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A SIMPLE CHARACTERIZATION OF THE TRACE-CLASS OF OPERATORS

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ABSTRACT. The trace-class (τc) of operators on a Hilbert space is characterized in terms of existence of certain centralizers.

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1. INTRODUCTION.

Not long ago Saworotnow [1] characterized the trace-class τA (see [2]) associated with an arbitrary H*-algebra A (see [3]) as well as the trace-class (τc) of operators (see [4]). Now, we shall show that, in the second case, there is a much simpler characterization.

We shall use the terminology and the notation of Saworotnow [1]. In particular, a trace algebra is a Banach *-algebra with a trace tr and with the following properties:

$$(1)\quad tr(xy) = tr(yx), \quad (2)\quad tr(x^*x) = n(x^*x), \quad (3)\quad n(x^*) = n(x)$$

$$(4)\quad |tr\ x| \leq n(x) \text{ and } (5)\quad x \neq 0 \text{ implies } x^*x \neq 0$$

where $x, y \in B$ and n( ) denotes the norm of B. It is also assumed that $n(xy) \leq n(x)n(y)$ for all $x, y \in B$.

2. MAIN RESULT.

THEOREM. Let B be a simple trace-algebra (see [1]). Assume that for each ι ∈ B there exists a (linear) centralizer U such that $Ua \geq 0$ (tr $x^*Uax \geq 0$ for each x B), $(Ux)^2 = a^*a$ and $n(a) = \mathrm{tr}Ua$. Then there exists a Hilbert space H such that B is isometric to the trace-class (τc) (see [4]) of operators on H.

PROOF. Let A be the H*-algebra associated with B (see [1]) and let Tr denote the trace on τA induced by A (see [2], p. 97). It follows from simplicity of B that the ideal

$$I = [\sum_{i=1}^{n} x_i y_i : x_i, y_i \in B]$$

is dense in B. Also the norm n( ) of B coincides on I with the norm τ( ) induced by A (see [2], p. 99):

$$n(\sum x_i y_i) = \mathrm{tr}(u \sum x_i y_i) = \sum \mathrm{tr}Ux_i y_i = \sum (y_i^*, Ux_i)$$
$$= \sum \mathrm{Tr}(Ux_i y_i) = \mathrm{Tr}U(\sum x_i y_i) = \mathrm{Tr}[\sum x_i y_i] = \tau(\sum x_i y_i)$$

where U denotes the centralizer associated with $\sum x_i y_i$. The equality $U \sum x_i y_i = [\sum x_i y_i]$ follows from the fact that the positive square root of the member $(\sum x_i y_i)^* \sum x_i y_i$ of A is unique. Thus we may conclude that B is identifiable with τA.

Now we can complete the proof as in Saworotnow [1], the proof of the corollary to Theorem 2.

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