega_3)_-) \ge J(\omega_1) + J(\omega_2) = c_1 + c_2.$$

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Department of Mathematics, University of North Texas, Denton, Texas 76203
E-mail address: acastro@unt.edu

Departamento de Matemáticas, Universidad Nacional de Colombia, Apartado Aéreo 3840, Medellín, Colombia.
E-mail address: jcossio@sigma.eafit.edu.co

Department of Mathematics, University of North Texas, Denton, Texas 76203

Current address: Department of Mathematics, Northern Arizona University, Flagstaff, AZ 86011-5717
E-mail address: John.Neuberger@NAU.Edu