A Simple Class of $U(N)$ Racah Coefficients and Their Application *

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Abstract. Using permutation group techniques, a general expression is derived for the special class of $U(N)$ Racah coefficients for which the representations $[f^1]$ and $[f^3]$ in the recoupling matrix for $[f^1] \times [f^2] \times [f^3] \to [f]$ are either both totally antisymmetric or both totally symmetric. For the totally antisymmetric case further specialization gives a simple expression for a $U(N)$ Racah coefficient which is needed in taking the average of the product of operators over the states of an irreducible representation of $U(N)$, where this result can be useful in the study of identical fermion systems by spectral distribution methods.

Introduction

In recent years the Wigner-Racah calculus for the unitary groups $U(N)$, with $N > 2$, has been brought to a state of development comparable to that for the angular momentum calculus, especially through the work of Biedenharn, Louck, and coworkers [1–7]. For multiplicity-free Wigner couplings, in particular, algebraic formulae for $U(N)$ Wigner coefficients can generally be read off directly from their diagrammatic pattern calculus [1]. Biedenharn and Louck advocate the view that there is a canonical structure for the $U(N)$ Wigner-Racah calculus which eliminates all free choices in the resolution of the multiplicity problem for the most general Wigner coupling. Except for phase there is therefore no arbitrariness in the definition of a $U(N)$ Wigner or Racah coefficient. For $N > 3$, however, explicit algebraic constructions for Wigner couplings involving the most general multiplicity structure have so far been limited to matrix elements of the simplest self-adjoint Wigner operators [2], which transform according to the $U(N)$ irreducible representation $[211...10]$. Louck and Biedenharn [2] also give the $U(N)$ Racah coefficient for the recoupling matrix for $[f] \times [11...10] \times [10...0] \to [f]$ in elegantly compact form. In the applications to physical problems $U(N)$ Racah coefficients are often more useful than the Wigner coefficients [8, 9]. Being independent of subgroup labels, Racah coefficients also have a simpler algebraic structure than the Wigner coefficients. Despite this fact general expressions for $U(N)$ Racah coefficients for arbitrary N have so far been limited to a few special cases. Even the Racah coefficients for the recoupling transformations for which all four Wigner couplings in the Racah recoupling process are free of multiplicity can not yet be written down directly from a simple pattern calculus, except for a limited number of special cases. When the $U(N)$ representations $[f^2]$ and $[f^3]$ are both totally symmetric (representations with

one-rowed tableaux) and are themselves coupled to a totally symmetric representation, Biedenharn *et al.* [3] show how the Racah coefficients for the recoupling matrix for $[f^1] \times [f^2] \times [f^3] \to [f]$ can be written down from an extended pattern calculus. This extended pattern calculus can also be used to evaluate certain "stretched" Racah coefficients [3] for 2-rowed representations $[f^1]$ with additional restrictions on $[f^2]$ and $[f^3]$. Results for other special cases have also been worked out. Jucys *et al.* [10] give a special class of Racah coefficient involving three totally symmetric representations. Moshinsky and Chacon [11] have evaluated Racah coefficients for the recoupling matrix for $[f^1] \times [f^2] \times [f^3] \to [f]$ where the representations $[f^1]$ and $[f^3]$ are both totally symmetric; but explicit results are limited to the case when the remaining representations in the recoupling transformation are at most 2- or 3-rowed, and their algebraic expression for the Racah coefficient is complicated by a summation over many indices. Moshinsky and Chacon make use of the intimate relationship between the unitary group $U(N)$ and the symmetric group. In a recent investigation [8] of spectroscopic problems involving the method of spectral distributions, this relationship has also been exploited to evaluate a summation relation for $U(N)$ Racah coefficients [12] which is independent of the $U(N)$ multiplicity structure and could therefore be evaluated by permutation group techniques. Any $U(N)$ Racah coefficient, for which the four Wigner couplings in the Racah recoupling transformation are all free of multiplicity, can be evaluated by similar techniques. Moshinsky and Chacon [11] relate their special class of $U(N)$ Racah coefficient to the matrix element of a permutation operator exchanging a *single* pair of particles of an n-particle state. By using n-particle states for a harmonic oscillator in N dimensions they gain totally symmetric representations of arbitrary length by associating with each particle an arbitrary number of oscillator quanta. By relating the $U(N)$ Racah coefficient to the matrix element of a more complicated permutation operator which exchanges one group of particles with a second group of particles, it is possible to calculate $U(N)$ Racah coefficients involving either totally symmetric or totally antisymmetric representations by simple permutation group techniques. The totally antisymmetric case will have useful applications to systems of identical fermions.

It is the purpose of this work to derive a general expression for the special class of $U(N)$ Racah coefficients for which the representations $[f^1]$ and $[f^3]$ in the recoupling matrix for $[f^1] \times [f^2] \times [f^3] \to [f]$ are either both totally antisymmetric or both totally symmetric. In Section 1 it is shown how such $U(N)$ Racah coefficients can be related to the matrix element of a permutation operator which exchanges one group of particles with a second group of particles. The evaluation of this matrix element by straightforward permutation group techniques is carried out in Section 2. Finally, in Section 3 additional specializations are made for the totally antisymmetric case, giving very simple expressions for a class of $U(N)$ Racah coefficients which may have useful applications for systems of identical fermions; particularly for problems in nuclear spectroscopy, using the methods of spectral distributions developed recently by French and collaborators [13]. These techniques involve the averaging of products of operators over the states of an irreducible representation of $SU(N)$, generally for large N. Since this averaging eliminates the dependence of matrix elements on $U(N)$ Wigner

coefficients, results can be written down in terms of $SU(N)$ Racah coefficients provided the operators of physical interest can be decomposed into their appropriate $SU(N)$ irreducible tensor components.

1. $U(N)$ Racah Coefficients and the Matrix Elements of Permutation Operators

It will be convenient to use a notation for the $U(N)$ Racah coefficient which is a straightforward generalization of that for the angular momentum case. The Racah coefficient in unitary form, or U-coefficient, is given by the recoupling matrix

$$U([f^1][f^2][f][f^3];[f^{12}]\varrho^{12}\varrho^{12,3};[f^{23}]\varrho^{23}\varrho^{1,23})$$
$$=\langle(([f^1]\times[f^2])[f^{12}]\varrho^{12}\times[f^3])[f] \tag{1}$$
$$\cdot\varrho^{12,3}|([f^1]\times([f^2]\times[f^3])[f^{23}]\varrho^{23})[f]\varrho^{1,23}\rangle.$$

Here, the $U(N)$ irreducible representation labels $[f^s]\equiv[f^s_{iN}]$ are given by the partition numbers f^s_{iN}, $i=1,...,N$, which specify the number of squares in the ith row of the Young tableau describing the representation $[f^s]$. Multiplicity labels ϱ^{st} are needed whenever the Wigner coupling of $[f^s]$ with $[f^t]$ can yield a specific representation $[f^{st}]$ with a d-fold multiplicity, with $d>1$. In the special case when both $[f^1]$ and $[f^3]$ are either totally symmetric, $[f^s]=[m0...0]\equiv[m]$, or totally antisymmetric, $[f^s]=[11...10...0]\equiv[1^m]$; $d=1$ for all couplings, and all four multiplicity labels $\varrho^{12},\varrho^{12,3},\varrho^{23},\varrho^{1,23}$ become unnecessary and can be omitted. It may be useful to relate the above notation to that of Biedenharn and Louck [2–7] who define the Racah invariant operator

$$\left\{\begin{pmatrix}[f^{23}]\\(\Gamma^{1,23})\end{pmatrix}\begin{pmatrix}(\Gamma^{23})\\[f^3]\\(\Gamma^{12,3})\end{pmatrix}\begin{pmatrix}[f^2]\\(\Gamma^{12})\end{pmatrix}\right\}, \tag{2}$$

where the U-coefficient of Eq. (1) is the matrix element of this operator connecting a state of irreducible representation $[f^1]$ (on the right) to a state $[f]$ (on the left). The labels Γ^{st} include both the multiplicity label ϱ^{st} and the shift indices, $\Delta^{st}_i=f^{st}_{iN}-f^s_{iN}$, which indicate how many of the squares of the tableau for $[f^t]$ have been added to the ith row of the tableau for $[f^s]$ to make the tableau for $[f^{st}]$. Since it will be convenient to keep representations $[f^1],[f^2],[f^3]$, on an equal footing, the notation of Eq. (1) will be preferable for this investigation. To save on notation it will also be convenient to illustrate all derivations with the case where $[f^1]$ and $[f^3]$ are totally antisymmetric; $[f^1]=[1^q]$, $[f^3]=[1^p]$. (Results for the totally symmetric case will be given subsequently.)

The first step in the derivation involves the composition of recoupling transformations illustrated by Fig. 1. Part (a) of the figure illustrates the straight-forward Racah recoupling transformation

$$|(([1^q]\times[f^2])[f^{12}]\times[1^p])[f]m\rangle \tag{3}$$
$$=\sum_{[f^{23}]}U([1^q][f^2][f][1^p];[f^{12}];[f^{23}])\,|([1^q]\times([f^2]\times[1^p])[f^{23}])[f]m\rangle,$$
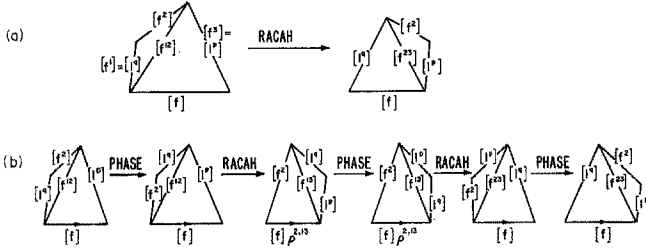
Fig. 1a and b

where m is a shorthand notation for a complete set of $U(N)$ subgroup labels. (The Gelfand labels f_{ij} with $i \le j = 1, 2, ..., N - 1$ could be used, for example [7]). The state vectors of Eq. (3) can also be related by the transformation of Part (b) of Fig. 1. The transformations marked "phase" in Fig. 1 are of the type

$$|([1^q] \times [f^2]) [f^{12}] m^{12}\rangle = (-1)^{\varphi([1^q][f^2],[f^{12}])} |([f^2] \times [1^q]) [f^{12}] m^{12}\rangle \quad (4)$$

involving an interchange in the order of the first and second representation in the Wigner coupling of the state vector. [Note, however, that the state vectors of the left and right hand sides of Eq. (4) differ only by a representation-dependent phase factor because one of the representations is totally antisymmetric, leading to a multiplicity-free Wigner coupling.] From

$$\langle(([1^q] \times [f^2]) [f^{12}] \times [1^p]) [f] m|([1^q] \times ([f^2] \times [1^p]) [f^{23}]) [f] m\rangle$$
$$= \sum_{[f^{13}]\varrho} \langle(([1^q] \times [f^2]) [f^{12}] \times [1^p]) [f] m|([f^2] \times ([1^q] \times [1^p]) [f^{13}]) [f] m, \varrho\rangle$$
$$\cdot \langle([f^2] \times ([1^q] \times [1^p]) [f^{13}]) [f] m, \varrho|([1^q] \times ([f^2] \times [1^p]) [f^{23}]) [f] m\rangle \quad (5)$$

we obtain the relation

$$U([1^q] [f^2] [f] [1^p]; [f^{12}]; [f^{23}])$$
$$= (-1)^{\varphi_1([1^q][f^2],[f^{12}])} (-1)^{\varphi_2([1^q][f^{23}],[f])}$$
$$\cdot \sum_{[f^{13}]\varrho} (-1)^{\varphi([1^q][1^p],[f^{13}])} U([f^2] [1^q] [f] [1^p]; [f^{12}]_{--}; [f^{13}]_{-}\varrho) \quad (6)$$
$$\cdot U([f^2] [1^p] [f] [1^q]; [f^{23}]_{--}; [f^{13}]_{-}\varrho)$$

The sum over $[f^{13}]$ and ϱ in Eq. (6) can now be related to the matrix element of a permutation operator which exchanges the particle indices of the two totally antisymmetric parts of the n-particle state vector $|[f] m\rangle$. To evaluate this matrix element it will be convenient to introduce n-particle state vectors

$$|[f] m; r_n r_{n-1} ... r_2 r_1\rangle \quad (7)$$

which are simultaneously base vectors for an irreducible representation of $U(N)$, (labeled by $[f]$ and m), and of the standard Young-Yamanouchi orthogonal representation of S_n, (with $n = \Sigma_i f_{iN}$), where $r_n r_{n-1} ... r_1$ is a standard Yamanouchi symbol[14]. It will further be useful to transform to non-standard representations of S_n [15, 16] in which one group of p particles and a second group of q particles

have the definite permutation symmetry $[1^p]$ and $[1^q]$. Such states are simple linear combinations of the states (7). E.g., the state vector in which the q particles labeled $n-p, n-p-1, \ldots, n-p-q+1$ belong to the totally antisymmetric representation of S_q is given by [15]

$$| [f] \, m; r_n \ldots r_{n-p+1} \{ b_1 b_2 \ldots b_q \}_{[1^q]} \, r_{n-p-q} \ldots r_1 \rangle \tag{8}$$

$$= \sum_{\mathscr{P}} (-1)^{\chi+1} \, \mathscr{P} \left[\frac{1}{q!} \prod_{i<j}^q \left(1 - \frac{1}{\tau_{b_i b_j}} \right) \right]^{\frac{1}{2}} | [f] \, m; r_n \ldots r_{n-p+1} b_1 b_2 \ldots b_q r_{n-p-q} \ldots r_1 \rangle \, ,$$

where the sum is over the $q!$ permutations $\mathscr{P}$ which permute the symbols $b_1 b_2 \ldots b_q$ in both the state vector and the coefficient, and where χ is even (odd) for even (odd) permutations $\mathscr{P}$. The coefficients are given in terms of the "axial distances", τ_{lk},

$$\tau_{lk} = f_l - f_k + k - l - \sigma(l) + \sigma(k) \, , \tag{9}$$

where f_k designates the number of squares in the kth row of the n-particle Young tableau $[f]$ ($f_k \equiv f_{kN}$), and where $\sigma(k) =$ number of times the label k occurs among the Yamanouchi symbols $r_n, \ldots, r_{n-p+1}$ preceding the b_j, [similarly for $\sigma(l)$]. In terms of such state vectors the sum of Eq. (6) can now be related to the matrix element of a permutation operator. The result is

$$\sum_{[f^{13}] \varrho} (-1)^{\varphi([1^q][1^p],[f^{13}])} \, U([f^2] \, [1^q] \, [f] \, [1^p]; [f^{12}]_{--}; [f^{13}]_{-} \varrho)$$

$$\cdot U([f^2] \, [1^p] \, [f] \, [1^q]; [f^{23}]_{--}; [f^{13}]_{-} \varrho) \tag{10}$$

$$= \langle [f] \, m; \{ b_1' b_2' \ldots b_q' \}_{[1^q]} \{ a_1' a_2' \ldots a_p' \}_{[1^p]} \ldots | P | [f] \, m \{ a_1 a_2 \ldots a_p \}_{[1^p]} \{ b_1 b_2 \ldots b_q \}_{[1^q]} \ldots \rangle \, ,$$

where it is assumed (without loss of generality) that $q \geq p$, and where $b_1' \ldots b_q' a_1' \ldots a_p'$ is a specific permutation of the symbols $a_1 \ldots a_p b_1 \ldots b_q$. The operator P is the permutation operator which exchanges the particle indices of the p-particle group with particle indices in the q-particle group

$$P = \mathscr{P}_p \mathscr{P}_{p-1} \ldots \mathscr{P}_2 \mathscr{P}_1 P_{n,n-q} P_{n-1,n-q-1} \ldots P_{n-p+1,n-q-p+1} \, . \tag{11a}$$

Here, $P_{n-k,n-q-k}$ is the transposition which interchanges the particle labeled number $n-k$ with particle $n-q-k$, and $\mathscr{P}_k$ is the cyclic interchange of particles numbered $n-k+1, n-p, n-p-1 \ldots, n-q+1$, with $k=1, \ldots, p; (q \geq p)$. Note that any $\mathscr{P}_k$ when acting to the left in the matrix element of Eq. (10) gives the simple factor $(-1)^{q-p}$, so that the operator P can in effect be replaced by

$$P = (-1)^{p(q-p)} \prod_{k=1}^p P_{n-k+1,n-q-k+1} \, . \tag{11b}$$

The derivation of Eq. (10) follows. The state vector of the right hand side can be expanded by successive $U(N)$ Wigner couplings

$$| [f] \, m; \{ a_1 \ldots a_p \}_{[1^p]} \{ b_1 \ldots b_q \}_{[1^q]} \, r_{n-p-q} \ldots r_1 \rangle$$

$$= \sum_{m^{12} m^3 m^2 m^1} \langle [f^{12}] \, m^{12} [1^p] \, m^3 | [f] \, m \rangle \langle [f^2] \, m^2 [1^q] \, m^1 | [f^{12}] \, m^{12} \rangle \tag{12}$$

$$\psi_{m^3}^{[1^p]}(n, n-1, \ldots n-p+1) \, \psi_{m^1}^{[1^q]}(n-p, n-p-1, \ldots n-p-q+1)$$

$$\cdot \psi_{m^2}^{[f^2]}(n-p-q, \ldots 2, 1) \, ,$$

where the uncoupled state vectors are written out as functions of the particle indices to make it easier to indicate explicitly their dependence on particle number. The coefficients $\langle [f^s] m^s [f^t] m^t | [f^{st}] m^{st} \rangle$ are full $U(N)$ Wigner coefficients. (Since one of the representations $[f^s], [f^t]$ is totally antisymmetric no multiplicity labels occur in these Wigner coefficients.) The representation $[f^{12}]$ is obtained from $[f]$ by removing squares from the rows labeled $a_1 a_2 \ldots a_p$ of the tableau for $[f]$, while $[f^2]$ is obtained by a further removal of squares from the rows labeled $b_1 b_2 \ldots b_q$ from the resultant tableau for $[f^{12}]$. The operator $\boldsymbol{P}$ acting on the n-particle functions of Eq. (12) yields

$$\boldsymbol{P} \psi_{m^3}^{[1^p]}(n, n-1, \ldots n-p+1)\, \psi_{m^1}^{[1^q]}(n-p, n-p-1, \ldots n-p-q+1)$$

$$= \psi_{m^1}^{[1^q]}(n, n-1, \ldots n-q+1)\, \psi_{m^3}^{[1^p]}(n-q, n-q-1, \ldots n-p-q+1) \quad (13)$$

$$= \sum_{[f^{13}] m^{13}} |([1^q] \times [1^p])\,[f^{13}]\, m^{13}\rangle \langle [1^q]\, m^1 [1^p]\, m^3 | [f^{13}]\, m^{13}\rangle .$$

After combining Eqs. (12) and (13), the sum over $U(N)$ subgroup labels m^1, m^3, m^{12} can be carried out by expressing the sum over the product of three $U(N)$ Wigner coefficients in terms of $U(N)$ Racah coefficients by

$$\sum_{m^1 m^{12} m^3} \langle [f^2]\, m^2 [1^q]\, m^1 | [f^{12}]\, m^{12} \rangle \langle [f^{12}]\, m^{12} [1^p]\, m^3 | [f]\, m \rangle$$

$$\cdot \langle [1^q]\, m^1 [1^p]\, m^3 | [f^{13}]\, m^{13} \rangle \qquad\qquad (14)$$

$$= \sum_{\varrho} U([f^2]\,[1^q]\,[f]\,[1^p]; [f^{12}]_{--}; [f^{13}]_{-\varrho}) \langle [f^2]\, m^2 [f^{13}]\, m^{13} | [f]\, m \rangle_{\varrho} .$$

Since the Wigner coupling $[f^2] \times [f^{13}]$ is in general not free of multiplicity, both the $U(N)$ Wigner and Racah coefficients of the right hand side of this relation are functions of the multiplicity label $\varrho (\equiv \varrho^{2,13})$, and the result involves a sum over this multiplicity label [2, 17]. The resultant action of the operator $\boldsymbol{P}$ on the n-particle state vector can then be expressed as

$$\boldsymbol{P} | [f]\, m; \{a_1 a_2 \ldots a_p\}_{[1^p]} \{b_1 b_2 \ldots b_q\}_{[1^q]} \ldots \rangle$$

$$= \sum_{[f^{13}] \varrho} U([f^2]\,[1^q]\,[f]\,[1^p]; [f^{12}]_{--}; [f^{13}]_{-\varrho})$$

$$\cdot \sum_{m^2 m^{13}} \langle [f^2]\, m^2 [f^{13}]\, m^{13} | [f]\, m \rangle_{\varrho} \qquad\qquad (15)$$

$$\cdot |([1^q] \times [1^p])\,[f^{13}]\, m^{13}\rangle\, |[f^2]\, m^2\rangle .$$

The state vector of the bra side of Eq. (10) can be expanded in similar fashion to give

$$\langle [f]\, m; \{b_1' b_2' \ldots b_q'\}_{[1^q]} \{a_1' a_2' \ldots a_p'\}_{[1^p]} \ldots |$$

$$= \sum_{[f^{13}] \varrho'} U([f^2]\,[1^p]\,[f]\,[1^q]; [f^{23}]_{-}; [f^{13}]_{-\varrho'})$$

$$\cdot \sum_{m^2 m^{13}} \langle [f^2]\, m^2 [f^{13}]\, m^{13} | [f]\, m \rangle_{\varrho'} \qquad\qquad (16)$$

$$\cdot (-1)^{\varphi([1^q][1^p],[f^{13}])} \langle ([1^q] \times [1^p])\,[f^{13}]\, m^{13} | \langle [f^2]\, m^2 | ,$$

where, in order to apply the analogue of Eq. (14), it was necessary to make use of a symmetry property of the (multiplicity-free) $U(N)$ Wigner coefficient

$$\langle [1^q] m^1 [1^p] m^3 | [f^{13}] m^{13} \rangle$$
$$= (-1)^{\varphi([1^q][1^p],[f^{13}])} \langle [1^p] m^3 [1^q] m^1 | [f^{13}] m^{13} \rangle \tag{17}$$

[cf. Eq. (14) and Fig. 1). Taking the overlap of (15) and (16), and finally making use of the orthonormality of the $U(N)$ Wigner coefficients

$$\sum_{m^2 m^{13}} \langle [f^2] m^2 [f^{13}] m^{13} | [f] m \rangle_\varrho \langle [f^2] m^2 [f^{13}] m^{13} | [f] m \rangle_{\varrho'} = \delta_{\varrho\varrho'}, \tag{18}$$

the matrix element of $\boldsymbol{P}$ is reduced to the sum over $[f^{13}]$ and ϱ of Eq. (10). Combining Eqs. (6) and (10) gives the final result

$$U([1^q] [f^2] [f] [1^p]; [f^{12}]; [f^{23}]) = (-1)^{\varphi_1([1^q][f^2],[f^{12}]) + \varphi_2([1^q][f^{23}],[f])}$$
$$\cdot (-1)^{p(q-p)} \langle [f] \{b'_1 \ldots b'_q\}_{[1^q]} \{a'_1 \ldots a'_p\}_{[1^p]} \ldots | \prod_{k=1}^{p} \tag{19}$$
$$\cdot P_{n-k+1, n-q-k+1} | [f] \{a_1 \ldots a_p\}_{[1^p]} \{b_1 \ldots b_q\}_{[1^q]} \ldots \rangle .$$

The phase factors φ_1 and φ_2, as always, are somewhat dependent on phase conventions. (It will be advantageous to postpone a specific choice of phase conventions until specific applications are made.)

In the Racah coefficient the representation $[f^{12}]$ is obtained from the representation $[f]$ by the removal of squares from the rows labeled $a_1 a_2 \ldots a_p$ of the Young tableau for $[f]$; that is, $f_i = f_i^{12} + \Delta_i$, with $\Delta_i = 1$ for $i = a_1, a_2, \ldots, a_p$; or, in the shorthand notation of Ref. [1],

$$[f] = [f^{12}] + \Delta(a_1 \ldots a_p); \quad \text{alternately} \quad [f^{12}] = [f(a_1 \ldots a_p)] .$$

The representation $[f^2]$ is obtained from $[f^{12}]$ by the further removal of squares from the rows labeled $b_1 b_2 \ldots b_q$ of the resultant tableau for $[f^{12}]$; that is

$$[f^{12}] = [f^2] + \Delta(b_1 \ldots b_q); \quad \text{or} \quad [f^2] = [f^{12}(b_1 \ldots b_q)] .$$

On the other hand, the representation $[f^{23}]$ is obtained from $[f]$ by the removal of squares from the rows labeled $b'_1 b'_2 \ldots b'_q$ of the tableau for $[f]$,

$$[f] = [f^{23}] + \Delta(b'_1 \ldots b'_q) ;$$

whereas, finally

$$[f^{23}] = [f^2] + \Delta(a'_1 \ldots a'_p) .$$

2. Matrix Element of the Permutation Operator

2.1. Preliminaries

Although the permutation operator in the basic matrix element of Eq. (19) appears rather complicated, it is possible to reduce its evaluation to the basic matrix elements of the transpositions $P_{m,m-1}$ in the standard Young-Yamanouchi

representation [4]

$$\langle [f]\dots r_m = j\, r_{m-1} = k\dots |P_{m,m-1}| [f]\dots r_m = j\, r_{m-1} = k\dots\rangle = \frac{1}{\tau_{jk}}$$

$$\langle [f]\dots r_m = k\, r_{m-1} = j\dots |P_{m,m-1}| [f]\dots r_m = j\, r_{m-1} = k\dots\rangle = \left[1 - \frac{1}{\tau_{jk}^2}\right]^{\frac{1}{2}},$$

(21)

where τ_{jk} is the "axial distance" between the squares labeled j and k in the m-particle tableau left after the particles labeled $n, n-1, \dots, m+1$ have been removed from the original n-particle tableau of shape $[f]$ [see Eq. (9)]. To evaluate the matrix element of a transposition operator $P_{m,m'}$, with $m' \neq m \pm 1$, between state vectors belonging to the non-standard representations of S_n utilized in Eqs. (8) through (19), it will be useful to partially expand such state vectors, e.g.

$$|\{a_1 a_2 \dots a_p\}_{[1^p]}\dots\rangle \tag{22}$$

$$= \sum_{i=1}^{p} (-1)^{i-1} \left[\frac{1}{p} \prod_{\substack{j=1 \\ j \neq i}}^{p} \left(1 - \frac{1}{\tau_{a_i a_j}}\right)\right]^{\frac{1}{2}} |a_i \{a_1 \dots a_{i-1} a_{i+1} \dots a_p\}_{[1^{p-1}]}\dots\rangle.$$

Straightforward repeated application of Eqs. (21) and (22) yields

$$P_{n-m,n-m-q}|r_n \dots r_{n-m+1} a\{b_1 b_2 \dots b_q\}_{[1^q]} r_{n-m-q-1}\dots\rangle$$

$$= |r_n \dots r_{n-m+1}\{b_1 b_2 \dots b_q\}_{[1^q]} a\dots\rangle \left[\prod_{j=1}^{q} \left(1 - \frac{1}{\tau_{ab_j}^2}\right)\right]^{\frac{1}{2}}$$

$$+ \sum_{i=1}^{q} (-1)^{i+1} |\dots r_{n-m+1}\{a\dots b_{i-1} b_{i+1}\dots b_q\}_{[1^q]} b_i\dots\rangle$$

(23)

$$\cdot \frac{1}{\tau_{ab_i}} \left[\prod_{\substack{j=1 \\ j \neq i}}^{q} \left(1 - \frac{1}{\tau_{ab_j}}\right)\left(1 - \frac{1}{\tau_{b_j b_i}}\right)\right]^{\frac{1}{2}}.$$

This is the basic relation needed for the evaluation of the matrix element of Eq. (19).

2.2. The Totally Antisymmetric Case

The final result (to be established by induction) is: With

$$\{b_1'\dots b_q'\} = \{a_{\lambda_1} a_{\lambda_2}\dots a_{\lambda_s}\dots b_{\mu_1-1} b_{\mu_1+1}\dots b_{\mu_2-1} b_{\mu_2+1}\dots b_{\mu_s-1} b_{\mu_s+1}\dots\},$$

and

$$\{a_1'\dots a_p'\} = \{a_{i_1} a_{i_2}\dots a_{i_{p-s}} b_{\mu_1} b_{\mu_2}\dots b_{\mu_s}\},$$

$$\cdot \langle [f] \{b_1'\dots b_q'\}_{[1^q]} \{a_1'\dots a_p'\}_{[1^p]}\dots|$$

$$\cdot \prod_{k=1}^{p} P_{n-k+1,n-q-k+1} |[f] \{a_1\dots a_p\}_{[1^p]} \{b_1\dots b_q\}_{[1^q]}\dots\rangle$$

(24)

$$= (-1)^{\Phi} F_s(a_{\lambda_1}\dots a_{\lambda_s}; b_{\mu_1}\dots b_{\mu_s})$$

$$\cdot \left[\prod_{\lambda,\mu} \prod_{i,j} \left(1 - \frac{1}{\tau_{a_\lambda a_i}}\right)\left(1 - \frac{1}{\tau_{a_\lambda b_j}}\right)\left(1 - \frac{1}{\tau_{a_i b_\mu}}\right)\left(1 - \frac{1}{\tau_{b_j b_\mu}}\right)\left(1 - \frac{1}{\tau_{a_i b_j}^2}\right)\right]^{\frac{1}{2}}.$$

Here, the indices $\boldsymbol{a}$ and $\boldsymbol{b}$ are split into two classes denoted by Greek and Roman-letter subscripts, respectively. The $a_{\lambda_t}(t = 1, ..., s)$ are those of the indices $\boldsymbol{a}$ which have been shifted from the p-particle group on the right hand side of the matrix element (24) to the q-particle group on the left hand side. Similarly, the $b_{\mu_t}(t = 1, ...s)$ are those of the indices $\boldsymbol{b}$ which have been *shifted* from the q-particle group on the right to the p-particle group on the left. The $a_{i_k}(k = 1, ..., p - s)$ on the other hand are those of the indices $\boldsymbol{a}$ which remain in the p-particle group on the left. Similarly, the $b_{j_l}(l = 1, ..., q - s)$ are those of the indices $\boldsymbol{b}$ which remain in the q-particle group on the left.

The result of Eq. (24) includes three factors:

(1) The square root factor. The products over λ, μ, i, j in this factor run over all indices $\lambda_t, \mu_t(t = 1, ..., s), i_k(k = 1, ..., p - s), j_l(l = 1, ...q - s)$. The square root includes: (i) all factors of the type $(1 - 1/\tau_{a_\lambda c})$ where a_λ runs over all shifted a_{λ_t}, whereas c runs over all unshifted indices, $c = a_{i_k}$ or $c = b_{j_l}$; (ii) all factors of the type $(1 - 1/\tau_{cb_\mu})$ where b_μ runs over all shifted indices b_{μ_t}, whereas the first index c runs over all unshifted indices $c = a_{i_k}$ or $c = b_{j_l}$; and (iii) all factors of the type $(1 - 1/\tau_{a_ib_j}^2)$, where the first index runs over all unshifted a_{i_k} whereas the second index runs over all unshifted b_{j_l}. (Note that this is the only type of term which survives for the special case $s = 0$.)

The antisymmetry of the state vectors requires $a_i \neq a_j$ for $j \neq i$, similarly $b_i \neq b_j$ for $j \neq i$. However, it may be important to note that a particular b_i could be equal to some a_k. In that case

$$\tau_{a_kb_i} = +1, \qquad \tau_{cb_i} = \tau_{ca_k} + 1, \qquad \tau_{b_ic} = \tau_{a_kc} - 1.$$

It might appear that the operator $\boldsymbol{P}$ acting on $|\{a_1...a_p\}_{[1^p]}\{b_1...b_q\}_{[1^q]}...\rangle$ could in this case make contributions to the matrix element (24) in two ways: (i) by producing a state $|\rangle$ in which b_i and a_k are both shifted, and (ii) by producing a $|\rangle$ in which b_i and a_k are both unshifted. Note, however, that the second state has a zero coefficient through the factor $(1 - 1/\tau_{a_kb_i}^2)$.

(2) The phase factor. The phase factor Φ is given by

$$\Phi = \chi_\lambda + \sum_{t=1}^{s} \mu_t + \mathrm{sp} + \tfrac{1}{2}s(s - 1) = \chi_\lambda + \chi_\mu + s(p + q), \tag{25}$$

where $\chi_\lambda = $ even (odd) for $a_{\lambda_1}a_{\lambda_2}...a_{\lambda_s}a_{i_1}...a_{i_{p-s}} = $ even (odd) permutation of $a_1 a_2...a_p$; while $\chi_\mu = $ even (odd) for $b_{j_1}...b_{j_{q-s}}b_{\mu_1}b_{\mu_2}...b_{\mu_s} = $ even (odd) permutation of $b_1 b_2...b_q$.

(3) The Functions $F_s(a_{\lambda_1}...a_{\lambda_s}; b_{\mu_1}...b_{\mu_s})$. These functions are symmetric in the s indices $a_{\lambda_1}...a_{\lambda_s}$ and in the s indices $b_{\mu_1}...b_{\mu_s}$ and have the form

$$F_s(a_{\lambda_1}...a_{\lambda_s}; b_{\mu_1}...b_{\mu_s}) = \sum_{m=0}^{s} (-1)^m \mathscr{F}_s(m) \tag{26}$$

with

$$\mathscr{F}_s(m) = \sum_{i_1 < i_2 < \cdots < i_m} f_s(\mu_{i_1}) f_s(\mu_{i_2})...f_s(\mu_{i_m}) \prod_{i=i_1}^{i_m} \prod_{\substack{j=1 \\ j \neq i_1...i_m}}^{s} \left(1 - \frac{1}{\tau_{b_{\mu_i}b_{\mu_j}}}\right), \tag{26a}$$

where

$$f_s(\mu_i) = \prod_{t=1}^{s} \left(1 - \frac{1}{\tau_{a_{\lambda_t} b_{\mu_i}}} \right). \tag{26b}$$

E.g.

$$\mathscr{F}_3(2) = f(\mu_1) f(\mu_2) \left(1 - \frac{1}{\tau_{b_{\mu_1} b_{\mu_3}}} \right) \left(1 - \frac{1}{\tau_{b_{\mu_2} b_{\mu_3}}} \right)$$
$$+ f(\mu_1) f(\mu_3) \left(1 - \frac{1}{\tau_{b_{\mu_1} b_{\mu_2}}} \right) \left(1 - \frac{1}{\tau_{b_{\mu_3} b_{\mu_2}}} \right) \tag{26c}$$
$$+ f(\mu_2) f(\mu_3) \left(1 - \frac{1}{\tau_{b_{\mu_2} b_{\mu_1}}} \right) \left(1 - \frac{1}{\tau_{b_{\mu_3} b_{\mu_1}}} \right).$$

With $m = 0$, $\mathscr{F}_s(0) = 1$. With $m = s$, no indices of type b_{μ_j} (with $j \neq i_1 \dots i_s$) exist, and

$$\mathscr{F}_s(s) = \prod_{i=1}^{s} f(\mu_i). \tag{26d}$$

2.3. Derivation

The derivation of Eq. (24) uses the method of induction. For $p = 1$ the result follows from Eq. (23). For $p = 1$, $s = 0$

$$\langle \{b_1 \dots b_q\}_{[1^q]} a \dots | P_{n, n-q} | a \{b_1 \dots b_q\}_{[1^q]} \dots \rangle = \left[\prod_{j=1}^{q} \left(1 - \frac{1}{\tau_{ab_j}^2} \right) \right]^{\frac{1}{2}} \tag{27}$$

with $F_0 = 1$, (the only a index and all b indices are of the unshifted type); while for $p = 1$, $s = 1$

$$\langle \{a \dots b_{\mu-1} b_{\mu+1} \dots\}_{[1^q]} b_\mu \dots | P_{n, n-q} | a \{b_1 \dots b_q\}_{[1^q]} \dots \rangle$$
$$= F_1(a; b_\mu) \left[\prod_{\substack{j=1 \\ j \neq \mu}}^{q} \left(1 - \frac{1}{\tau_{ab_j}} \right) \left(1 - \frac{1}{\tau_{b_j b_\mu}} \right) \right]^{\frac{1}{2}} \tag{28}$$

with $F_1(a; b_\mu) = \mathscr{F}_1(0) - \mathscr{F}_1(1) = 1 - (1 - 1/\tau_{ab_\mu}) = 1/\tau_{ab_\mu}$.

Assuming the result for arbitrary $p - 1$; that is, assuming the result (24) for the matrix element of the operator

$$\prod_{k=2}^{p} P_{n-k+1, n-q-k+1} \quad \text{acting on functions of the type} \quad |r_n \{\dots\}_{[1^{p-1}]} \{\dots\}_{[1^q]} \dots \rangle,$$

and using Eq. (22), we obtain

$$\prod_{k=1}^{p} P_{n-k+1, n-q-k+1} | \{a_1 \dots a_p\}_{[1^p]} \{b_1 \dots b_q\}_{[1^q]} \dots \rangle$$

$$= P_{n, n-q} \sum_{M=1}^{p} (-1)^{M+1} \left[\prod_{\substack{L=1 \\ L \neq M}}^{p} \frac{1}{p} \left(1 - \frac{1}{\tau_{a_M a_L}} \right) \right]^{\frac{1}{2}} \sum_{s=0}^{p-1} \sum_{\lambda, \mu} F_s(\lambda_1 \dots \lambda_s; \mu_1 \dots \mu_s) \tag{29}$$

$$\cdot (-1)^{\Phi(s, p-1)} [\dots]^{\frac{1}{2}}$$

$$\cdot |a_M \{a_{\lambda_1} \dots a_{\lambda_s} \dots b_{\mu_1-1} b_{\mu_1+1} \dots b_{\mu_s-1} b_{\mu_s+1} \dots\}_{[1^q]} \{a_{i_1} \dots a_{i_{p-s-1}} b_{\mu_1} \dots b_{\mu_s}\}_{[1^{p-1}]} \dots \rangle,$$

where the sum over λ is over all possible combinations $a_{\lambda_1}\ldots a_{\lambda_s}$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_s$, (similarly for the sum over μ); but where (a's with Greek or lower case Roman subscripts) $\neq a_M$. The factor $[\ldots]^{\frac{1}{2}}$ has the form of the square root factor of (24) but the products over a_λ's and a_i's exclude the specific value a_M.

Now, using Eq. (23) for $P_{n,n-q}$ and the inverse of (22) applied to states of symmetry $[1^q]$, we obtain

$$\langle \{a_{\lambda_1}\ldots a_{\lambda_s} a_{\lambda_{s+1}}\ldots b_{\mu_{s+1}-1}b_{\mu_{s+1}+1}\ldots\}_{[1^q]}\,\{a_{i_1}\ldots a_{i_{p-s-1}}b_{\mu_1}\ldots b_{\mu_s}b_{\mu_{s+1}}\}_{[1^p]}\ldots|$$

$$\cdot \prod_{k=1}^{p} P_{n-k+1,n-q-k+1}|\{a_1\ldots a_p\}_{[1^p]}\,\{b_1\ldots b_q\}_{[1^q]}\ldots\rangle$$

$$= \left(\sum_{t=1}^{s+1}\left\{\frac{1}{\tau_{a_{\lambda_t}b_{\mu_{s+1}}}}\prod_{\substack{r=1\\r\neq t}}^{s+1}\left(1 - \frac{1}{\tau_{a_{\lambda_r}b_{\mu_{s+1}}}}\right)\left(1 - \frac{1}{\tau_{a_{\lambda_t}a_{\lambda_r}}}\right)\right.\right. \tag{30}$$

$$\left.\left.\cdot F_s(a_{\lambda_1}\ldots a_{\lambda_{t-1}}a_{\lambda_{t+1}}\ldots;b_{\mu_1}\ldots b_{\mu_s})\right\}\right)[\ldots]^{\frac{1}{2}}(-1)^{\Phi(s+1,p)},$$

where $[\ldots]^{\frac{1}{2}}$ is now the full square root factor of Eq. (24), including the indices λ_{s+1}, μ_{s+1}.

The final phase is that appropriate for indices p and $s+1$. The factor $(-1)^{M+1}$ in Eq. (29) assures that the χ_λ-part of the phase factor remains unchanged. The operator $P_{n,n-q}$, acting on the state vector of Eq. (29), introduces a phase which depends on the ordering of the b_i [see Eq. (23)]. If it is assumed for the moment that $\mu_\sigma < \mu_{s+1} < \mu_{\sigma+1}$, $b_{\mu_{s+1}}$ in the state vector of Eq. (29) is preceded by s symbols a_λ; but σ of the symbols b_μ preceding $b_{\mu_{s+1}}$ are missing. The action of $P_{n,n-q}$ thus produces a state

$$|\{a_{\lambda_1}\ldots a_{\lambda_s}\ldots b_{\mu_{s+1}-1}b_{\mu_{s+1}+1}\ldots\}_{[1^q]}\,b_{\mu_{s+1}}\{a_{i_1}\ldots a_{i_{p-s-1}}b_{\mu_1}\ldots b_{\mu_s}\}_{[1\,p-1]}\ldots\rangle$$

with a phase $(-1)^\Phi$, with

$$\Phi = \Phi(s, p-1; \mu_1\ldots\mu_s) + \mu_{s+1} + 1 + s - \sigma$$

$$= \chi_\lambda + \sum_{t=1}^{s+1}\mu_t + (s+1)p + \tfrac{1}{2}(s+1)s + (s-p+1-\sigma) = \Phi(s+1,p) + (s-p+1-\sigma).$$

The extra phase factor $(s-p+1-\sigma)$ is precisely the factor needed to bring $b_{\mu_{s+1}}$ to its proper position in the state of symmetry $[1^p]$, [cf. Eq. (22)].

To carry out the sum over t in Eq. (30), it will be convenient to rewrite

$$\frac{1}{\tau_{a_{\lambda_t}b_{\mu_{s+1}}}}\prod_{\substack{r=1\\r\neq t}}^{s+1}\left(1 - \frac{1}{\tau_{a_{\lambda_r}b_{\mu_{s+1}}}}\right) = \prod_{r=1}^{s+1}\left(1 - \frac{1}{\tau_{a_{\lambda_r}b_{\mu_{s+1}}}}\right)\left[1 - \left(1 - \frac{1}{\tau_{\bar{a}_{\lambda_t}b_{\mu_{s+1}}}}\right)\right]$$

$$= f_{s+1}(\mu_{s+1})\left[1 - \left(1 - \frac{1}{\tau_{\bar{a}_{\lambda_t}b_{\mu_{s+1}}}}\right)\right], \tag{31}$$

where we have also introduced the shorthand notation

$$\tau_{\bar{a}b} = \tau_{ab} - 1, \quad \text{similarly} \quad \tau_{a\bar{b}} = \tau_{ab} + 1.$$

(Note that $\tau_{ba} = -\tau_{ab}$, and that $\tau_{\bar{a}\bar{b}} = \tau_{ab}$; a bar over a subscript of τ can be used to denote the previous removal of one square from the corresponding row of the Young tableau.) In the expansion of $F_s(\ldots a_{\lambda_{t-1}} a_{\lambda_{t+1}} \ldots; b_{\mu_1} \ldots b_{\mu_s})$ in terms of $f_s(\mu_i)$, it will also be convenient to rewrite

$$f_s(\mu_i) = \prod_{r \neq t} \left(1 - \frac{1}{\tau_{a_{\lambda_r} b_{\mu_i}}}\right) = f_{s+1}(\mu_i) \left[2 - \left(1 - \frac{1}{\tau_{\bar{a}_{\lambda_t} b_{\mu_i}}}\right)\right] \tag{32}$$

where the product Π_r in f_{s+1} now includes the index $r = t$. Finally, we need the identity

$$\sum_{i=1}^{s} \prod_{\substack{j=1 \\ j \neq i}}^{s} \left(1 - \frac{1}{\tau_{ij}}\right) = s \tag{33}$$

(which follows from simple contour integration, see Eq. (22) of Ref. [12]), and leads to the further identities

$$\sum_{t=1}^{s+1} \prod_{\substack{r=1 \\ r \neq t}}^{s+1} \left(1 - \frac{1}{\tau_{a_{\lambda_t} a_{\lambda_r}}}\right)\left(1 - \frac{1}{\tau_{\bar{a}_{\lambda_t} b_{\mu_i}}}\right) = (s+2) - \frac{1}{f_{s+1}(\mu_i)} \tag{34}$$

and

$$\sum_{t=1}^{s+1} \prod_{\substack{r=1 \\ r \neq t}}^{s+1} \left(1 - \frac{1}{\tau_{a_{\lambda_t} a_{\lambda_r}}}\right) \prod_{i=1}^{k} \left(1 - \frac{1}{\tau_{\bar{a}_{\lambda_t} b_{\mu_i}}}\right)$$

$$= (s+k+1) - \sum_{i=1}^{k} \prod_{\substack{j=1 \\ j \neq i}}^{k} \left(1 - \frac{1}{\tau_{b_{\mu_i} b_{\mu_j}}}\right) \frac{1}{f_{s+1}(\mu_i)}. \tag{35}$$

With these identities the sum over t in Eq. (30) can now be carried out. It is important to note that the functions $f_{s+1}(\mu_i)$ are symmetric in the λ_r, $r = 1, \ldots, s+1$, and can be taken outside the summation. In the expansion of F_s in terms of the $f_s(\mu_i)$ and subsequent summation over t in Eq. (30) two types of terms must be considered:

(1) Terms independent of $f_{s+1}(\mu_{s+1})$. Such terms arise only through the second, [i.e. the $(1 - 1/\tau_{\bar{a}b})$-factor of Eq. (31)] and the second terms of the right hand sides of Eqs. (34), (35), with $i = s + 1$. The general term of this type in $\mathscr{F}_{s+1}(m)$ becomes

$$\sum_{i_1 < i_2 < \cdots < i_m}^{s} (-1)^m f_{s+1}(\mu_{i_1}) f_{s+1}(\mu_{i_2}) \ldots f_{s+1}(\mu_{i_m}) \prod_{i=i_1}^{i_m} \prod_{\substack{j=1 \\ j \neq i_1, \ldots i_m}}^{s} \left(1 - \frac{1}{\tau_{b_{\mu_i} b_{\mu_j}}}\right)$$

$$\cdot \left\{2^m - 2^{m-1} \sum_{i=i_1}^{i_m} \left(1 - \frac{1}{\tau_{b_{\mu_{s+1}} b_{\mu_i}}}\right) + \cdots + 2^{m-k}(-1)^k \sum_{j_1 < j_2 < \cdots < j_k = i_1}^{i_m} \prod_{j=j_1}^{j_k} \left(1 - \frac{1}{\tau_{b_{\mu_{s+1}} b_{\mu_j}}}\right)\right.$$

$$\left. + \cdots (-1)^m \prod_{i=i_1}^{i_m} \left(1 - \frac{1}{\tau_{b_{\mu_{s+1}} b_{\mu_i}}}\right)\right\}. \tag{36}$$

[Here, the kth term in $\{\ldots\}$ arises from $\mathscr{F}_s(m)$ in the expansion, via Eq. (26), of the F_s of Eq. (30); the factors 2^{m-k} and the remaining k factors coming from $m - k$ factors of 2 and k factors of $-(1 - 1/\tau)$ in the expansion of the m factors $f_s(\mu_i)$ in terms of $f_{s+1}(\mu_i)$ by means of Eq. (32).] The terms in $\{\ldots\}$ can be summed by expanding all products in powers of $1/\tau_{b_{\mu_i} b_{\mu_{s+1}}}$, (where it was convenient to use

$\tau_{jk} = -\tau_{kj}$). The coefficient of

$$\sum_{j_1 < j_2 < \cdots < j_k = i_1}^{i_m} \prod_{j=j_1}^{j_k} \frac{1}{\tau_{b_{\mu_j} b_{\mu_{s+1}}}} \quad \text{is} \quad (-1)^k \left[(-1)^{m-k} \sum_{l=0}^{m-k} (-2)^l \binom{m-k}{l} \right] = (-1)^k$$

(using the binomial expansion), so that

$$\{\ldots\} = \prod_{i=i_1}^{i_m} \left(1 - \frac{1}{\tau_{b_{\mu_i} b_{\mu_{s+1}}}} \right),$$

and Eq. (36) becomes

$$(-1)^m \sum_{i_1 < i_2 < \cdots < i_m} \prod_{i=i_1}^{i_m} f_{s+1}(\mu_i) \prod_{\substack{j=1 \\ j \neq i_1, i_2, \ldots i_m}}^{s+1} \left(1 - \frac{1}{\tau_{b_{\mu_i} b_{\mu_j}}} \right), \tag{37}$$

the required form for $\mathscr{F}_{s+1}(m)$.

(2) The second type of term is that which includes the factor $f_{s+1}(\mu_{s+1})$. The general term of this type in $\mathscr{F}_{s+1}(m)$ becomes

$$\sum_{i_1 < i_2 < \cdots < i_{m-1}}^{s} f_{s+1}(\mu_{i_1}) \cdots f_{s+1}(\mu_{i_{m-1}}) f_{s+1}(\mu_{s+1}) \prod_{i=i_1}^{i_{m-1}} \prod_{\substack{j=1 \\ j \neq i_1 \ldots i_{m-1}}}^{s} \left(1 - \frac{1}{\tau_{b_{\mu_i} b_{\mu_j}}} \right)$$

$$\cdot (-1)^{m-1} \left\{ 2^{m-1}[(s+1) - (s+2)] + \cdots + 2^{m-1-k}(-1)^k \binom{m}{k} [(s+k+1) - (s+k+2)] \right.$$

$$\left. + \cdots + (-1)^{m-1}[(s+m) - (s+m+1)] \right\}$$

$$+ \sum_{l=1}^{s} \sum_{\substack{i_1 < i_2 < \cdots < i_{m-1} \\ l \neq i_1 \ldots i_{m-1}}}^{s} f_{s+1}(\mu_{i_1}) \cdots f_{s+1}(\mu_{i_{m-1}}) f_{s+1}(\mu_{s+1}) \prod_{\substack{i=i_1 \\ j \neq i_1 \ldots i_{m-1} \\ \text{also } j \neq l}}^{i_{m-1}} \prod_{j=1}^{s} \left(1 - \frac{1}{\tau_{b_{\mu_i} b_{\mu_j}}} \right)$$

$$\cdot \left(1 - \frac{1}{\tau_{b_{\mu_l} b_{\mu_j}}} \right) (-1)^m \left[1 - \left(1 - \frac{1}{\tau_{b_{\mu_l} b_{\mu_{s+1}}}} \right) \right] \tag{38}$$

$$\cdot \left\{ 2^{m-1} + \cdots + 2^{m-1-k}(-1)^k \sum_{\substack{j_1 < j_2 < \cdots j_k = i_1 \\ j \neq l}}^{i_{m-1}} \prod_{j=j_1}^{j_k} \left(1 - \frac{1}{\tau_{b_{\mu_i} b_{\mu_j}}} \right) + \cdots \right\},$$

where the first term arises from $\mathscr{F}_s(m-1)$ in the expansion [via Eq. (26)], of F_s of Eq. (30); the two terms $(s+k+1)$ and $(s+k+2)$ in each square bracket arising from the factors f_{s+1} and $-f_{s+1}(1 - 1/\tau_{ab})$ of Eq. (31) through the first terms of the right hand sides of Eqs. (34) and (35). The bracket $\{\ldots\}$ of this first term of (38) sums to -1 via the binomial expansion. The second term of Eq. (38) arises from $\mathscr{F}_s(m)$ in the expansion of F_s, where this expansion contributes an extra factor $f_{s+1}(\mu_l)$ which is subsequently cancelled by one of the second terms of the right hand sides of Eqs. (34) and (35). The factors 1 and $-(1 - 1/\tau_{b_{\mu_l} b_{\mu_{s+1}}})$ arise through the factors f_{s+1} and $-f_{s+1}(1 - 1/\tau_{\bar{a}b})$ in Eq. (31), respectively. The bracket $\{\ldots\}$ in the second term of (38) sums to

$$\prod_{i=i_1}^{i_{m-1}} \left(1 - \frac{1}{\tau_{b_{\mu_i} b_{\mu_l}}} \right)$$

[cf. Eq. (37)]. After taking the symmetric functions in the $s - m + 1$ indices $j = 1, \ldots, s, j \neq i_1, \ldots, i_{m-1}$

$$\prod_{\substack{i=i_1}}^{i_{m-1}} \prod_{\substack{j=1 \\ j \neq i_1 \ldots i_{m-1}}}^{s} \left(1 - \frac{1}{\tau_{b_{\mu_i} b_{\mu_j}}}\right)$$

outside the summation over l, this summation over $s - m + 1$ terms can be carried out via (32) to give

$$\sum_{\substack{l=1 \\ l \neq i_1 \ldots i_{m-1}}}^{s} \prod_{\substack{j=1 \\ j \neq i_1 \ldots i_{m-1}}}^{s} \left(1 - \frac{1}{\tau_{b_{\mu_i} b_{\mu_j}}}\right)\left[1 - \left(1 - \frac{1}{\tau_{b_{\mu_i} b_{\mu_{s+1}}}}\right)\right]$$

$$= \{(s - m + 1) - (s - m + 2)\} + \prod_{\substack{j=1 \\ j \neq i_1 \ldots i_{m-1}}}^{s} \left(1 - \frac{1}{\tau_{b_{\mu_{s+1}} b_{\mu_j}}}\right). \tag{39}$$

The first term (in curly brackets), $\{= -1\}$, cancels the first term of Eq. (38) so that the combination of all terms of Eq. (38) finally yields

$$(-1)^m \sum_{i_1 < i_2 < \cdots i_{m-1}}^{s} \prod_{i=i_1}^{i_{m-1}} f_{s+1}(\mu_i) f_{s+1}(\mu_{s+1}) \prod_{\substack{j=1 \\ j \neq i_1 \ldots i_{m-1}}}^{s} \left(1 - \frac{1}{\tau_{b_{\mu_i} b_{\mu_j}}}\right)\left(1 - \frac{1}{\tau_{b_{\mu_{s+1}} b_{\mu_j}}}\right)$$

$$\tag{40}$$

which has the required form for $\mathscr{F}_{s+1}(m)$, and together with (37) gives the full expression for $\mathscr{F}_{s+1}(m)$. The inductive proof is therefore completed.

2.4. The Totally Symmetric Case

The derivation for the recoupling matrix for $[f^1] \times [f^2] \times [f^3] \to [f]$ for which $[f^1]$ and $[f^3]$ are both totally symmetric representations proceeds in almost identical fashion. If $[1^p]$ and $[1^q]$ are replaced by totally symmetric representations $[p]$ and $[q]$; the analogues of Eqs. (8), (22), and (23) can be obtained by replacing all factors $(1 - 1/\tau) \to (1 + 1/\tau)$. Also, all terms in the summations of these eqs. now become positive. [Note, however, that $(1 - 1/\tau^2) \to (1 - 1/\tau^2)$.]

The final result is

$$U([q] [f^2] [f] [p]; [f^{12}]; [f^{23}]) = (-1)^{\varphi_1([q][f^2][f^{12}]) + \varphi_2([q][f^{23}][f])}$$

$$\cdot \left[\prod_{\lambda,\mu} \prod_{i,j} \left(1 + \frac{1}{\tau_{a_\lambda a_i}}\right)\left(1 + \frac{1}{\tau_{a_\lambda b_j}}\right)\left(1 + \frac{1}{\tau_{a_i b_\mu}}\right)\left(1 + \frac{1}{\tau_{b_j b_\mu}}\right)\left(1 - \frac{1}{\tau_{a_i b_j}^2}\right)\right]^{\frac{1}{2}} \tag{41}$$

$$\cdot F_s'(\lambda_1 \ldots \lambda_s; \mu_1 \ldots \mu_s),$$

where

$$[f] = [f^{12}] + \Delta(a_1 \ldots a_p); \quad (\text{alternately } [f^{12}] = [f(a_1 \ldots a_p)])$$
$$[f^{12}] = [f^2] + \Delta(b_1 \ldots b_q)$$
$$[f] = [f^{23}] + \Delta(a_{\lambda_1} \ldots a_{\lambda_s} b_{j_1} \ldots b_{j_{q-s}}) \tag{41a}$$
$$[f^{23}] = [f^2] + \Delta(a_{i_1} \ldots a_{i_{p-s}} b_{\mu_1} \ldots b_{\mu_s}),$$

where the indices $\lambda_t, \mu_t, a_i, b_j$ are to be interpreted as for Eqs. (19) and (24); but where now

$$F_s'(\lambda_1 \ldots \lambda_s; \mu_1 \ldots \mu_s) = (-1)^s \sum_{m=0}^{s} \mathscr{F}_s'(m) \; (-1)^m \qquad (41\,\text{b})$$

with

$$\mathscr{F}_s'(m) = \sum_{i_1 < i_2 < \cdots i_m} \prod_{i=i_1}^{i_m} f_s'(\mu_i) \prod_{\substack{j=1 \\ j \neq i_1 \cdots i_m}}^{s} \left(1 + \frac{1}{\tau_{b_{\mu_i} b_{\mu_j}}}\right) \qquad (41\,\text{c})$$

and

$$f_s'(\mu_i) = \prod_{t=1}^{s} \left(1 + \frac{1}{\tau_{a_{\lambda_t} b_{\mu_i}}}\right). \qquad (41\,\text{d})$$

The extra factor $(-1)^s$ in (41 b) comes about because the analogue of Eq. (31), viz.

$$\frac{1}{\tau_{a_{\lambda_t} b_{\mu_{s+1}}}} \prod_{\substack{r=1 \\ r \neq t}}^{s+1} \left(1 + \frac{1}{\tau_{a_{\lambda_r} b_{\mu_{s+1}}}}\right) = -f_{s+1}'(\mu_{s+1}) \left[1 - \left(1 + \frac{1}{\tau_{a_{\lambda_t} \bar{b}_{\mu_{s+1}}}}\right)\right] \qquad (42)$$

now contains an extra factor of -1.

For the totally symmetry case, it is now possible to have some $a_j = a_i$, or $b_j = b_i$, as well as some $a_i = b_k$; so that it may be useful to give a few examples. We shall use the case $[p] = [q] = [2]$ for purposes of illustration:

(i) With $\{a_1 a_2\} \{b_1 b_2\} = \{ij\} \{kl\}$; $\{b_1' b_2'\} \{a_1' a_2'\} = \{kl\} \{ij\}$; $i \neq j \neq k \neq l$, all indices are of the unshifted type and

$$U([2] [f(ijkl)] [f] [2]; [f(ij)] [f(kl)]$$
$$= (-1)^{\varphi_1 + \varphi_2} \left[\left(1 - \frac{1}{\tau_{ik}^2}\right)\left(1 - \frac{1}{\tau_{il}^2}\right)\left(1 - \frac{1}{\tau_{jk}^2}\right)\left(1 - \frac{1}{\tau_{jl}^2}\right)\right]^{\frac{1}{2}}, \qquad (43)$$

where $[f(ij)]$, e.g., indicates the representation with a tableau obtained by removing one square each from rows i and j of the tableau for $[f]$.

(ii) With $\{a_1 a_2\} \{b_1 b_2\} = \{ij\} \{kl\}$; $\{b_1' b_2'\} \{a_1' a_2'\} = \{ij\} \{kl\}$ $i \neq j \neq k \neq l$, on the other hand, all indices are of the shifted type. The square root factor of Eq. (41) is $+1$ (there are no unshifted indices); and, except for phase, the U-coefficient is given by the factor $F_2'(ij; kl)$:

$$U([2] [f(ijkl)] [f] [2]; [f(ij)] [f(ij)])$$
$$= (-1)^{\varphi_1 + \varphi_2} \frac{[1 + \tau_{ik} + \tau_{jl} + \tau_{ik}\tau_{jl} + \tau_{jk}\tau_{il}]}{\tau_{ik}\tau_{il}\tau_{jk}\tau_{jl}}. \qquad (44)$$

(iii) In the special case when $j = i$: $\tau_{jm} = \tau_{\bar{i}m} = \tau_{im} - 1$, $\tau_{mj} = \tau_{m\bar{i}} = \tau_{mi} + 1$; and (case (ii) reduces to

$$U([2] [f(iikl)] [f] [2]; [f(ii)] [f(ii)])$$
$$= (-1)^{\varphi_1 + \varphi_2} \frac{2}{(\tau_{ik} - 1)(\tau_{il} - 1)}, \qquad (45)$$

where representation $[f(ii)]$ is obtained from $[f]$ by the removal of 2 squares from row i.

(iv) Finally, with $\{a_1 a_2\}\{b_1 b_2\} = \{ij\}\{il\}$; $\{b'_1 b'_2\}\{a'_1 a'_2\} = \{il\}\{ij\}$, l and j are unshifted indices, while $\tau_{a_1 b_1} = \tau_{ii} = +1$; $F'_1(i;i) = -\left[1 - \left(1 + \dfrac{1}{\tau_{ii}}\right)\right] = +1$; and

$$U([2][f(iijl)][f][2];[f(ij)][f(il)])$$

$$= (-1)^{\varphi_1 + \varphi_2}\left[\left(1 + \frac{1}{\tau_{ij}}\right)\left(1 + \frac{1}{\tau_{il}}\right)\left(1 + \frac{1}{\tau_{j\bar{i}}}\right)\left(1 + \frac{1}{\tau_{l\bar{i}}}\right)\left(1 - \frac{1}{\tau_{jl}^2}\right)\right]^{\frac{1}{2}} \quad (46)$$

$$= (-1)^{\varphi_1 + \varphi_2}\left[\frac{(\tau_{ij}+1)(\tau_{ij}-2)}{\tau_{ij}(\tau_{ij}-1)}\frac{(\tau_{il}+1)(\tau_{il}-2)}{\tau_{il}(\tau_{il}-1)}\left(1 - \frac{1}{\tau_{jl}^2}\right)\right]^{\frac{1}{2}}.$$

(Note that in this case the two indices i are considered to be *shifted* indices. It might appear that there should also have been a contribution from a term where the operator P acting on $|\{ij\}_{[2]}\{il_{[2]}...\rangle$ leaves the i's in the *unshifted* positions; but such a term is multiplied by the factor $[(1 - 1/\tau_{ii}^2)]^{\frac{1}{2}}$ which is now identically zero.)

3. Applications

The totally antisymmetric case will have useful applications to systems of identical fermions; particularly for problems in nuclear spectroscopy employing the methods of spectral distributions developed recently by French and collaborators [13]. In a configuration of identical nucleons (neutrons *or* protons) involving a large part of the nuclear shell model space, (several shell model orbits with angular momenta $j_1, j_2, ...$), the relevant unitary group $SU(N)$, with $N = (2j_1 + 1) + (2j_2 + 1) + \cdots$, may involve large values of N. However, the irreducible representations for n-particle states are restricted to be of the simple totally antisymmetric type $[1^n]$. The single particle creation (and annihilation) operators a_i^+ (and a_i), with $i = 1, ..., N$ transform according to the representations $[1]$ and $[1^{N-1}]$ of $SU(N)$. Operators $(a_{i_1}^+...a_{i_p}^+)$ and $(a_{j_1}...a_{j_h})$ transform according to the representations $[1^p]$ and $[1^{N-h}]$; whereas a k-body operator

$$(a_{i_1}^+ \times a_{i_2}^+ \times \cdots a_{i_p}^+)(a_{j_1} \times a_{j_2} \times \cdots a_{jh}) \quad (47)$$

with $k = \frac{1}{2}(h + p)$ in general contains all two-columned representations $[f_0] = [2^x 1^{N-h+p-2x}]$, with $(p - h, 0)_> \leq x \leq (p, N - h)_<$. (Here the concept of a k-body operator has been generalized [13] to the case with $p \neq h$.) It will be useful to expand any k-body operator in terms of $U(N)$ irreducible unit tensor operators, defined by

$$T_{m_0}^{[f_0]}(h, p) = \sum_{m_p \bar{m}_h} \langle [1^{N-h}]\,\bar{m}_h [1^p]\, m_p| [f_0]\, m_0\rangle (a^+...a^+)_{m_p}^{[1^p]}(a...a)_{\bar{m}h}^{[1^{N-h}]} \quad (48)$$

where the coefficients are $U(N)$ Wigner coefficients, and the $U(N)$ subgroup labels m_p are again given in a shorthand notation which now includes the angular momentum quantum numbers and all additional labels necessary to specify the p-particles states.

The basic operation of the spectral distribution technique involves the averaging of dynamical operators, or products of operators, over the complete set of states of an irreducible representation of SU(N), e.g.

$$\langle O_1 O_2 \rangle^n = \frac{1}{\dim[1^n]} \sum_m \langle [1^n] \, m | O_1 O_2^+ | [1^n] \, m \rangle$$

$$= \frac{1}{\dim[1^n]} \sum_{m,m'} \langle [1^n] \, m | O_1 | [1^{n'}] \, m' \rangle \langle [1^n] \, m | O_2 | [1^{n'}] \, m' \rangle \,, \tag{49}$$

where the dimension of the n-particle representation is

$$\dim[1^n] = N!/(N-n)!\,n!\,.$$

If the operators can be expanded in terms of irreducible unit tensors of the type (48), the problem is reduced to the evaluation of the matrix element of such an operator

$$\langle [1^n] \, m | T_{m_0}^{[f_0]}(h,p) | [1^{n'}] \, m' \rangle = \sum_{m_p \bar{m}_h m''} \langle [1^{N-h}] \, \bar{m}_h [1^p] \, m_p | [f_0] \, m_0 \rangle$$

$$\cdot \langle [1^n] \, m | (a^+ \ldots a^+)_{m_p}^{[1^p]} | [1^{n''}] \, m'' \rangle \langle [1^{n''}] \, m'' | (a \ldots a)_{\bar{m}_h}^{[1^{N-h}]} | [1^{n'}] \, m' \rangle \,, \tag{50}$$

where n'' is restricted to $n'' = n - p$ by the nature of the operators; $(n' = n'' + h)$. The matrix element of $(a^+ \ldots a^+)_{m_p}^{[1^p]}$, (essentially an $n \to (n-p)$-particle coefficient of fractional parentage) is, except for an n, p-dependent factor, given by a simple $U(N)$ Wigner coefficient [18]

$$\langle [1^n] \, m | (a^+ \ldots a^+)_{m_p}^{[1^p]} | [1^{n''}] \, m'' \rangle = \left[\frac{n!}{(n-p)!} \right]^{\frac{1}{2}} \langle [1^{n''}] \, m'' [1^p] \, m_p | [1^n] \, m \rangle \,. \tag{51}$$

The matrix element of $(a \ldots a)_{\bar{m}_h}^{[1^{N-h}]}$ can also be expressed in terms of a $U(N)$ Wigner coefficient

$$\langle [1^{n''}] \, m'' | (a_{i_1} \times a_{i_2} \times \cdots a_{i_h})_{\bar{m}_h}^{[1^{N-h}]} | [1^{n'}] \, m' \rangle$$

$$= (-1)^{\frac{1}{2}h(h-1)+\eta_h} \langle [1^{n'}] \, m' | (a_{i_1}^+ \times a_{i_2}^+ \times \cdots a_{i_h}^+)_{m_h}^{[1^h]} | [1^{n''}] \, m'' \rangle$$

$$= (-1)^{\frac{1}{2}h(h-1)+\eta_h} \left[\frac{n'!}{(n'-h)!} \right]^{\frac{1}{2}} \langle [1^{n''}] \, m'' [1^h] \, m_h | [1^{n'}] \, m' \rangle \,, \tag{52}$$

where we have first used hermitean conjugation and subsequently complex conjugation in the repeated Wigner couplings of representations [1] with $U(N)$ subgroup labels i_1, i_2, ..., i_h. The phase factor, η_h, is associated with this conjugation process. It is the phase factor associated with the conjugation of the final h-particle state

$$|[1^h] \, m_h \rangle^* = (-1)^{\eta_h} |[1^{N-h}] \, \bar{m}_h \rangle \,. \tag{53}$$

In order to bring the $U(N)$ Wigner coefficient of Eq. (52) into a form which makes it possible to carry out the m-sums of Eq. (50), it is necessary to use a symmetry property of the $U(N)$ Wigner coefficient

$$\langle [1^{n''}] \, m'' [1^h] \, m_h | [1^{n'}] \, m' \rangle = (-1)^{\eta_h + \chi} \left[\frac{\dim[1^{n'}]}{\dim[1^{n''}]} \right]^{\frac{1}{2}} \langle [1^{n'}] \, m' [1^{N-h}] \, \bar{m}_h | [1^{n''}] \, m'' \rangle \,, \tag{54}$$
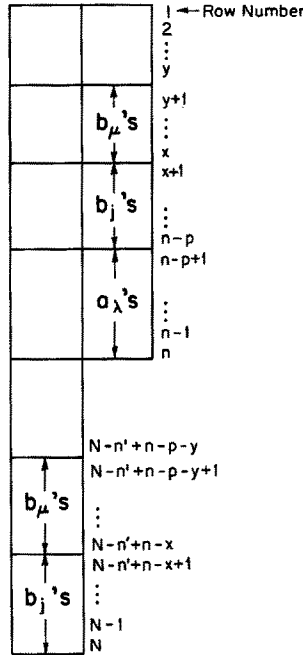
Fig. 2

where the phase factor must include the conjugation phase factor, η_h, and an additional representation-dependent phase factor χ which, as always, is dependent on phase conventions. [Note that the Wigner coefficients of Eq. (54) involve multiplicity-free $U(N)$ Wigner couplings.] To make the final result as independent of phase conventions as possible, it will be useful to eliminate the phase factor χ, by expressing the above dimensional and phase factors in terms of an $SU(N)$ Racah coefficient with the scalar representation in the 23 position, $([f^{23}] = [0] \equiv [1^N]$ for $SU(N))$.

$$U([1^{n'}] [1^{N-h}] [1^{n'}] [1^h]; [1^{n''}]; [0]) = (-1)^\chi \left[\frac{\dim[1^{n''}]}{\dim[1^{n'}]\dim[1^h]} \right]^{\frac{1}{2}} \quad (55)$$

which follows from Eqs. (14) and (54). With the use of Eqs. (51)–(55) it is now possible to carry out the m-sums of Eq. (50) and express the final matrix element in terms of a product of a single $U(N)$ Wigner and Racah coefficient by means of Eq. (14)

$$\langle [1^n] \, m | \, T_{m_0}^{[f_0]}(h, p) | \, [1^{n'}] \, m' \rangle = \langle [1^{n'}] \, m' [f_0] \, m_0 | \, [1^n] \, m \rangle \qquad (56)$$

$$\cdot (-1)^{\frac{1}{2}h(h-1)} \left[\frac{n! \, n'! \, (N-h)! \, h!}{(n-p)! \, (n'-h)! \, N!} \right]^{\frac{1}{2}} \frac{U([1^{n''}] [1^{N-h}] [1^{n'}] [1^p]; [1^{n-p}]; [f_0])}{U([1^{n''}] [1^{N-h}] [1^{n'}] [1^h]; [1^{n'-h}]; [0])} \, ,$$

where the dependence on $U(N)$ subgroup labels sits entirely in the single $U(N)$ Wigner coefficient. Since the $SU(N)$ Racah coefficients can be calculated explicitly, the dependence on p and h is effectively factored out of the above matrix element.

Equation (56) therefore serves as a reduction formula which reduces the evaluation of an irreducible tensor $T^{[f_0]}$ of arbitrary p and h to the evaluation of the simplest such operator: the operator with $p = x$, assuming $p \leq N - h$, (otherwise $x = N - h$), where $[f_0] = [2^x 1^{N-h+p-2x}]$. (This reduction process has already been exploited by French *et al.* [13].)

The SU(N) Racah coefficients needed for Eq. (56) follow from the present investigation. In general, the SU(N) Racah coefficients needed for identical fermion spectroscopy are those involving the action of an operator (transforming according to a 2-columned representation of SU(N)) on an n'-particle state [1-columned representation of SU(N)] to make an n-particle state via an $(n - p)$-particle parent. The final n-particle state, a result of the coupling $[1^n] \times [f_0]$, thus corresponds to the 2-columned U(N) representation $[2^n 1^{N-n}]$, (with a completed first column of length N, equivalent to $[1^n]$ in SU(N)). Identical particle spectroscopy thus leads to SU(N) Racah coefficients of the general type

$$U([1^{n'}] \, [2^y 1^{N-n'+n-p-2y}] \, [1^n] \, [1^p]; [1^{n-p}]; [2^x 1^{N-n'+n-2x}]) \tag{57}$$

with $x \geq y$, $p \geq x - y$, $y \geq n - p$, required by the nature of the couplings. The evaluation of this coefficient via Eqs. (19) and (24) is slightly different for the two cases $(n - p) \geq x$, $(n - p) < x$. Fig. 2 illustrates the row labels a_λ, a_i, b_μ, b_j for the 2-columned U(N) tableau $[2^n 1^{N-n}]$ for the case $(n - p) \geq x$. The evaluation of the U-coefficient (57) involves the following identification of the row indices

$$\{a_1 \ldots a_p\} = \{n, n - 1, \ldots, n - p + 1\}$$

$$\{b_1 \ldots b_{q=n'}\} = \{n - p, n - p - 1, \ldots, y + 1, N, N - 1, \ldots, N - n' + n - p - y + 1\}$$

$$\{b_1' \ldots b_{q=n'}'\} = \{n, n - 1, \ldots x + 1, N, N - 1, \ldots, N - n' + n - x + 1\} \tag{58}$$

$$\{a_1' \ldots a_p'\} = \{x, x - 1, \ldots y + 1, N - n' + n - x, N - n' + n - x - 1, \ldots N - n'$$
$$+ n - p - y + 1\},$$

where the only axial distances needed are of the type

$$\tau_{n-i, n-j} = (i - j); \quad \tau_{N-k, N-l} = (k - l); \quad \tau_{n-i, N-k} = (N - n + i - k + 1). \tag{59}$$

Note that $a_{\lambda_t} = (n - t + 1)$, with $t = 1, \ldots, p$, so that

$$\left(1 - \frac{1}{\tau_{a_{\lambda_t} a_{\lambda_{t-1}}}}\right) = 0,$$

and the evaluation of $F_{s=p}$ of Eq. (24) proceeds most economically via Eq. (30), since the sum over t collapses to the single term with $t = 1$.

To evaluate the U-coefficient it is now also necessary to make a specific choice of phase conventions for the U(N) Wigner coupling of the type $[f] \times [1^q] \to [f']$. With the Biedenharn-Louck [1] phase conventions (which amount to generalized Condon and Shortley phase conventions, see Eq. (38) of Ref. [1]), the phase factors $\varphi([f] [1^q], [f'])$ of Eqs. (4) and (19) become

$$(-1)^{\varphi([f][1^q],[f'])} = (-1)^{\frac{1}{2}q(q+1) + \sum\limits_{i=1}^{N} (f_i' - f_i)i} = (-1)^{\frac{1}{2}q(q+1) + \sum\limits_{t=1}^{q} b_t} \tag{60}$$

where

$$[f'] = [f] + \Delta(b_1 \ldots b_q).$$

The final result for the SU(N) Racah coefficient of type (57) is

$$U([1^{n'}] \, [2^y 1^{N-n'+n-p-2y}] \, [1^n] \, [1^p]; [1^{n-p}]; [2^x 1^{N-n'+n-2x}])$$

$$= (-1)^{N(x+y+p)+p(n'+y)+x(y+1)} \tag{61}$$

$$\cdot \left[\frac{p!(n'-n+p+y)!(n-x)!(N+1-y)!(N-n)!}{(x-y)!(p+y-x)!(n-p-y)!(N+1-x)!(N-n+p)!} \right.$$

$$\left. \frac{(N-n'+n-x-p-y)!(N-n'-y)![N+n-n'-2x+1]}{(N-n'-x)!(n'-n+x)!(N+n-n'-x-y+1)!} \right]^{\frac{1}{2}} .$$

[The derivation for the case $(n-p) < x$ via Eqs. (19) and (24) proceeds somewhat differently, involving symmetric functions F_s with $s = (n-x)$ rather than $s = p$; but the final result (61) is independent of the condition $(n-p) \gtrless x$.] It may also be useful to note that a U-coefficient for a recoupling transformation for which all four Wigner couplings are multiplicity-free is invariant under conjugation of all SU(N) irreducible representations, so that

$$U([1^{N-n'}] \, [2^{N-y-a} 1^a] \, [1^{N-n}] \, [1^{N-p}]; [1^{N-n+p}]; [2^{N-x-b} 1^b])$$

$$= U([1^{n'}] \, [2^y 1^a] \, [1^n] \, [1^p]; [1^{n-p}]; [2^x 1^b]) \tag{62}$$

which may be particularly useful for the case $n' < p$.

The special case $y = 0$, (with $n' = n - p + h$), furnishes the U-coefficients needed for Eq. (56), and leads to the explicit result

$$\langle [1^n] \, m | \, T_{m_0}^{[f_0]}(h, p) | [1^n] \, m' \rangle = (-1)^{(N+1)x+(N+n-1)(p-h)+\frac{1}{2}h(h-1)}$$

$$\cdot \langle [1^n] \, m' [f_0] \, m_0 | [1^n] \, m \rangle \tag{63}$$

$$\left[\frac{p!\,h!\,(N+1)!\,(N-h-x)!\,n!\,(n-x)!\,(N-n)!\,[N+p-h-2x+1]}{x!(p-x)!(h-p+x)!(N+1-x)!(N+1+p-h-x)![(n-p)!]^2(N-n-h-x+p)!} \right]^{\frac{1}{2}}$$

$$\equiv \langle [1^n] \, m' [f_0] \, m_0 | [1^n] \, m \rangle \, K(n, N, h, p, x) .$$

With this result the average of the product of two operators over the states of an irreducible representation of SU(N) can be carried out. Since only the scalar [SU(N)-invariant piece] of a product of operators $O_1 O_2^+$ can make a contribution to the average, $\langle O_1 O_2 \rangle^n$, this average can be different from zero only if O_1 and O_2 have pieces of the same SU(N) irreducible tensor character. If O_1 and O_2 in Eq. (48) are operators of type $T_{m_0}^{[f_0]}(h, p)$ and $T_{m_0}^{[f_0']}(h', p')$, with $[f_0'] = [f_0] = [2^x 1^{N-h+p-2x}]$; $m_0 = m_0'$; $h' - p' = h - p$,

$$\langle O_1 O_2 \rangle^n = \frac{1}{\dim [1^n]} \sum_{m, m'} \langle [1^n] \, m' [f_0] \, m_0 | [1^n] \, m \rangle^2$$

$$\cdot K(n, N, h, p, x) \, K(n, N, h', p', x) , \tag{63'}$$

where the sums over SU(N) subgroup labels can be carried out [using symmetry properties such as (17) and (54)], to give

$$\langle O_1 O_2 \rangle^n = \frac{1}{\dim [f_0]} \, K(n, N, h, p, x) \, K(n, N, h', p', x) . \tag{64}$$

With

$$\dim[f_0] = \dim[2^x 1^{N-h+p-2x}] = \frac{(N-h+p-2x+1)}{(N+1)} \frac{(N+1)!}{x!(N+1-x)!}$$
$$\cdot \frac{(N+1)!}{(N+1-h+p-x)!(h-p+x)!} \tag{65}$$

this leads to

$$\langle T^{[f_0]}(h,p) \, T^{[f_0]}(h',p') \rangle^n = (-1)^{\frac{1}{2}h(h-1)+\frac{1}{2}h'(h'-1)} \tag{66}$$
$$\cdot \frac{n!(n-x)!(N-n)!}{(n-p)!(n-p')!(N-n-h+p-x)!} \left[\frac{p!\,p'!\,h!\,h'!\,(N-h-x)!\,(N-h'-x)!}{(p-x)!\,(p'-x)!\,N!\,N!} \right]^{\frac{1}{2}} \cdot$$

This is a result which (in somewhat different form) has already been derived by Chang, French, and Thio [13] with*out* the detailed use of $U(N)$ Racah algebra. (Cf. Eq. (3.11) of Ref. [13]. To establish the connection between the notation of Ref. [13] and the more conventional $U(N)$ notation of this investigation, note that the symbols v, μ, and k of Ref. [13] are related to x, h, and p by $x = v + \mu$; $h = k + \mu$; $p = k - \mu$.)

Although results such as (66) can be derived by the simpler methods of Chang, French, and Thio [13], the detailed use of $U(N)$ Racah algebra will make it possible to generalize such results to matrix elements of more complicated operators or to more complicated operator averages.

Consider, for example, the average

$$\langle O_1 O_2 O_4^+ O_3^+ \rangle^n = \frac{1}{\dim[1^n]} \sum_{mm'} \langle [1^n] \, m | O_1 O_2 | [1^{n''}] \, m' \rangle \langle [1^n] \, m | O_3 O_4 | [1^{n''}] \, m' \rangle \tag{67}$$

over the complete set of states of $[1^n]$. If $O_1, O_2, \ldots$ are k-body operators of the type (47), with $k_1 = \frac{1}{2}(h_1 + p_1)$, $k_2 = \frac{1}{2}(h_2 + p_2)$, ..., it will be convenient to couple the operators in the order $([1^{p_1}] \times ([1^{N-h_1}] \times ([1^{p_2}] \times [1^{N-h_2}])))$, from right to left, and define the basic tensor operators

$$T\{[1^{p_1}] \times \{[1^{N-h_1}] \times (T^{[f_2]}(h_2, p_2))\} \, [f']\}_{m_0}^{[f_0]}$$
$$= \sum_{m_{p_1} \, m_{h_1} \, m_2 m'} \langle [f_2] \, m_2 [1^{N-h_1}] \, m_{h_1} | [f'] \, m' \rangle \langle [f'] \, m' [1^{p_1}] \, m_{p_1} | [f_0] \, m_0 \rangle \tag{68}$$
$$\cdot (a^+ \ldots a^+)_{m_{p_1}}^{[1^{p_1}]} (a \ldots a)_{m_{h_1}}^{[1^{N-h_1}]} T_{m_2}^{[f_2]}(h_2, p_2),$$

where $T_{m_2}^{[f_2]}(h_2, p_2)$ is an irreducible unit tensor operator of the type defined by Eq. (48). Also, $[f_2]$, $[f']$, and $[f_0]$ must be restricted to 2-columned representations in order to have non-zero matrix elements between n-particle states which belong to $SU(N)$ representations of type $[1^n]$; that is

$$[f_2] = [2^{x_2} 1^{N-h_2+p_2-2x_2}]; \qquad [f'] = [2^{x'} 1^{N-h_2+p_2-h_1-2x'}]$$
$$[f_0] = [2^{x_0} 1^{N-h_2+p_2-h_1+p_1-2x_0}]$$

with $x' \leq x_2$, $x_0 \geq x'$. Using the techniques outlined above, the matrix elements of such an operator can be expressed in terms of $SU(N)$ Racah coefficients and a

single $U(N)$ Wigner coefficient [which carries the dependence on the $U(N)$ subgroup labels]

$$\langle [1^n]\, m\,|\, T\{[1^{p_1}] \times \{[1^{N-h_1}] \times (T^{[f_2]}(h_2, p_2))\}\, [f']\}_{m_0}^{[f_0]}\,|\, [1^{n'}]\, m'\rangle$$

$$= \langle [1^{n'}]\, m'\, [f_0]\, m_0\,|\, [1^n]\, m\rangle\, (-1)^{\frac{1}{2}h_1(h_1-1)+\frac{1}{2}h_2(h_2-1)}$$

$$\cdot \left[\frac{n''!\, n'!\, n!\, n''!\, (N-h_2)!\, h_2!\, (N-h_1)!\, h_1!}{(n''-p_2)!\, (n'-h_2)!\, (n-p_1)!\, (n''-h_1)!\, N!\, N!}\right]^{\frac{1}{2}} \tag{69}$$

$$\cdot \frac{U([1^{n''}]\, [1^{N-h_2}]\, [1^{n''}]\, [1^{p_2}];\, [1^{n''-p_2}];\, [f_2])\, U([1^{n'}]\, [f_2]\, [1^{n-p_1}]\, [1^{N-h_1}];\, [1^{n''}];\, [f'])}{U([1^{n''}]\, [1^{N-h_2}]\, [1^{n'}]\, [1^{h_2}];\, [1^{n'-h_2}];\, [0])\, U([1^{n''}]\, [1^{N-h_1}]\, [1^{n''}]\, [1^{h_1}];\, [1^{n''-h_1}];\, [0])}$$

$$\cdot U([1^{n'}]\, [f']\, [1^n]\, [1^{p_1}];\, [1^{n-p_1}];\, [f_0])$$

with $n'' = n - p_1 + h_1$, $n' = n'' - p_2 + h_2$, where the U-coefficients can all be expressed as simple functions of n, N, h, p, x by means of Eqs. (61) and (62). Operator averages of the type $\langle O_1 O_2 O_4^+ O_3^+ \rangle^n$ can therefore be carried out, provided operator products $O_i O_j$ can be expanded in terms of appropriately coupled $SU(N)$ irreducible tensor operators.

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