A Simple Algorithm for Computing the Generalized Inverse of a Matrix

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The generalized inverse of a matrix is important in analysis because it provides an extension of the concept of an inverse which applies to all matrices. It also has many applications in numerical analysis, but it is not widely used because the existing algorithms are fairly complicated and require considerable storage space. A simple extension has been found to the conventional orthogonalization method for inverting nonsingular matrices, which gives the generalized inverse with little extra effort and with no additional storage requirements. The algorithm gives the generalized inverse for any m by n matrix A, including the special case when $m = n$ and A is nonsingular and the case when $m > n$ and rank $(A) = n$. In the first case the algorithm gives the ordinary inverse of A. In the second case the algorithm yields the ordinary least squares transformation matrix $(A^TA)^{-1}A^T$ and has the advantage of avoiding the loss of significance which results in forming the product A^TA explicitly.

The generalized inverse is an important concept in matrix theory because it provdes an extension of the concept of an inverse which applies to all matrices. Penrose [1] showed that for any $m \times n$ complex matrix A there exists a unique $n \times m$ matrix X which satisfies the following relations:

$$AXA = \mathrm{A} \tag{1}$$

$$XAX = X \tag{2}$$

$$(AX)^H = AX \tag{3}$$

$$(XA)^H = XA. \tag{4}$$

These four relations are often called Penrose's Lemmas, and the matrix X is said to be the generalized inverse of A and is often denoted by A^I. In the special case where $m = n$ and A is nonsingular, this generalized inverse is simply the ordinary inverse of A. Also, in the special case where $m \geq n$ and the columns of A are linearly independent, we can write

$$A^I = (A^HA)^{-1}A^H. \tag{5}$$

It is an easy matter to see that this matrix satisfies all of Penrose's Lemmas. It is important in numerical analysis because it solves the problem of minimizing the distance

$$\rho(x) = |b - Ax|,$$

where b is a given vector in m-space and A is a given $m \times n$ matrix with $m \geq n$ and linearly independent columns.

More generally, if A is any $m \times n$ matrix and b is any vector in m-space, there may exist many vectors x which minimize the distance $\rho(x)$, but the vector defined by

$$x = A^Ib \tag{6}$$

is the shortest of all such vectors. The problem of finding the vector x of shortest length $|x|$ which minimizes the distance $\rho(x)$ may be referred to as the generalized least squares problem. It is solved by the generalized inverse.

Suppose that the matrix A can be partitioned in the following manner:

$$A = (R, S) \tag{7}$$

where R is an $(m \times k)$-matrix $(k \leq n)$ with linearly independent columns and S is an $(m \times (n - k))$-matrix whose columns are linear combinations of the columns of R.

THEOREM I. $R^IR = I$. (8)

PROOF. The columns of R are linearly independent. Therefore, by (5), $R^I = (R^HR)^{-1}R^H$. Hence $R^IR = (R^HR)^{-1}R^HR = I$.

THEOREM II. *The matrix S has a unique factorization in the form*

$$S = RU \tag{9}$$

and the matrix U is given by

$$U = R^IS. \tag{10}$$

Proof that the factorization exists. Suppose

$$S = (s_{k+1}, s_{k+2}, \cdots, s_n).$$

Each column of S is a linear combination of the columns of R. Therefore $s_i = Ru_i$ from some vector u_i, $i = k+1$,

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$\cdots, n$. Hence

$$S = (Ru_{k+1}, Ru_{k+2}, \cdots, Ru_n)$$
$$= R(u_{k+1}, u_{k+2}, \cdots, u_n),$$

i.e., $S = RU$ where $U = (u_{k+1}, u_{k+2}, \cdots, u_n)$.

Proof that the factorization is unique. It has just been shown that $S = RU$ for some U. Therefore $R^I S = R^I RU = U$ since by (8), $R^I R = I$. Thus $U = R^I S$.

We now show that the problem of computing the generalized inverse of any matrix A can be reduced to the problem of computing the generalized inverse of a matrix of the form $A' = (R, S)$, where R and S are matrices of the same form as the R and S of eq. (7).

THEOREM III. *If P_{ij} is any nth order elementary permutation matrix then* $(AP_{ij})^I = P_{ij}A^I$.

The truth of this theorem can easily be demonstrated by showing that the matrix $P_{ij}A^I$ does actually satisfy Penrose's Lemmas. Furthermore it is easy to see that if $P_1, P_2, \cdots, P_r$ is any finite set of elementary permutation matrices, then

$$(AP_1P_2 \cdots P_r)^I = P_r \cdots P_2P_1A^I. \tag{11}$$

Thus if A is any $m \times n$ matrix it can be reduced to the form $A' = AP_1P_2 \cdots P_r = (R, S)$ with all the linearly dependent columns (if any) occurring last. Then by eq. (11), $P_r \cdots P_2P_1A^I = (R, S)^I$ and hence

$$A^I = P_1P_2 \cdots P_r(R, S)^I. \tag{12}$$

Thus it is now necessary only to consider the problem of computing the generalized inverse of matrices of the form $A = (R, S)$.

To obtain an expression for the generalized inverse in terms of the matrices R and U, we appeal to the least squares property of A^I. Let us confine ourselves for the time being to real matrices A. The results which we shall obtain can easily be generalized to the complex case. Consider the system

$$As = b, \tag{13}$$

where b is any vector in the column space of A; i.e., the system will have exact solutions. In this case all the least squares solutions will have the property $\rho(s) = | b - As | = 0$, and the shortest such s is given by $s = A^I b$.

Consider for a moment the problem of minimizing the length s with the restriction $As = b$. This is equivalent to minimizing $| s |^2 = s^T s$ with the restriction $As = b$. Assume that A has a partitioning (R, S) with all the linearly dependent columns last and partition S as follows:

$$s = \begin{pmatrix} x \\ y \end{pmatrix},$$

where x is a vector of order k and y is a vector of order $n-k$. Then the problem is to minimize the quantity

$$(x^T, y^T) \begin{pmatrix} x \\ y \end{pmatrix} = x^T x + y^T y,$$

with the restriction that

$$(R, S) \begin{pmatrix} x \\ y \end{pmatrix} - b = 0,$$

or simply

$$Rx + Sy - b = 0.$$

Let us apply the method of Lagrange multipliers. Set

$$L = x^T x + y^T y + z^T[Rx + Sy - b],$$

where z is the vector of parameters to be eliminated. Since by eqs. (9) and (10), $S = RU$ where $U = R^I S$, we can write

$$L = x^T x + y^T y + z^T[Rx + RUy - b].$$

Differentiating L with respect to each element of the vectors x and y and setting these derivatives equal to zero gives

$$\frac{\partial L}{\partial x} = 2x + R^T z = 0 \tag{14}$$

$$\frac{\partial L}{\partial y} = 2y + U^T R^T z = 0 \tag{15}$$

where $\partial L/\partial x$ is the vector whose elements are the derivatives of L with respect to the corresponding elements of x and $\partial L/\partial y$ has a similar interpretation. Adding the restriction

$$Rx + RUy - b = 0, \tag{16}$$

enables us to eliminate the vector z and solve for x and y. Premultiplying eq. (14) by U^T gives

$$2U^T x + U^T R^T z = 0.$$

Combining this result with eq. (15) gives $2y = 2U^T x$ or

$$y = U^T x. \tag{17}$$

If we now substitute the expression for y into eq. (16), we have

$$Rx + RUU^T x = b,$$
$$R(I + UU^T)x = b.$$

Now, by Theorem I, $R^I R = I$. Therefore

$$(I + UU^T)x = R^I b.$$

The matrix $(I + UU^T)$ is a symmetric positive definite matrix and hence is nonsingular. Therefore

$$x = (I + UU^T)^{-1} R^I b. \tag{18}$$

Substituting this value for x into eq. (17) gives

$$y = U^T (I + UU^T)^{-1} R^I b. \tag{19}$$

Now eqs. (18) and (19) lead to the conjecture that of all the vectors s satisfying the restriction

$$\rho(s) = | As - b | = 0,$$

the s of minimal length is given by

$$s = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (I + UU^T)^{-1}R^I b \\ U^T(I + UU^T)^{-1}R^I b \end{pmatrix} = \begin{pmatrix} (I + UU^T)^{-1}R^I \\ U^T(I + UU^T)^{-1}R^I \end{pmatrix} b.$$

Furthermore, since the required s is given by

$$s = A^I b,$$

it can be conjectured that the generalized inverse is given by

$$A^I = \begin{pmatrix} (I + UU^T)^{-1}R^I \\ U^T(I + UU^T)^{-1}R^I \end{pmatrix}. \tag{20}$$

It is a simple matter to verify that this matrix actually satisfies Penrose's Lemmas and is actually the generalized inverse of A. In fact, if A is any complex matrix with a partitioning of the form (R, S) then A^I is given by

$$A^I = \begin{pmatrix} (I + UU^H)^{-1}R^I \\ U^H(I + UU^H)^{-1}R^I \end{pmatrix}. \tag{21}$$

Thus we have an expression for A^I in terms of R^I and U. The remaining problem is to compute R^I and U. For this purpose, let us briefly review the Gramm-Schmidt orthogonalization process.

If $\{a_1, a_2, \cdots, a_n\}$ is any set of linearly independent vectors in m-space ($m \geq n$), then this set can be replaced by an orthonormal set $\{q_1, q_2, \cdots, q_n\}$ in the following manner:

(i) $$q_1 = \frac{a_1}{|a_1|}$$

(ii) $$c_2 = a_2 - (a_2^H q_1)q_1$$

$$q_2 = \frac{c_2}{|c_2|}$$

(iii) $$c_3 = a_3 - (a_3^H q_1)q_1 - (a_3^H q_2)q_2$$

$$q_3 = \frac{c_3}{|c_3|}$$

$$\vdots$$

Continue in this manner, at each step (i) forming c_i from a_i and the previous q's and then normalizing c_i to length 1 to get q_i. After n such steps the result is a set $q_1, q_2, \cdots, q_n$ of orthonormal vectors. i.e.,

$$q_i^H q_j = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

In particular, if the vectors a_i are the columns of an $m \times n$ matrix A, then the above process replaces A with a matrix Q satisfying

$$Q^H Q = I. \tag{22}$$

Since each q_i depends only on a_i and the previous q_j, the columns of A can be replaced one column at a time as illustrated in the following schematic diagram:

$$\begin{aligned} A &= (a_1, a_2, a_3, \cdots, a_n) \\ &\xrightarrow{(1)} (q_1, a_2, a_3, \cdots, a_n) \\ &\xrightarrow{(2)} (q_1, q_2, a_3, \cdots, a_n) \\ &\quad \vdots \\ &\xrightarrow{(n)} (q_1, q_2, q_3, \cdots, q_n). \end{aligned}$$

Furthermore, in this scheme each new column q_i is obtained from a linear combination of the vector a_i and the previous q's. Hence the columns of A are orthogonalized by a series of elementary column operations. If when carrying out this process on the columns of A we begin with the nth order identity matrix and perform the same elementary column operations on it, a matrix Z is obtained such that

$$AZ = Q. \tag{23}$$

If $m = n$, then the matrix A is nonsingular and this process provides a method for computing the inverse of A. Beginning with A and the nth order identity, we apply the Gramm-Schmidt process to obtain the matrices Z and Q. This process is illustrated schematically by the diagram:

$$\begin{pmatrix} A \\ \overline{I} \end{pmatrix} \xrightarrow{G\text{-}S} \begin{pmatrix} Q \\ \overline{Z} \end{pmatrix}.$$

Now by eq. (22) $Q^H Q = I$ or $Q^{-1} = Q^H$ and by eq. (23) Z is a matrix satisfying $AZ = Q$. Hence

$$A^{-1} = ZQ^H. \tag{24}$$

Thus if A is nonsingular its inverse can be computed by the Gramm-Schmidt orthogonalization process.

We now extend this method to compute the generalized inverse of an arbitrary complex matrix A.

In general, the columns of A will not be linearly independent, and the Gramm-Schmidt orthogonalization process will not work for a linearly dependent set of vectors. If we try to apply it to such a linearly dependent set in which the first k vectors are linearly independent but the $(k + 1)$-th vector is a linear combination of the previous k, it will successfully orthogonalize the first k vectors, but upon calculating c_{k+1}, we will find

$$c_{k+1} = a_{k+1} - \sum_{i=1}^{k} (a_{k+1}^H q_i)q_i = 0.$$

Thus the process breaks down upon encountering a linearly dependent vector. Although the columns of A will in general be linearly dependent, we have seen that it can just as well be assumed that A has a partitioning in the form $A = (R, S)$ with all the linearly dependent columns last.

Therefore, let us carry out a modified Gramm-Schmidt process in the following manner: apply the normal orthogonalization process to the columns of R and continue over the columns of S in the same manner except that as each vector becomes zero no normalization step is performed. If we carry out this process and, beginning with the nth order identity matrix, carry out exactly the same elementary column operations on it, we have

$$\begin{bmatrix} R & S \\ \hline I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \xrightarrow{G\text{-}S} \begin{bmatrix} Q & 0 \\ \hline Z & X \\ 0 & I_{n-k} \end{bmatrix}$$

where

$$(R, S) \begin{pmatrix} Z & X \\ 0 & I_{n-k} \end{pmatrix} = (Q, 0) \tag{25}$$

and Q is a matrix with the property $Q^H Q = I$.

Note that the $(n - k)$-order identity matrix in the lower righthand corner of the bookkeeping matrix remains unchanged by the process. This is because all the columns of S become zero when the process is applied to them, thus essentially zeroing any terms that might change the I_{n-k} when the same elementary column operations are applied to the bookkeeping matrix.

From eq. (25) it can be seen that

$$RZ = Q \tag{26}$$

and

$$RX + S = 0.$$

Rearranging the latter equation gives $RX = -S$, and by eq. (8), $X = -R^I S$. Since by eq. (10) $U = R^I S$ we have

$$X = -U. \tag{27}$$

Thus the matrix U comes out of the bookkeeping matrix; i.e.,

$$\begin{bmatrix} R & S \\ \hline I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \xrightarrow{G\text{-}S} \begin{bmatrix} Q & 0 \\ \hline Z & -U \\ 0 & I_{n-k} \end{bmatrix}$$

Also it is easy to see that R^I is given by

$$R^I = ZQ^H. \tag{28}$$

To verify this one need only note that by eq. (26) $R = QZ^{-1}$, and if this expression is used for R, then the matrix ZQ^H does actually satisfy Penrose's Lemmas and hence must be R^I.

Recall that the expression for A^I was by eq. (21),

$$A^I = \begin{pmatrix} (I + UU^H)^{-1} R^I \\ U^H (I + UU^H)^{-1} R^I \end{pmatrix}$$

and now we have a method for obtaining U and R^I. The only remaining problem is the evaluation of the expressions $(I + UU^H)^{-1}$ and $U^H(I + UU^H)^{-1}$.

For this purpose, note that the former term can be rewritten

$$(I + UU^H)^{-1} = I - U(U^H U + I)^{-1} U^H$$

and the latter term,

$$U^H (I + UU^H)^{-1} = (U^H U + I)^{-1} U^H.$$

These two expressions are easily verified matrix identities and making these substitutions in the expression for the generalized inverse gives

$$A^I = \begin{pmatrix} [I - U(U^H U + I)^{-1} U^H] R^I \\ (U^H U + I)^{-1} U^H R^I \end{pmatrix}. \tag{29}$$

Now recall that upon completion of the orthogonalization process, the matrix

$$\begin{pmatrix} -U \\ I \end{pmatrix}$$

appeared as the last $(n - k)$ columns of the bookkeeping matrix. Obviously this matrix has linearly independent columns; so its columns can be orthogonalized by the Gramm-Schmidt process. If we carry along a bookkeeping matrix, then

$$\begin{pmatrix} -U \\ I_{n-k} \\ \hline I_{n-k} \end{pmatrix} \xrightarrow{G\text{-}S} \begin{pmatrix} Y \\ W \\ \hline P \end{pmatrix}$$

where

$$\begin{pmatrix} -U \\ I_{n-k} \end{pmatrix} P = \begin{pmatrix} Y \\ W \end{pmatrix}.$$

Clearly, by the above relationship $W = P$ and $Y = -UP$.

Thus there is no need to carry along a bookkeeping matrix since the matrix W of the result contains the same information that the bookkeeping matrix would. So

$$\begin{pmatrix} -U \\ I \end{pmatrix} \xrightarrow{G\text{-}S} \begin{pmatrix} -UP \\ P \end{pmatrix},$$

where the columns of the result are orthogonal; i.e.,

$$\begin{pmatrix} -UP \\ P \end{pmatrix}^H \begin{pmatrix} -UP \\ P \end{pmatrix} = I,$$

or

$$(-P^H U^H, P^H) \begin{pmatrix} -UP \\ P \end{pmatrix} = I.$$

Carrying out the indicated multiplications gives

$$P^H U^H UP + P^H P = I,$$

and factoring out the P^H and the P gives

$$P^H (U^H U + I) P = I.$$

Now, P is a matrix which could be obtained from an identity matrix by elementary column operations and therefore must be nonsingular. Hence

$$(U^H U + I) = (P^H)^{-1} P^{-1},$$

whence

$$(U^H U + I)^{-1} = PP^H. \tag{30}$$

Also,

$$I - U(U^H U + I)^{-1} U^H = I - UPP^H U^H,$$

or

$$I - U(U^H U + I)^{-1} U^H = I - (UP)(UP)^H. \quad (31)$$

Thus we can substitute the expressions on the right of eqs. (30) and (31) into eq. (29) to obtain

$$A^I = \begin{pmatrix} [I - (UP)(UP)^H]R^I \\ PP^H U^H R^I \end{pmatrix}. \quad (32)$$

And substituting the value for R^I given by eq. (28) and

```
      SUBROUTINE GINV2 (A,U,AFLAG,ATEMP,MR,NR,NC)
C
C     THIS ROUTINE CALCULATES THE GENERALIZED INVERSE OF A
C         AND STORES THE TRANSPOSE OF IT IN A.
C       MR#FIRST DIMENSION NO. OF A.
C       NR # NO. OF ROWS IN A
C       NC # NO. OF COLUMNS IN A
C       U IS THE BOOKKEEPING MATRIX.
C       AFLAG AND ATEMP ARE TEMPORARY WORKING STORAGE.
C
      DIMENSION A(MR,NC),U(NC,NC),AFLAG(NC),ATEMP(NC)
      DO  10  I # 1,NC
      DO  5   J # 1,NC
    5 U(I,J) # 0.0
   10 U(I,I)#1.0
      FAC # DOT(MR,NR,A,1,1)
      FAC#1.0/SQRT(FAC)
      DO  15  I # 1,NR
   15 A(I,1)#A(I,1)*FAC
      DO  20  I # 1,NC
   20 U(I,1)#U(I,1)*FAC
      AFLAG(1)#1.0
C
C     DEPENDENT COL TOLERANCE FOR N BIT FLOATING POINT FRACTION
C
      N # 27
      TOL # (10. * 0.5**N)**2
      DO  100  J # 2,NC
      DOT1 # DOT(MR,NR,A,J,J)
      JM1#J-1
      DO 50 L#1,2
      DO 30 K#1,JM1
   30 ATEMP(K) # DOT(MR,NR,A,J,K)
      DO 45 K#1,JM1
      DO  35  I # 1,NR
   35 A(I,J)#A(I,J)-ATEMP(K)*A(I,K)*AFLAG(K)
      DO  40  I # 1,NC
   40 U(I,J)#U(I,J)-ATEMP(K)*U(I,K)
   45 CONTINUE
   50 CONTINUE
      DOT2 # DOT(MR,NR,A,J,J)
      IF((DOT2/DOT1) - TOL)  55,55,70
   55 DO 60 I#1,JM1
      ATEMP(I)#0.0
      DO 60 K#1,I
   60 ATEMP(I)#ATEMP(I)+U(K,I)*U(K,J)
      DO  65  I # 1,NR
      A(I,J)#0.0
      DO 65 K#1,JM1
   65 A(I,J)#A(I,J)-A(I,K)*ATEMP(K)*AFLAG(K)
      AFLAG(J)#0.0
      FAC # DOT(NC,NC,U,J,J)
      FAC#1.0/SQRT(FAC)
      GO TO 75
   70 AFLAG(J)#1.0
      FAC#1.0/SQRT(DOT2)
   75 DO  80  I # 1,NR
   80 A(I,J)#A(I,J)*FAC
      DO  85  I # 1,NC
   85 U(I,J)#U(I,J)*FAC
  100 CONTINUE
      DO 130 J#1,NC
      DO 130 I#1,NR
      FAC#0.0
      DO 120 K#J,NC
  120 FAC#FAC+A(I,K)*U(J,K)
  130 A(I,J)#FAC
      RETURN
      END


      FUNCTION DOT(MR,NR,A,JC,KC)
C
C     COMPUTES THE INNER PRODUCT OF COLUMNS JC AND KC
C     OF MATRIX A.
C
      DIMENSION A(MR,1)
      DOT#0.0
      DO  5  I # 1,NR
    5 DOT # DOT + A(I,JC)*A(I,KC)
      RETURN
      END
```

FIG. 1.

rearranging the bottom submatrix gives

$$A^I = \begin{pmatrix} [I - (UP)(UP)^H]ZQ^H \\ P(UP)^H ZQ^H \end{pmatrix}. \quad (33)$$

We now have a simple scheme for computing the generalized inverse.

Beginning with the matrix (R, S) and an identity matrix, we can illustrate the scheme as follows:

$$\begin{pmatrix} R & S \\ \hline I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \xrightarrow{G\text{-}S} \begin{pmatrix} Q & 0 \\ \hline Z & -U \\ 0 & I \end{pmatrix} \xrightarrow{G\text{-}S\binom{-U}{I}} \begin{pmatrix} Q & 0 \\ \hline Z & -UP \\ 0 & P \end{pmatrix}.$$

We would then have all the information necessary to compute the generalized inverse of A from eq. (33).

Thus we have a simple extension of the Gramm-Schmidt method for computing the generalized inverse.

In carrying out this algorithm on a computer, all the calculations could be performed in the space of the matrix A itself plus the space required for an $n \times n$ bookkeeping matrix. It is clear that all the orthogonalization required to reduce these matrices to the form

$$\begin{pmatrix} Q & 0 \\ \hline Z & -UP \\ 0 & P \end{pmatrix}$$

can be done in this space. We could then form the product $(UP)^H Z$ in the space of the zero submatrix in the lower lefthand corner of the bookkeeping matrix to get

$$\begin{pmatrix} Q & 0 \\ \hline Z & -UP \\ (UP)^H Z & P \end{pmatrix}$$

We could then form the product $[(UP)^H ZQ^H]^H$ in the 0–submatrix of $(Q, 0)$ and then restore the 0–submatrix in the lower lefthand part of the bookkeeping matrix to get

$$\begin{pmatrix} Q & [(UP)^H ZQ^H]^H \\ \hline Z & -UP \\ 0 & P \end{pmatrix}.$$

We then would only need to perform the product

$$\begin{pmatrix} Z & -UP \\ 0 & P \end{pmatrix} \begin{pmatrix} Q^H \\ (UP)^H ZQ^H \end{pmatrix} = \begin{pmatrix} ZQ^H - (UP)(UP)^H ZQ^H \\ P(UP)^H ZQ^H \end{pmatrix} = A^I$$

The transpose of this product can be formed in the space originally occupied by $(Q, (UP)^H ZQ^H)$. Thus the net result of carrying out the algorithm is to replace the matrix A by the transpose of A^I.

A FORTRAN subroutine for carrying out the algorithm is given in Figure 1. The program does not carry out the algorithm explicitly in that it avoids permuting the columns to obtain the form (R, S), and as each linearly dependent column becomes zero in the orthogonalization process, it is immediately replaced by a corrected column. The net result, however, is the same as would be obtained

(*Continued on page 387*)

Option C The longitudinal center line of the feed holes shall be located within 0.300 inch maximum from the two-track edge *and* 0.395 inch maximum from the three-track edge of the tape. The distance from this center line to either edge shall not vary by more than .006 inch (total variation) within any 6 inch length of tape.

To help clarify the above options, a sketch is submitted indicating the tolerance from the feed hole to the guided edge (Figure 2).

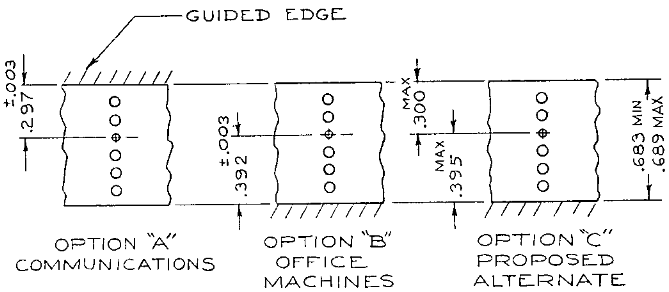

FIG. 2

Pertinent factors relating to the three options are as follows:

Option A

1. This type of dimensioning of paper tape punches has been standard in the communications industry for the past 75 years. To change the guiding of these machines would be economically impractical.
2. This type of dimensioning permits variation in feed hole location of 0.006 inch from the two hole edge and 0012 inch from the three hole edge of the tape.
3. Tapes punched according to this standard are sensed equally well by communication and business machine readers which guide on the sprocket holes only.
4. Tapes punched according to this standard are sensed well by readers which guide only on the two hole edge of the tape. There may be some loss of sensing margin (due to Item 2 above) when such tapes are passed through a reader which guides only in the three hole edge.
5. This method of dimensioning differs from that used in EIA RS-227 for one inch paper tape but conforms to that used in thousands of domestic and foreign machines in use and manufacture today.

Option B

1. This type of dimensioning of paper tape punches has been standard in the business machine industry for two decades. To change the guiding edge of these machines would be economically impractical.
2. This type of dimensioning permits a variation in feed hole location of 0.012 inch from the two hole edge and 0.006 inch from the three hole edge of the tape.
3. Tapes punched according to this standard are sensed equally well by communications and business machine readers which guide on the sprocket holes only.
4. Tapes punched according to this standard are sensed well by readers which guide only on the three hole edge of the tape. There may be some loss of sensing margin (due to Item 2 above) when such tapes are passed through a reader which guides only on the two hole edge.
5. This method of dimensioning is the same as used in EIA RS-227 for one inch paper tape and would permit a reader which guides on only the three hole edge of the tape to read both $1\frac{1}{16}$ inch and 1 inch tape with equal margins.

Option C

1. This type of dimensioning offers a compromise between Option A and B. It recognizes the present and continuing existence of tape perforators producing tape in accordance with both Option A and Option B conventions.
2. This type of dimensioning permits a variation in feed hole location of up to 0.012 inch from either the two hole or the three hole edge of the tape.
3. Tapes punched according to this standard are sensed equally well by communications and business machine readers which guide on the sprocket holes only.
4. This method of dimensioning requires that readers which guide only on one edge of the tape be designed to accommodate tapes guided on either edge during preparation. The number of readers which guide only on one edge is small and the design problems encountered in such a reader to allow for the possible maximum 0.012 inch variation (Item 2 above) are considered minimal.
5. A purpose of option "C" is to prevent interference between the tape and the tape guide in readers containing both a feed wheel and a narrow tape guide. However, preferred practice in the design of readers with feed wheels is to make the tape guide wide enough to assure locating the tape by the feed wheel only, at the sensing pins. The purpose of the guide, then, is to facilitate insertion of the tape into the reader and to prevent excessive skew.

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in carrying out the algorithm in the manner described above. In the interest of accuracy the program reorthogonalizes each column after it is first orthogonalized. This is a standard technique in carrying out the Gramm-Schmidt orthogonalization.

A number of publications have appeared in the past few years which are also concerned with methods for computing the generalized inverse. Pyle [2] discusses a method for finding the generalized inverse of an arbitrary $m \times n$ complex matrix A with $m \leq n$ in which the Gramm-Schmidt process is applied to the columns of A^H and then to the columns of A if rank $(A) \leq m$. Ben Israel and Wersan [3] describe elimination methods in which the elimination process is applied to the symmetric product A^HA or the symmetric product of some submatrix of A. It is important to note that all these methods, including that of the authors, depend upon the correct determination of the rank of the matrix. In [4] Golub discusses the strategy of using the generalized inverse to solve least squares problems when the matrix is deficient in rank or poorly conditioned.

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