A Simple Accurate Method for Generating Autocorrelated Nakagami-*m* Envelope Sequences

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***Abstract*— We propose a simple accurate method for generating autocorrelated Nakagami-*m* envelope sequences. The method allows for arbitrary values of fading parameter and nonisotropic fading scenarios. In essence, Nakagami-*m* samples are first drawn and then rearranged to match the Nakagami-*m* autocorrelation. The rearrangement is made in accordance with the rank statistics of an underlying Rayleigh reference sequence with the desired autocorrelation. Examples illustrate the excellent performance of the new method.**

***Index Terms*— Nakagami-*m* fading channels, simulation.**

I. Introduction

MUCH attention has been given to the Nakagami-*m* distribution for its flexibility, mathematical ease, and good fit to measured fading data. However, few published works address the issue of the Nakagami-*m* simulation. This is partly due to the lack of a well-established dynamic model for Nakagami-*m* fading channels. Such a model was not specified when the Nakagami-*m* distribution was proposed [1]. As a result, most simulators lay hold of particular assumptions to accomplish the temporal correlation of the Nakagami-*m* channel (cf. [2] and references therein).

The Nakagami-*m* envelope autocorrelation has been recently derived on a physical basis in [3] and [4]. Both derivations are rooted in the well-accepted model of the squared Nakagami-*m* envelope as the sum of i.i.d. squared Rayleigh envelopes [1], [5]. Indeed, a general envelope crosscorrelation between Nakagami-*m* fading processes with time-space-frequency diversity is given in [3], whereas the special case of nil space-frequency diversity, corresponding to the Nakagami-*m* envelope autocorrelation, is addressed in [4].

In this Letter, we present a simple accurate method for generating autocorrelated Nakagami-*m* envelope sequences, allowing for arbitrary values of fading parameter and non-isotropic fading scenarios. In essence, samples matching the Nakagami-*m* distribution are first drawn and then rearranged to match the Nakagami-*m* autocorrelation. The rearrangement is made in accordance with the rank statistics of an underlying Rayleigh reference sequence with the desired autocorrelation. Motivation to this approach is given along the text. More importantly, several examples are shown that illustrate the strikingly excellent performance of the new method.

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We have compared our method to the "approximation method" in [2] and empirically found that the simulation results of both methods are indeed very close to each other, although in [2] only the isotropic scenario has been considered. As for the approaches used, in [2], for each desired value of the fading parameter m, a non-linear optimization algorithm must first be run in order to obtain the coefficients of an approximate inverse Nakagami-*m* distribution. Our method inherently complies with the exact Nakagami-*m* distribution, and no numerical fitting is required. Comparisons to other existing methods have been discarded for their inherent limitations (cf. [2] for a complete discussion).

II. The Nakagami-*m* Model Revisited

The probability density function (PDF) of the Nakagami-*m* envelope R is given by [1]

$$f_R(r) = \frac{2m^m r^{2m-1}}{\Gamma(m)\Omega^m} \exp\left(-\frac{mr^2}{\Omega}\right) \tag{1}$$

and, based on the model of the squared Nakagami-*m* envelope as the sum of m i.i.d. squared Rayleigh envelopes [1], [5], the Nakagami-*m* envelope autocorrelation function (ACF) is found as [3, Eq. (1)] or [4, Eq. (25)]

$$R_R(\tau) = \frac{\Omega\ \Gamma^2\left(m+\frac{1}{2}\right)}{m\Gamma^2(m)}\ {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; m; \rho_2(\tau)\right) \tag{2}$$

where $\Omega = E[R^2]$ is the mean power, $m = \Omega^2/V[R^2]$ is the Nakagami-*m* fading parameter, $\Gamma(\cdot)$ is the gamma function, ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function, and $\rho_2(\tau)$ is the autocorrelation coefficient (ACC) of each underlying squared Rayleigh envelope. ($E[\cdot]$ denotes mean, $V[\cdot]$ variance.)

In this work, we address the general case of nonisotropic fading scenarios, for which the distribution of the angle of arrival (AOA) of the multipath waves is nonuniform. A plausible model for the directional AOA is the parametric Von Mises/Tikhonov distribution [6]. For this model, the squared Rayleigh ACC is obtained as [6]

$$\rho_2(\tau) = \left|\frac{I_0\left(\sqrt{\kappa^2-(2\pi f_D\tau)^2+j4\kappa f_D\tau\cos\mu}\right)}{I_0(\kappa)}\right|^2 \tag{3}$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind and zeroth order, f_D is the maximum Doppler frequency in Hz, μ represents the mean direction of the AOA, and κ controls the beamwidth. In particular, for $\kappa = 0$, we have the isotropic scenario with uniform AOA, for which $\rho_2(\tau) = J_0^2(2\pi f_D\tau)$.

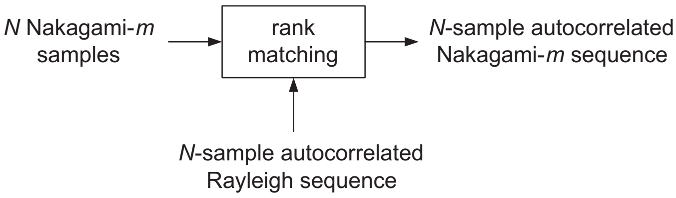

Fig. 1. The Nakagami-*m* envelope simulator.

III. THE NAKAGAMI-*m* SIMULATOR

Our aim is to generate an N-sample sequence matching the Nakagami-*m* PDF (1) and ACF (2), for an arbitrary value of fading parameter m. In principle, the standard approach of decomposing the Nakagami-*m* signal into m Rayleigh elements could be used [5], but this approach clearly applies to m integer only. In this section, we present a simple, general, accurate method for generating autocorrelated Nakagami-*m* sequences with m arbitrary. (Although in [1] the fading parameter is restricted to $m > 1/2$, the method proposed here indeed applies to any $m > 0$.)

Due to the lack of a dynamic Nakagami-*m* model for m arbitrary, our approach is to circumvent the traditional simulation paradigm, in which the output samples are generated in a single-shot manner, already fulfilling both the static (PDF) and the dynamic (ACF) requirements of the process. Instead, we propose i) first to draw N samples matching the Nakagami-*m* PDF and ii) then to rearrange these samples to match the Nakagami-*m* ACF. Of course, this approach inherently complies with the PDF requirements, since any rearrangement does not affect the distribution of the samples.

The first task—to draw Nakagami-*m* samples—is simple and can be accomplished by well-established methods of random generation, such as the percentile transformation method or the rejection method. Indeed, most commercial software packages have built-in routines for the generation of gamma distributed samples (e.g., the `gamrnd(·)` Matlab function), whose square root yields the desired Nakagami-*m* samples. On the other hand, the second task—to rearrange the Nakagami-*m* samples appropriately—is rather complex. One out of $N!$ possible sample arrangements must be found that fits the analytical ACF accurately. In principle, an exhaustive comparison between all of the arrangements could be used to find the best-fitted one, but, for practical purposes (large N), such a brute-force approach has a prohibitive computational cost. Alternatively, a simple solution is derived next.

To gain inspiration, note that, for a given set of N samples, each one of the $N!$ possible sample arrangements can be uniquely specified by an N-length statistical rank vector, whose ith element gives the statistical rank of the ith sample in the arrangement[1]. Thus, our second task can be reformulated as to find an appropriate N-length statistical rank vector to define the ordering of the N Nakagami-*m* samples drawn. In doing this, we need to investigate how each parameter of the Nakagami-*m* fading model affects the statistical rank vector of the sequences.

[1]Statistical rank is the ordinal number of a value in a list arranged in a specified (decreasing or increasing) order.

The mean power Ω has no effect on the rank statistics, for it is simply a scaling factor of the samples. On the other hand, it seems to be very difficult (if not impossible) to establish an exact analytical relationship between the fading parameter m—the number of underlying Rayleigh clusters composing the Nakagami-*m* signal [5]—and the rank statistics of the Nakagami-*m* envelope. In fact, we shall not answer this question here. Instead, in order to support the simulation scheme to be proposed, we argue that the influence of m on the Nakagami-*m* rank statistics is negligible, as follows.

Several rank metrics can be used to investigate the impact of m on the Nakagami-*m* rank statistics. In particular, the Spearman rank autocorrelation coefficient is a very representative, widely-used rank metric [7], and shall be considered here. The Spearman rank autocorrelation coefficient, say $\rho_S(\tau)$, measures the strength of association between samples at different time instances, and is known to be well approximated by the usual ACC, say $\rho(\tau)$, i.e., [7]

$$\rho_S(\tau) \approx \rho(\tau) \tag{4}$$

On the other hand, for Nakagami-*m* fading, it is known that the ACC $\rho(\tau)$ of the envelope is closely approximated by the ACC of the squared envelope[2], and that the latter equals the ACC $\rho_2(\tau)$ of each underlying squared Rayleigh envelope, so that [1, Eq. (139)]

$$\rho(\tau) \approx \rho_2(\tau) \tag{5}$$

for any m. Then, for Nakagami-*m* fading, it follows from (4) and (5) that

$$\rho_S(\tau) \approx \rho_2(\tau) \tag{6}$$

irrespective of m. This is a paramount result: the fading parameter m has a negligible impact on the Spearman rank autocorrelation coefficient of the envelope. Therefore, the rank statistics of the Nakagami-*m* process are expected to be loosely dependent on m as well.

The above suggests that the Nakagami-*m* rank statistics for m arbitrary can be well approximated by those for a given reference value of m. For convenience, we choose the reference $m = 1$ (Rayleigh). More specifically, in order to match the Nakagami-*m* ACF, we propose to rearrange the N Nakagami-*m* envelope samples in accordance with the statistical rank vector of an underlying N-sample Rayleigh envelope reference sequence with the desired ACC $\rho_2(\tau)$, generated by existing methods. In other words, each Nakagami-*m* sample must be placed in the same position occupied by the Rayleigh sample with the same statistical rank of that Nakagami-*m* sample. We call this approach rank matching, and its implementation is indeed straightforward in most commercial software packages. In Matlab, for instance, having first obtained the N Nakagami-*m* samples, say the vector `nakagami`, by using the `gamrnd(·)` function, and the N-sample autocorrelated Rayleigh reference sequence, say the vector `rayleigh`, by means of any existing Rayleigh simulator, the commands

```
[rayleigh,index]=sort(rayleigh);
nakagami=sort(nakagami);
```

[2]Indeed, this is true for the ACC of any integer power of the Nakagami-*m* envelope [1, Eq. (139)].

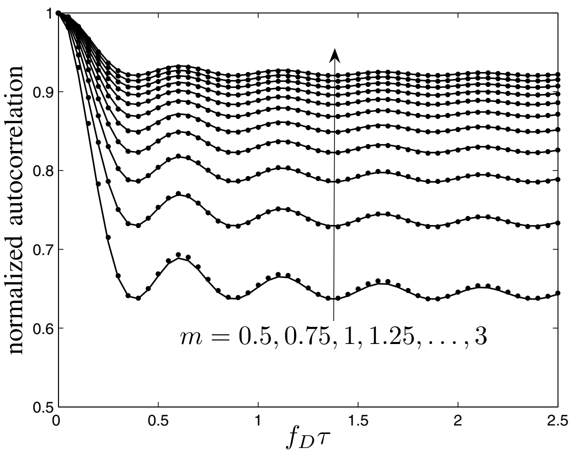

Fig. 2. Analytical and simulated normalized Nakagami-*m* envelope autocorrelations for isotropic scenario, $\kappa = 0$ (analytical: solid; simulated: dot).

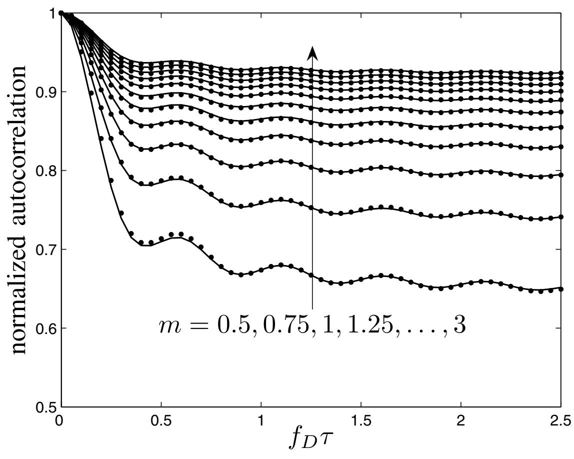

Fig. 3. Analytical and simulated normalized Nakagami-*m* envelope autocorrelations for slightly nonisotropic scenario, $\kappa = 1$ (analytical: solid; simulated: dot).

```
nakagami(index)=nakagami;
```

produce the desired N-sample rank-matched autocorrelated Nakagami-*m* sequence `nakagami`.

The proposed Nakagami-*m* simulation scheme is summarized in Fig. 1. The new scheme is simple, general, and provides excellent results, as shall be seen from sample simulation results.

IV. Simulation Results

The analytical and simulated normalized Nakagami-*m* envelope ACFs are shown in Figs. 2, 3, and 4, for isotropic ($\kappa = 0$), slightly nonisotropic ($\kappa = 1$), and nonisotropic ($\kappa = 2$) fading scenarios, respectively, and for several different values of fading parameter. In the examples, $N = 2^{20}$ and $\mu = 0$. The required Rayleigh reference sequences satisfying (3) have been generated using the method in [8]. Note the excellent match in all of the cases. The slightly poorer results for $m < 1$ are expected, for (5) deteriorates in this range. Since the proposed simulator inherently complies with the Nakagami-*m* distribution, PDF comparisons have been omitted.

V. Conclusions

A new simulation paradigm was introduced to derive a simple, general, accurate method for generating autocorrelated Nakagami-*m* envelope sequences with correct statistical properties. The method applies to arbitrary values of fading parameter and nonisotropic fading scenarios.

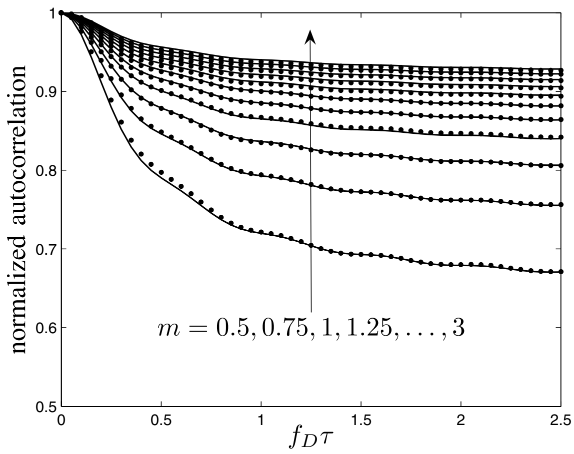

Fig. 4. Analytical and simulated normalized Nakagami-*m* envelope autocorrelations for nonisotropic scenario, $\kappa = 2$ (analytical: solid; simulated: dot).

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