A Simple, Combinatorial Algorithm for Solving SDD Systems in Nearly-Linear Time

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Joint work with Jonathan Kelner, Aaron Sidford and Zeyuan Allen Zhu

MIT MATHEMATICS

CSAIL

Problem Definition: SDD Systems

PROBLEM INPUT: $A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n$.

$$Ax = b$$

SQUARE SYSTEM OF LINEAR EQUATIONS

GOAL: solve for x

Restriction: A is **S**ymmetric **D**iagonally-**D**ominant

Problem Definition: SDD Systems

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$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

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GOAL: solve for x

Restriction: A is **S**ymmetric **D**iagonally-**D**ominant

- Symmetry: $A^{\mathrm{T}} = A$
- Diagonal Dominance:
  for all rows (and columns) i, diagonal entry dominates

$$a_{ii} \geq \sum_{j \neq i} |a_{ij}|$$

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WHY SDD ? Practically important subcase of positive semidefinite matrices

A **IS SDD** $\longrightarrow$ $A \succeq 0$

Easy-to-identify null space

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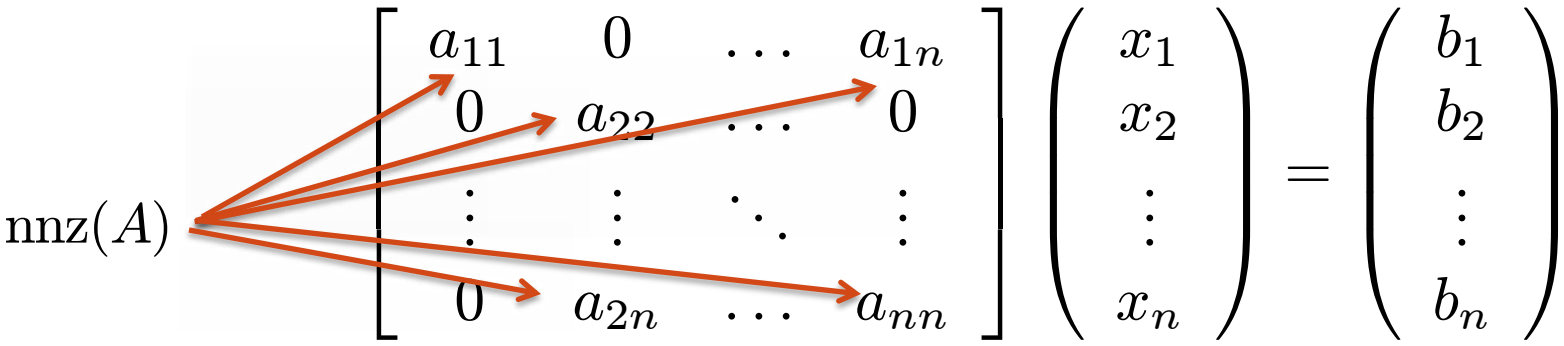


GOAL: solve for x

- approximately: $\|x - x^*\|_A \cdot \quad \epsilon \|x^*\|_A$

- in nearly-linear time in the sparsity $\mathrm{nnz}(A)$ and $\log(1/\epsilon)$, i.e.

$$\text{RunTime} = O\left(\mathrm{nnz}(A) \cdot \mathrm{polylog}(n) \cdot \log(1/\epsilon)\right)$$

Reduction to Laplacian Systems

SDD SYSTEM:

$$Ax = b$$

[Gremban'96]

LAPLACIAN SYSTEM:

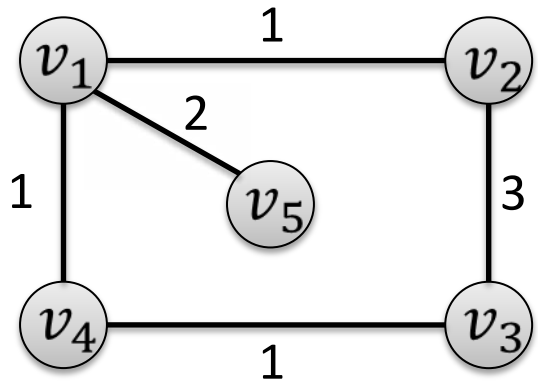
$$Lv = \chi$$

Reduction preserves approximation and sparsity

Graph Laplacian

$G = (V, E, w)$ weighted undirected graph with n vertices and m edges

GRAPH G



GRAPH LAPLACIAN $L(G)$

$$L(G) = \begin{bmatrix} 4 & -1 & 0 & -1 & -2 \\ -1 & 4 & -3 & 0 & 0 \\ 0 & -3 & 4 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

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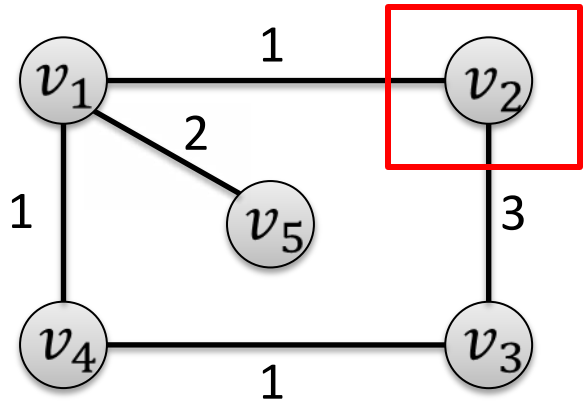

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Degree of vertex 2

Graph Laplacian

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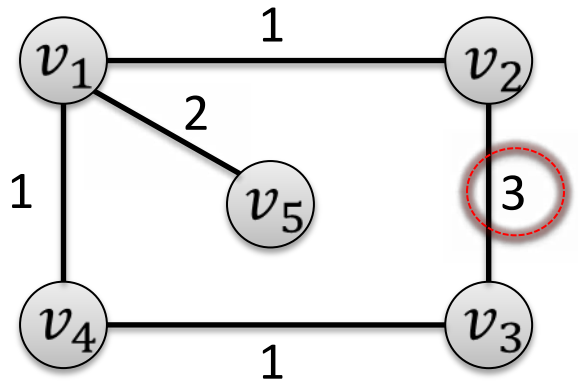
GRAPH G



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$$L(G) = \begin{bmatrix} 4 & -1 & 0 & -1 & -2 \\ -1 & 4 & -3 & & \\ 0 & -3 & 4 & & \\ -1 & 0 & -1 & & \\ -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

- Weight of edge {2,3}

Laplacian as Sum of Edges

$$L(G) = \sum_{e=\{i,j\}\in E} w_e \begin{array}{c} \\ \\ \\ i \\ \\ j \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} & & i & & j & & \\ \end{array} \\ \left(\begin{array}{ccccccc} 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & \dots & 1 & \dots & -1 & \dots & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & \dots & -1 & \dots & 1 & \dots & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{array}\right) \end{array}$$

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$$\begin{array}{c} \\ \\ \\ i \\ \\ \\ j \\ \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccccc} \quad & \quad & i & \quad & j & \quad & \quad \end{array} \\
\left(\begin{array}{ccccccc}
0 & \dots & 0 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & \dots & 1 & \dots & -1 & \dots & 0 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & \dots & -1 & \dots & 1 & \dots & 0 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & \dots & 0 & \dots & 0
\end{array}\right)
\end{array}$$

Edge Matrix L_e

Laplacian as Sum of Edges

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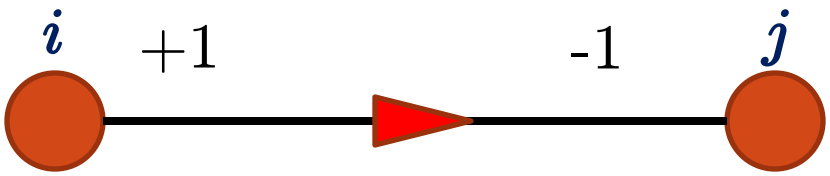
\end{array}$$

Edge Matrix L_e has rank 1

Laplacian as Sum of Edges

$$L(G) = \sum_{e=\{i,j\}\in E} w_e \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ i \\ \\ j \\ \\ \end{matrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \end{matrix} \begin{matrix} \\ \\ i \\ \\ \\ \\ \end{matrix} \begin{matrix} \\ \\ \\ \\ j \\ \\ \end{matrix} \begin{pmatrix} 0 & \cdots & 1 & \cdots & -1 & \cdots & 0 \end{pmatrix}^{\mathrm{T}}$$

$$\chi_e \qquad\qquad \chi_e^{\mathrm{T}}$$



ARBITRARY ORIENTATION OF EDGES: WILL BE USEFUL WHEN DISCUSSING FLOWS

More Graph Matrices: Incidence Matrix

$G = (V, E, w)$ weighted undirected graph with n vertices and m edges

$$B^{\mathrm{T}} = \begin{pmatrix} & \\ \text{——} & \chi_{e_1} & \text{——} \\ \text{——} & \chi_{e_2} & \text{——} \\ & \vdots & \\ \text{——} & \chi_{e_i} & \text{——} \\ & \vdots & \\ \text{——} & \chi_{e_m} & \text{——} \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \end{matrix} \begin{matrix} \\ \\ \\ \\ \\ \\ m \end{matrix}$$

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$$L = \sum_{e=\{i,j\} \in E} w_e \chi_e \chi_e^{\mathrm{T}} = B^T W B$$

W ↓ $\mathrm{diag}(w)$

More Graph Matrices: Incidence Matrix

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$$L = \sum_{e=\{} w_e \chi_e \chi_e^{\mathrm{T}} = B^T W B$$

$W^{1/2} B$ is square root of L

$\mathrm{diag}(w)$

Action of Incidence Matrix

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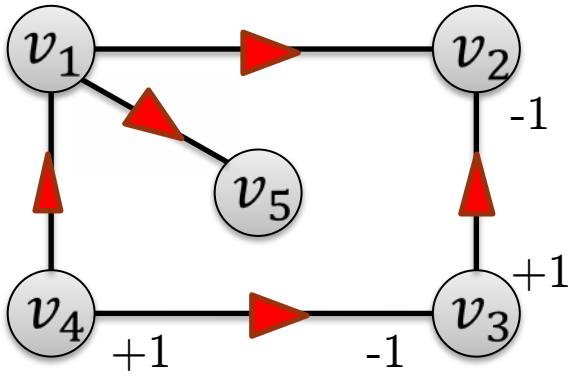


Action of B^T on a vector $f \in \mathrm{R}^m$:

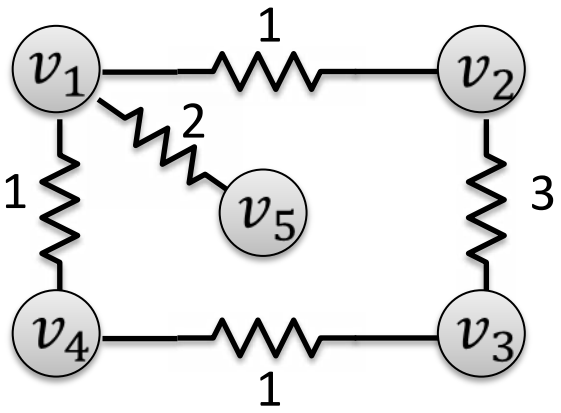
$$(B^T f)_i = \text{flow out of i} - \text{flow into of i} = \text{net flow into graph at i}$$

Action of B on a vector $v \in \mathrm{R}^n$:

$$(Bv)_{e=(i,j)} = v_i - v_j = \text{change in } v \text{ along } (i,j)$$

Graphs as Electrical Circuits

Edge conductances w_e

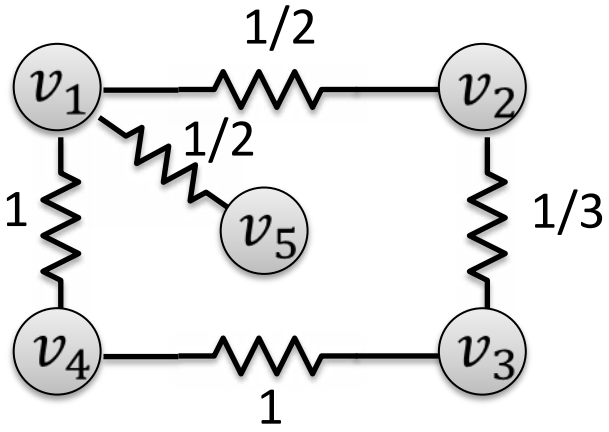


Graphs as Electrical Circuits

Edge conductances w_e

Edge resistances r_e

$$r_e = 1/w_e$$



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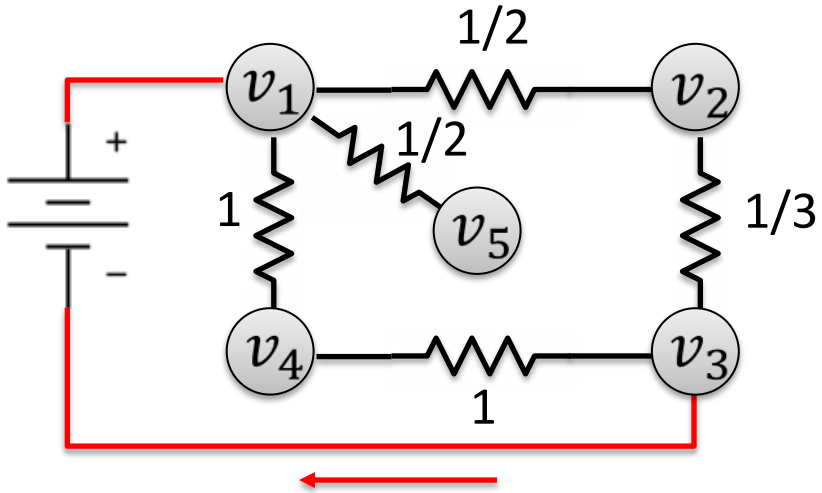

one unit of electrical flow
Into vertex 1 and out of 3

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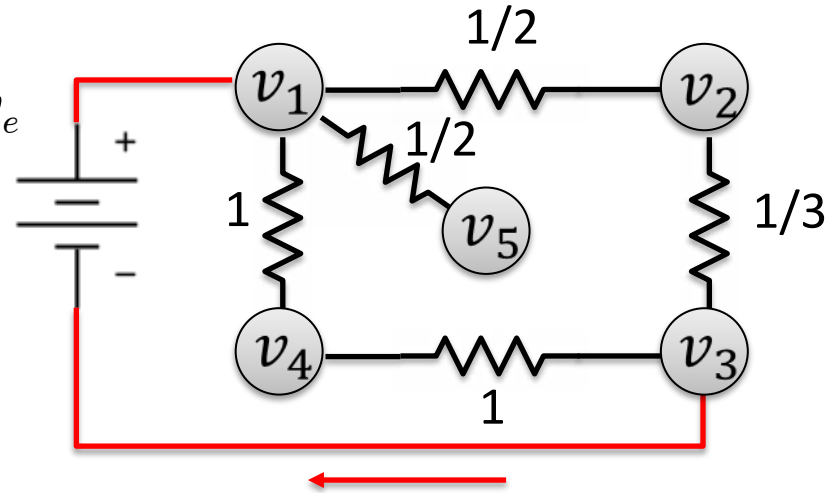

Q: What is resulting electrical flow on the graph?

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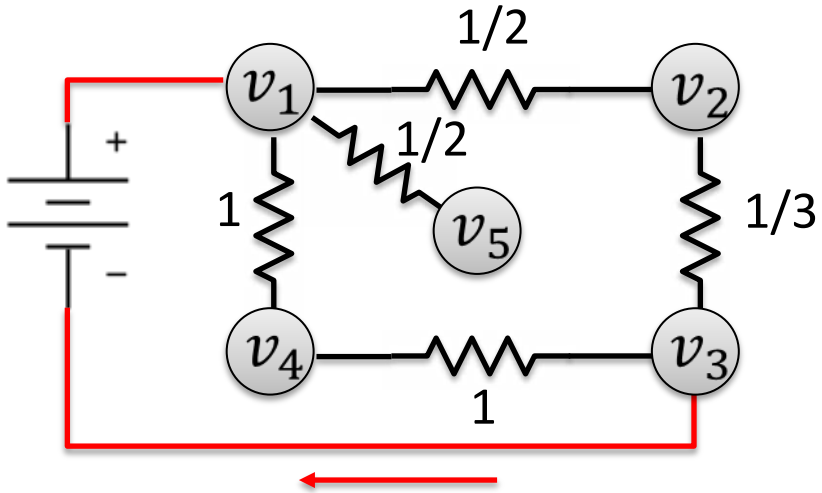

one unit of electrical flow
Into vertex 1 and out of 3

1. Ohm's Law: For every edge $e = (i, j) \in E$:

$$f_{(i,j)} = \frac{v_i - v_j}{r_{ij}}$$

2. Kirchoff's Conservation Law: For every vertex $i \in V$:

$$\text{flow out of } i - \text{flow into } i = \text{net flow into network at } i$$

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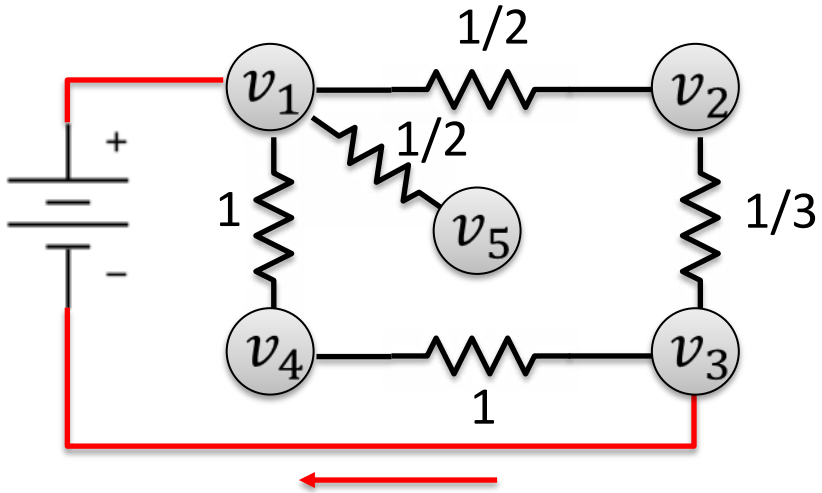

1. Ohm's Law: For every edge $e = (i, j) \in E$:

$$f_{(i,j)} = \frac{v_i - v_j}{r_{ij}} \longrightarrow f = R^{-1}Bv = WBv$$

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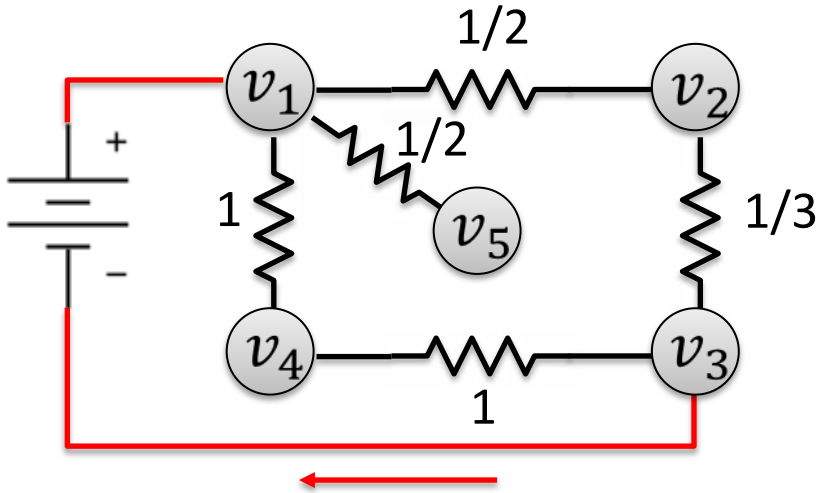

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$$B^T f = e_s - e_t$$

Example above: Current source s is vertex 1. Current sink t is vertex 3

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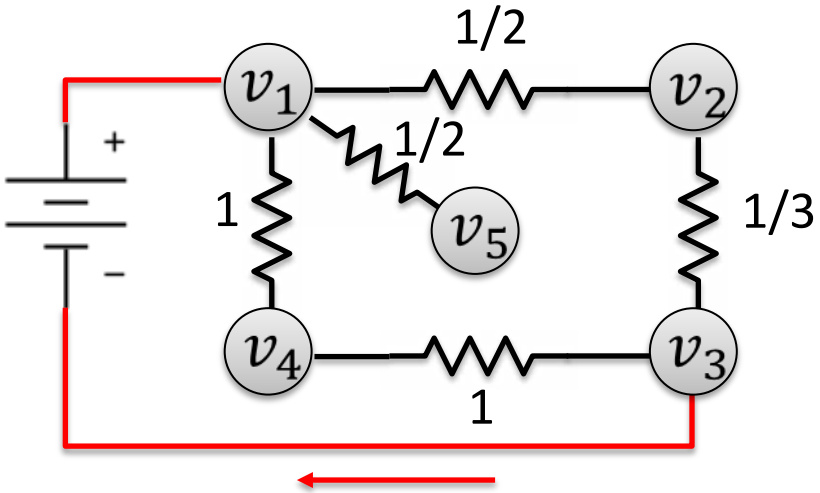


one unit of electrical flow
Into vertex 1 and out of 3

1. Ohm's Law:

$$f = R^{-1}Bv = WBv$$

2. Kirchoff's Conservation Law:

$$B^T f = \chi$$

General net flow vector χ allowed:

$$\chi^{\mathrm{T}}\vec{1} = 0$$

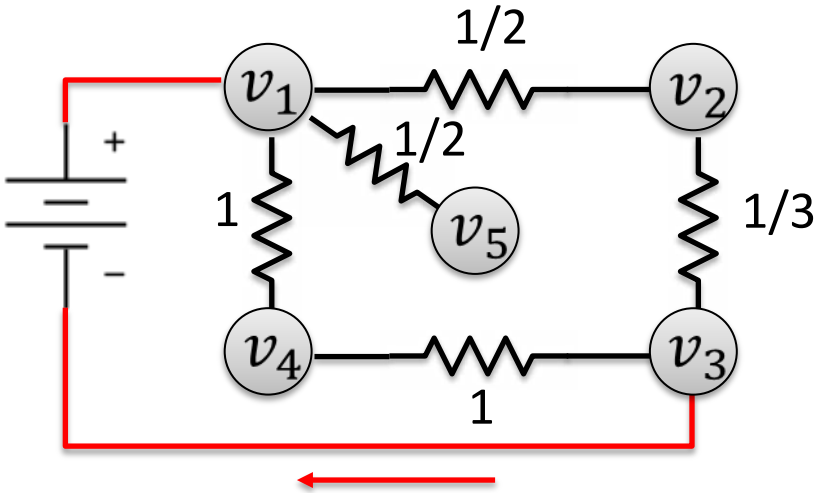
Example above: Current source s is vertex 1. Current sink t is vertex 3

Laplacian Systems as Electrical Problems

Edge conductances w_e

Edge resistances r_e

$r_e = 1/w_e$



General net flow vector χ :

$$\chi^{\mathrm{T}}\vec{1} = 0$$

Q: What is resulting electrical flow on the graph?

$$\left.\begin{array}{c} f = R^{-1}Bv = WBv, \\ B^{\mathrm{T}}f = \chi \end{array}\right\} \rightarrow B^{T}WBv = Lv = \chi$$

Laplacian Systems as Electrical Problems

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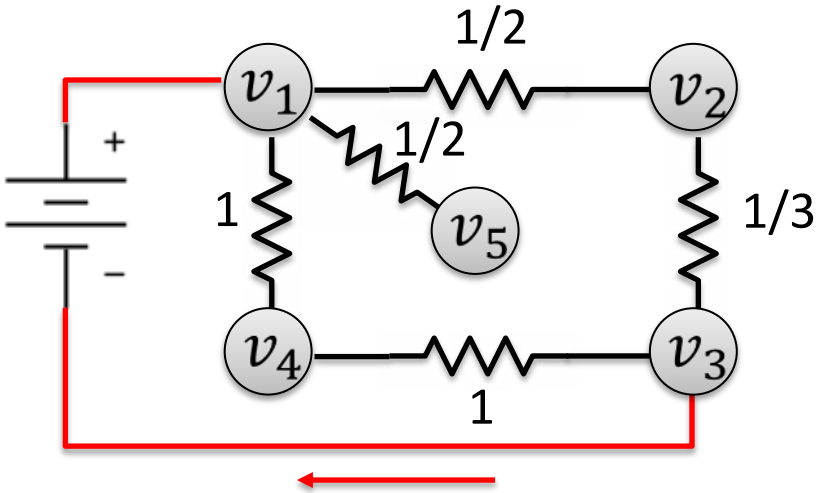

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$$Lv = \chi$$

Given input currents χ, finding voltage is equivalent to solving Laplacian system of linear equations

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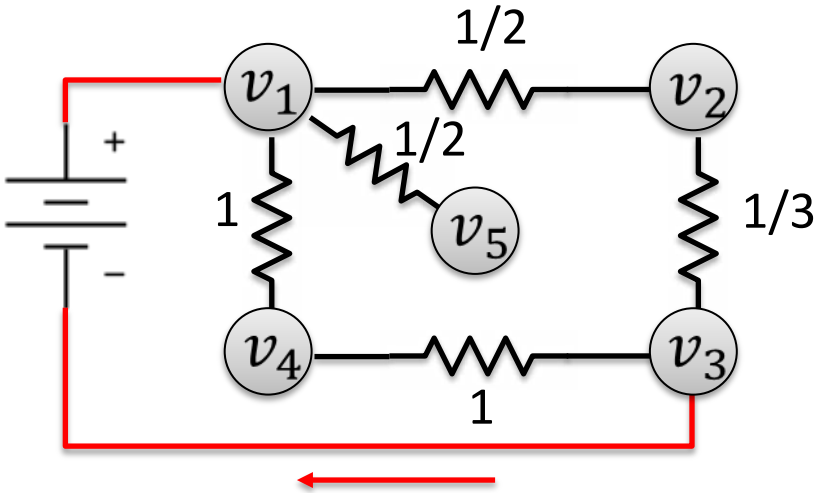

one unit of electrical flow
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General net flow vector χ :

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Pseudo-inverse

$$Lv = \chi \longrightarrow v = L^{+}\chi$$

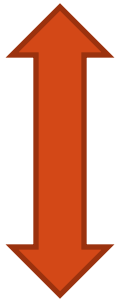
Given input currents χ, finding voltage is equivalent to solving Laplacian system of linear equations

Energy Intepretation

$$Lv = \chi$$

Optimality condition

$$\min_{x \perp \vec{1}} \quad \tfrac{1}{2} \cdot x^{\mathrm{T}} L x - x^{\mathrm{T}} \chi$$



Equivalent up to scaling

$$\min_{x \perp \vec{1}} \quad \tfrac{1}{2} \cdot x^{\mathrm{T}} L x$$

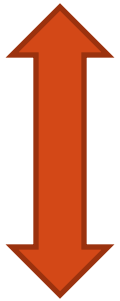
$$\text{s.t.} \quad x^{\mathrm{T}} \chi = 1$$

Energy Intepretation

$$Lv = \chi$$

Optimality condition

$$\min_{x \perp \vec{1}} \quad \frac{1}{2} \cdot x^{\mathrm{T}} L x - x^{\mathrm{T}} \chi$$



Equivalent up to scaling

$$\min_{x \perp \vec{1}} \quad \frac{1}{2} \cdot x^{\mathrm{T}} L x = \sum_{(i,j) \in E} \frac{(x_i - x_j)^2}{r_{ij}}$$

$$\text{s.t.} \qquad x^{\mathrm{T}} \chi = 1$$

Minimize energy for a fixed voltage gap

Why it matters

- Direct Applications
  - Modeling electrical networks
  - Simulating random walks
  - PageRank

$$L^{+}\chi = \sum_{t=0}^{\infty} \left(\frac{I+W}{2}\right)^{t} \chi$$

Aggregate behavior of lazy random walk started at χ

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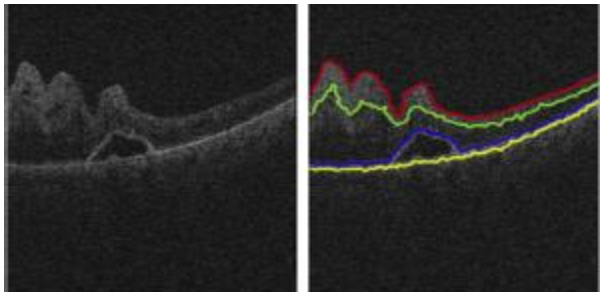
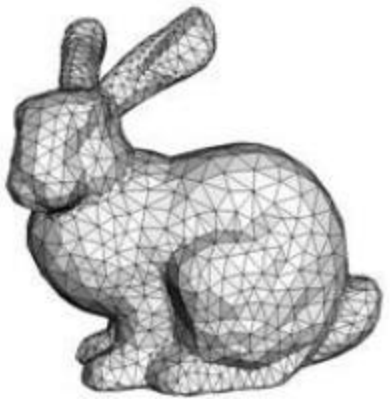


Numerical Applications

- Finite element [BHV08]
- Matrix exponential [OSV12]
- Largest eigenvalue [ST12]
- Image Smoothing



Why it matters: Faster Graph Algorithms

"The Laplacian Paradigm"

- Maximum flow [CKM+11,LRS13,KOLS12]
- Multicommodity flow [KMP12, KOLS12]
- Random spanning trees [KM09]
- Graph sparsification [SS11]
- Lossy flow, min-cost flow [DS08]
- Balanced partitioning [OSV12]
- Oblivious routing [KOLS12]
- … and more

Highlights of Previous Work

Direct solvers for general matrices

- Gaussian elimination $O(n^3)$
- Strassen alg $O(n^{2.807})$
- …
- Williams alg $O(n^{2.373})$

Iterative methods for PSD matrices

- Conjugate Gradient
- Chebyshev Method
- $O(nm)$

On various subclasses

- Multigrid
- Many many others…

For SDD/ Laplacians

- $O(m^{1.75})$ started by [Vaidya'90]
- …
- Spielman-Teng O(m polylog n)
- Sped up by [KMP'12] to $\tilde{O}(m \log n)$

Our Result

Solve $Lx = b$ in time $\tilde{O}\left(m \log^2 n \log \frac{1}{\epsilon}\right)$

- Very different approach
- Simple and intuitive algorithm
- Proof fits on a single blackboard
- Easily shown to be numerically stable

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Solve $Lx = b$ in time $\tilde{O}\left(m \log^2 n \log \frac{1}{\epsilon}\right)$

- Very different approach
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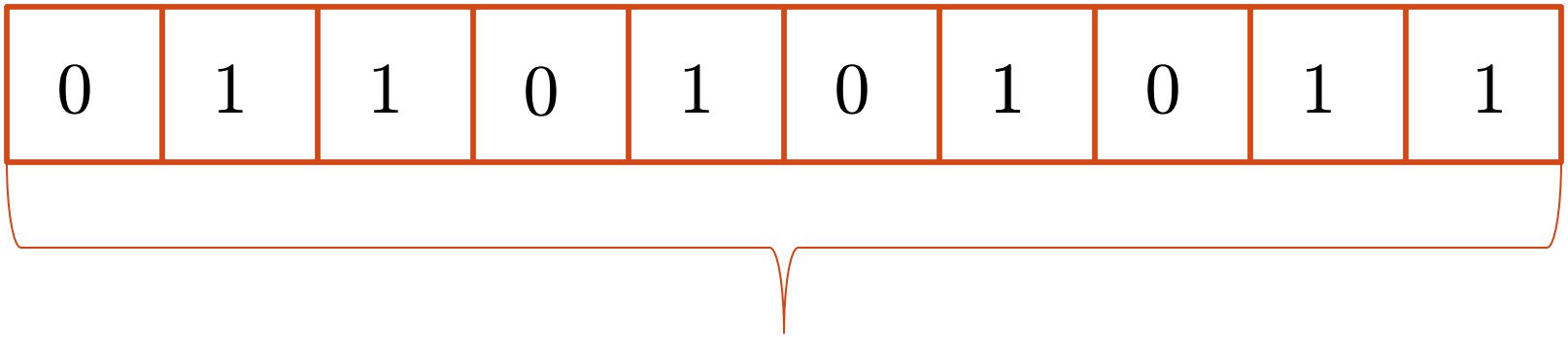
- Proof fits on a single blackboard
- Easily shown to be numerically stable

Previous methods are not as stable, need $\log^c n$-bit precision
(although $\tilde{O}(m \log n)$ is achieved, an extra $\log^c n$ factor is hidden)
In unit-cost RAM model, our algorithm is faster.

Which computational model?

Q: How do we account for running time of arithmetic operations?

A: Constant time for operations on word-size sequence of bits.



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Size of Word: Polynomial in size of input

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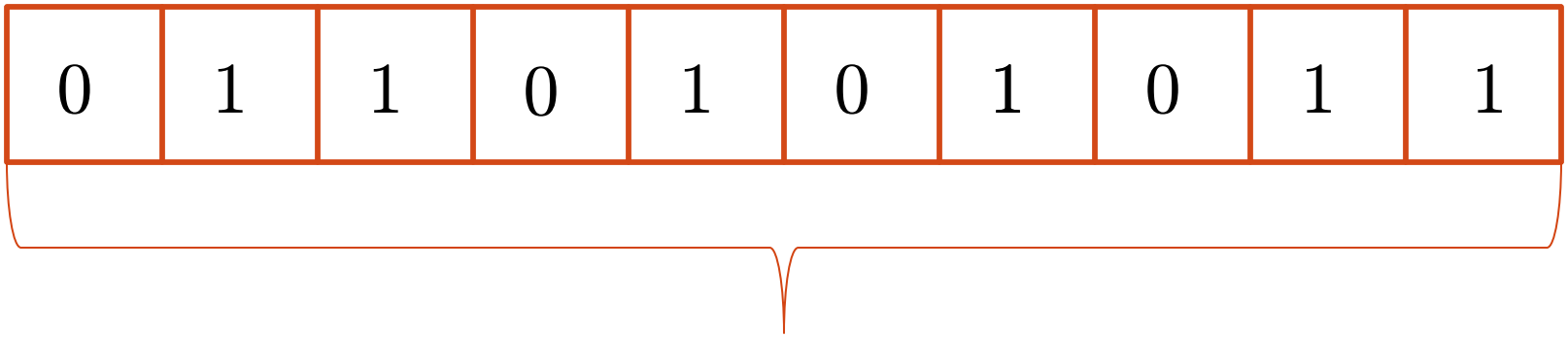

Previous works: $O(\log^c n)$

Size of Word: Polynomial in size of input

Our work: $O(\log n)$

UNIT-COST RAM MODEL:
MORE REALISTIC MODEL

Size of Word: Linear in size of input

The Philosophy of This Talk

PROBLEM REPRESENTATION

ITERATIVE METHOD

The Philosophy of This Talk

PROBLEM REPRESENTATION

Not all representations created equally:
e.g. eigenvalue decompostion

Good representation often requires **combinatorial insight**

ITERATIVE METHOD

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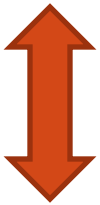
ITERATIVE METHOD

Leverage large body of techniques in **continuous optimization**

- Regularization
- Gradient Descent
- Accelerated Gradient Descent
- Randomized Coordinate Descent

The Philosophy of This Talk

PROBLEM REPRESENTATION



ITERATIVE METHOD

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Leverage large body of techniques in **continuous optimization**

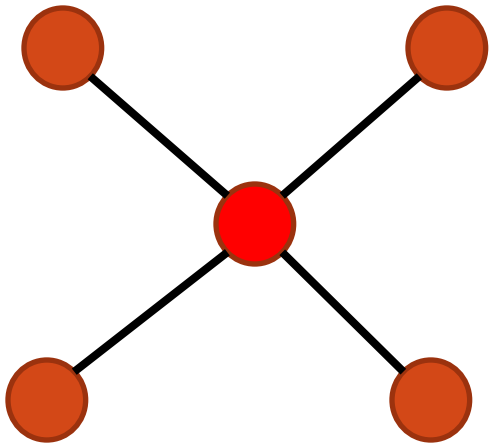
- Regularization
- Gradient Descent
- Accelerated Gradient Descent
- Randomized Coordinate Descent

Efficiency and simplicity of algorithm relies on **combining these two tools** in the right way

Techniques and Challenges in the Solution of Laplacian Systems

Changing Representation: Gaussian Elimination

Simple Graphic Interpretation:



$$L = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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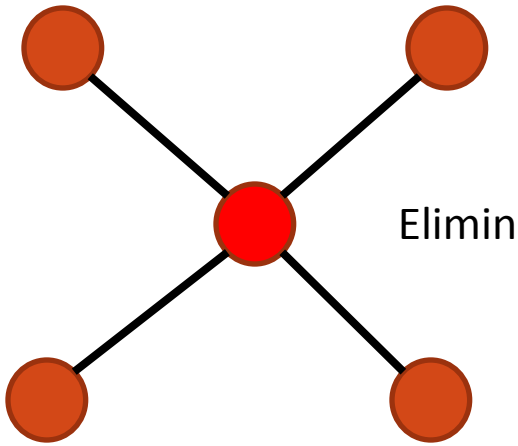

Eliminate (pivot) this node

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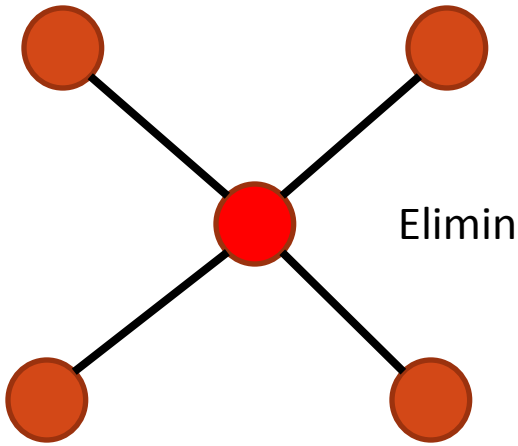

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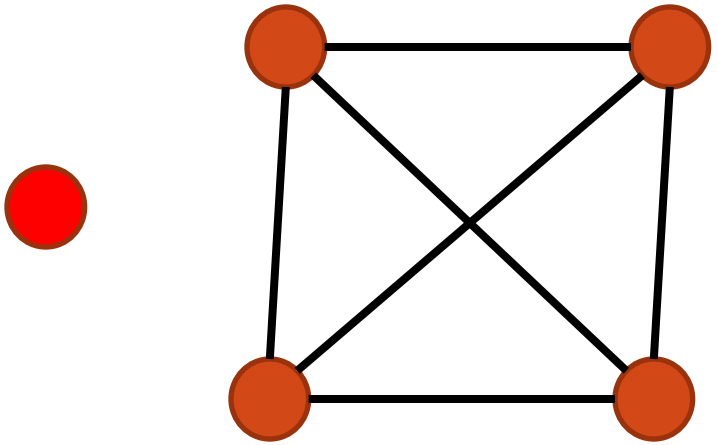
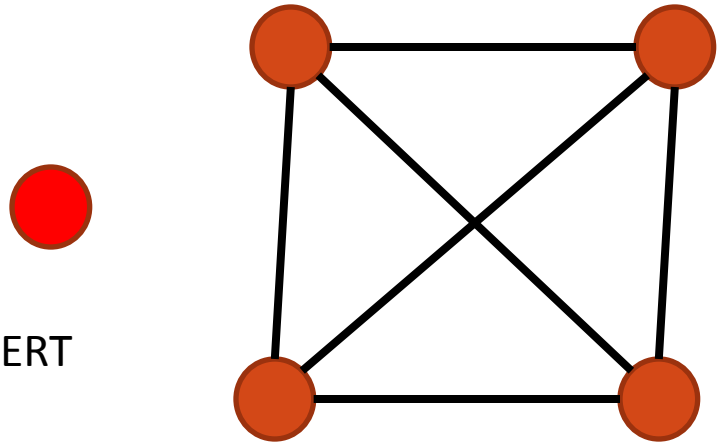


$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & -1 & \frac{3}{4} \end{pmatrix}$$

→ Laplacian on n-1 vertices

Cholesky Decomposition

Simple Graphic Interpretation:



EASY TO INVERT

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Advantages:

- Gives exact algorithm
- Computes inverse matrix (can then be used on any b)
- Can choose elimination order

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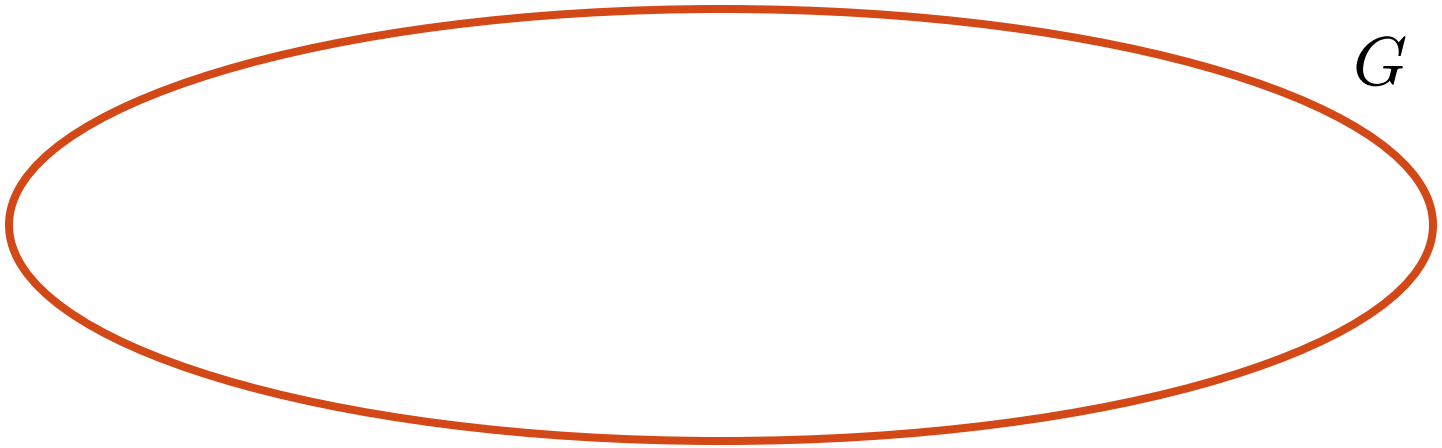
Disadvantages:

- Intermediate Laplacians can be very dense

SPARSE **DENSE**

Gaussian Elimination and Nested Dissection

Combinatorial arguments can help preserve sparsity by selecting good order:



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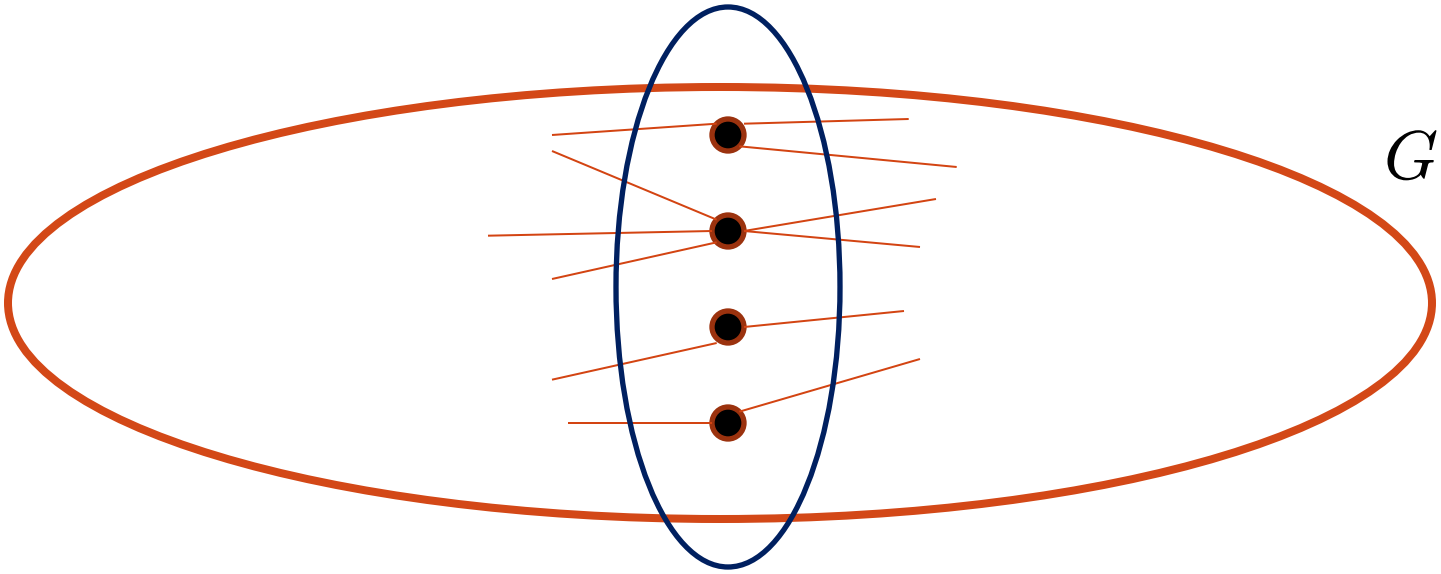

Small Vertex Separator

Gaussian Elimination and Nested Dissection

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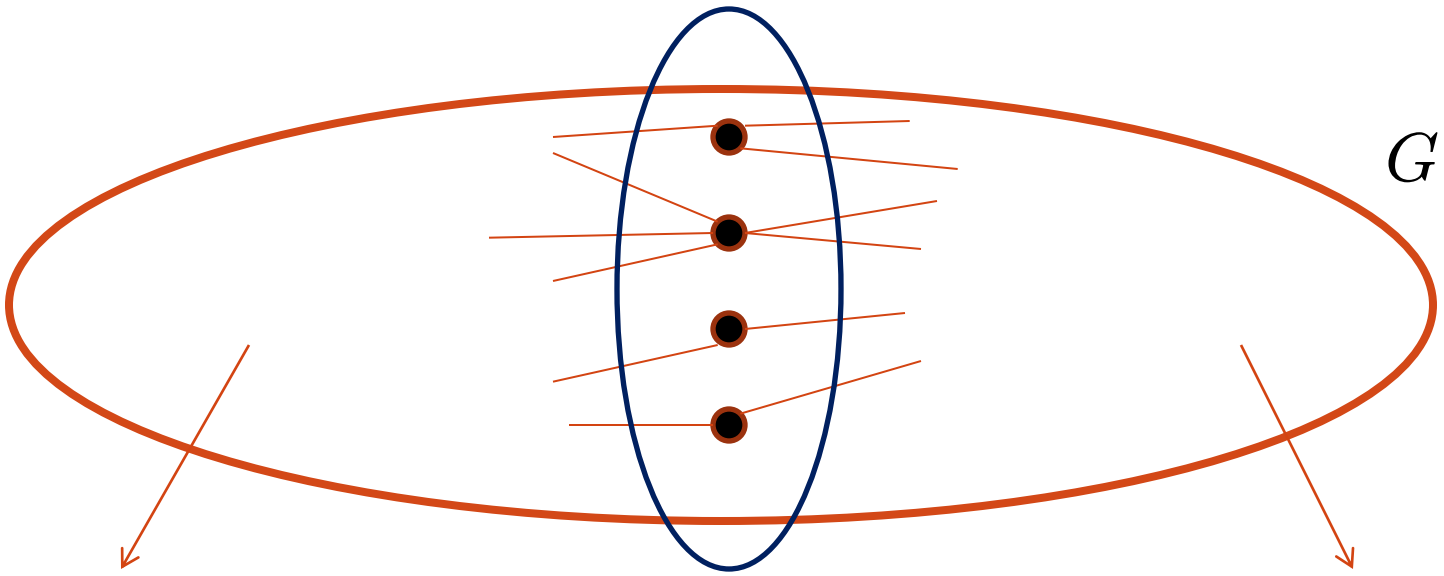

Eliminate recursively | Small Vertex Separator | Eliminate recursively

Gaussian Elimination and Nested Dissection

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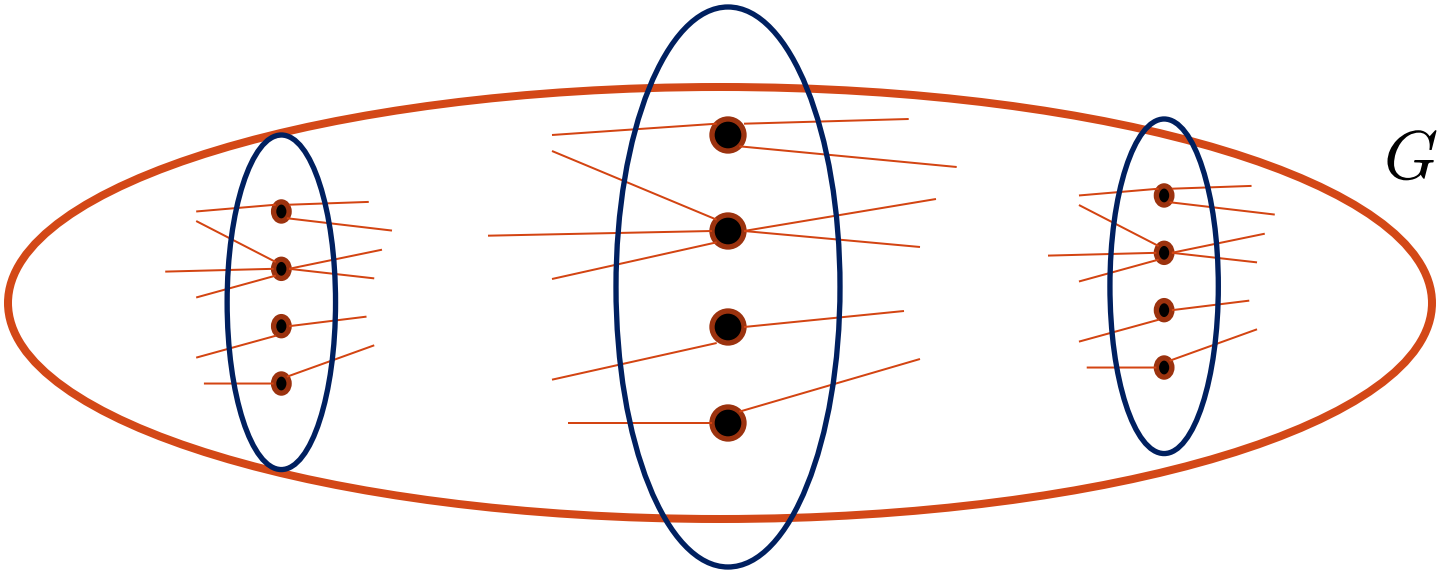

Size of existing separators bounds sparsity
Works well if small separators always exist (and are easy to find)

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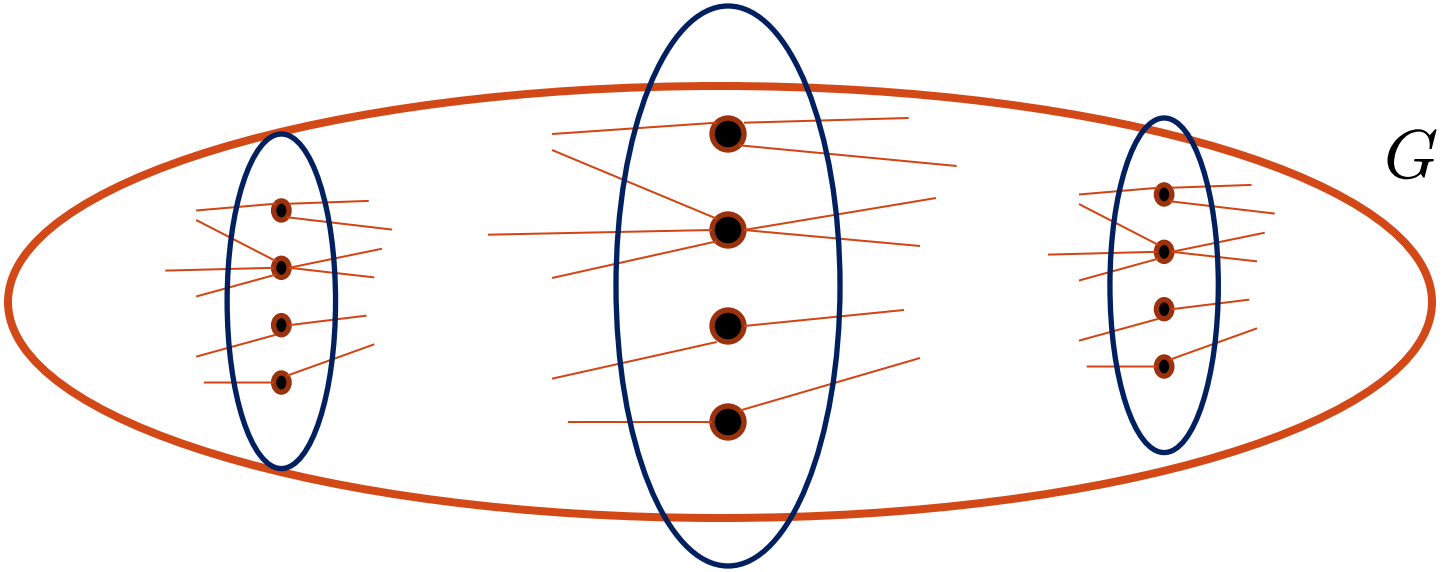
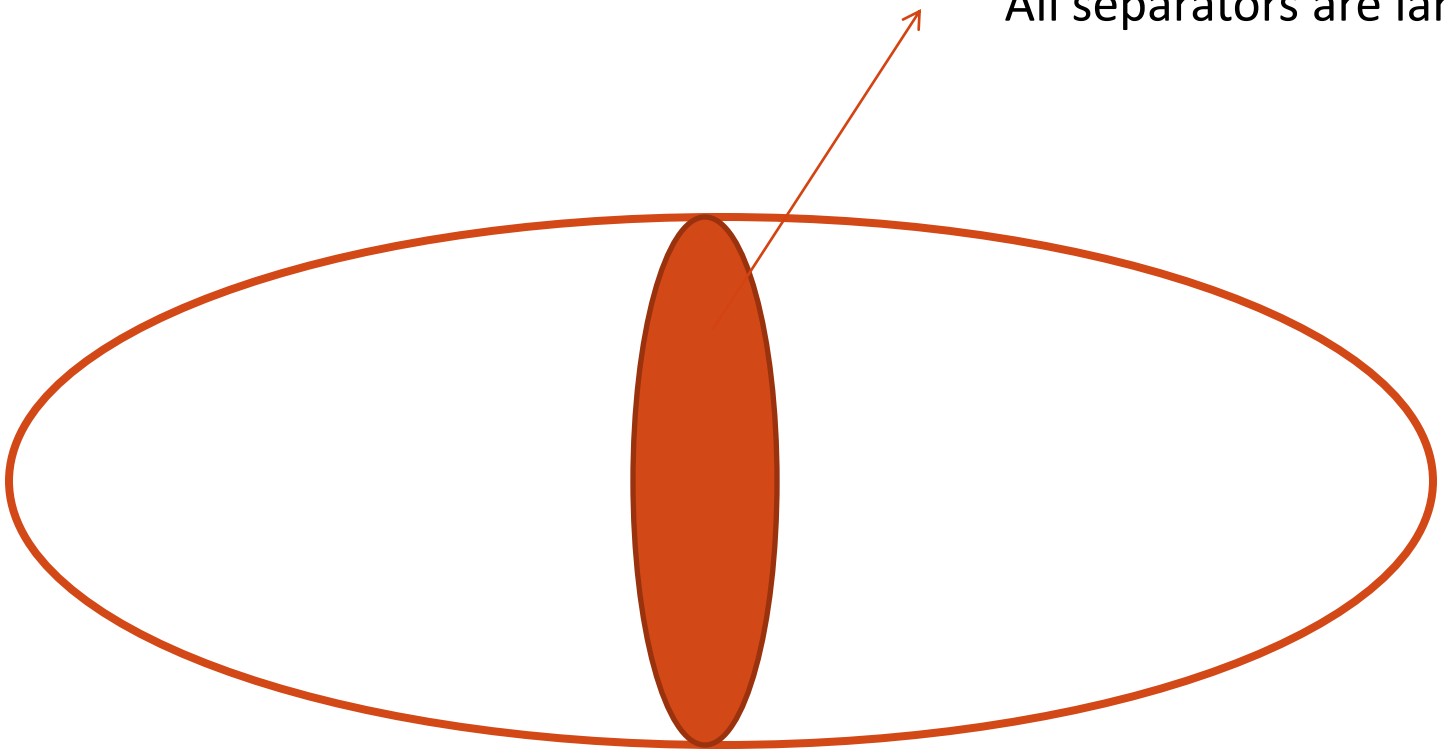


Size of existing separators bounds sparsity
Works well if small separators always exist (and are easy to find)
E.G.: Planar Graphs

Gaussian Elimination: the bad case

G sparse expander

All separators are large



Any elimination order requires producing a dense graph.

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- Gives exact algorithm
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- Can choose elimination order to minimize running time

Disadvantages:

- Intermediate Laplacians can be very dense
- **Very slow** in worst case:

$$O(n^3) \longrightarrow O(n^w)$$

FAR FROM NEARLY LINEAR!

Gaussian Elimination: the good case

Easily linear time for some graphs:

PATHS

Gaussian Elimination: the good case

Easily linear time for some graphs:

Eliminate in time $\mathrm{O}(1)$

PATHS

Gaussian Elimination: the good case

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PATHS $O(n)$ time

Gaussian Elimination: the good case

Easily linear time for some graphs:

PATHS $O(n)$ time

This works more in general for trees
by **recursively eliminating a leaf.**

Iterative Methods: Gradient Descent

Consider energy interpretation:

$$\min_{x \perp \vec{1}} \quad \frac{1}{2} \cdot x^{\mathrm{T}} L x - x^{\mathrm{T}} \chi$$

This is a **convex optimization problem** on which we can apply gradient descent techniques.

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$$\nabla f(x) = Lx - \chi$$

$$\nabla^2 f(x) = L$$

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$$\lambda_2 D \preceq \nabla^2 f(x) \preceq 2D$$

Use degree norm $\| \cdot \|_D$

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Construct iterative solutions: $x^{(0)}, x^{(1)}, x^{(2)}, \ldots, x^{(t)}, \ldots$

$$x^{(t+1)} = x^{(t)} - hD^{-1}\nabla f(x^{(t)})$$

Step length

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By standard gradient descent analysis $h = 1/2$
For quadratic function, it can be optimized at every step.

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.

UNRAVEL RECURSION TO OBTAIN TRUNCATED SERIES: t steps of random walk

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Iterations necessary to converge to ϵ-approximate solution:

$$T = O\left(\frac{2}{\lambda_2} \log\left(\frac{n}{\epsilon} \right) \right)$$

Bad Example for Gradient Descent

PATH: $\lambda_2 = \frac{1}{n^2}$

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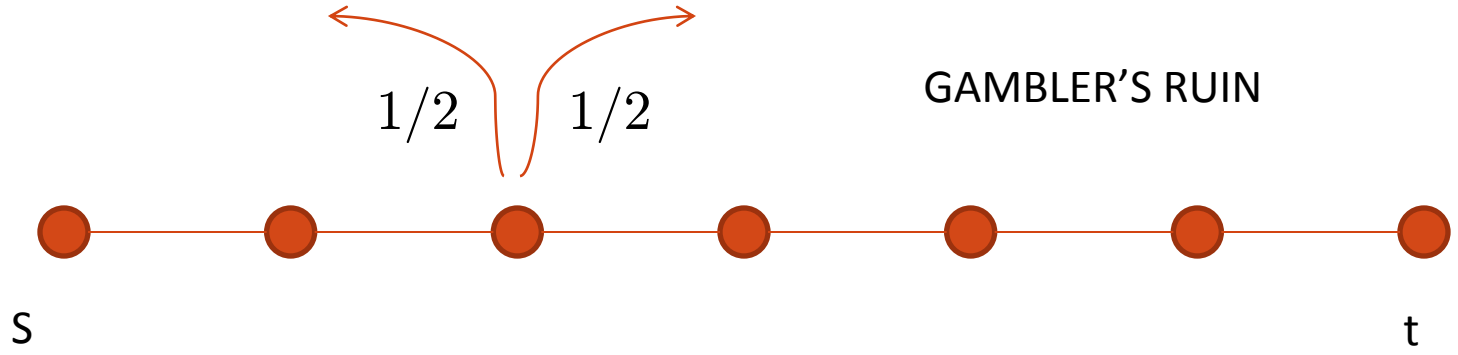

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$$\mathrm{RunTime} = O\left(mn^2\log\left(\frac{n}{\epsilon}\right)\right)$$

ESSENTIALLY TIGHT

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This is a **convex optimization problem** on which we can apply gradient descent techniques.

Accelerated gradient techniques achieve better convergence:

CHEBYSHEV'S ITERATION **CONJUGATE GRADIENT**

Improved iteration count:

$$T = O\left(\sqrt{\frac{2}{\lambda_2}} \log\left(\frac{n}{\epsilon}\right)\right)$$

Still no luck, but …

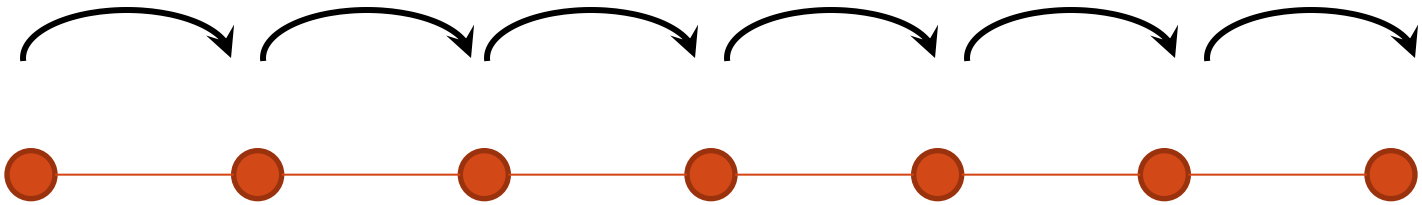
PATH: $\lambda_2 = \frac{1}{n^2}$ $\qquad T = O\left(n \log\left(\frac{n}{\epsilon}\right)\right)$

$$\text{RunTime} = O\left(mn \log\left(\tfrac{n}{\epsilon}\right)\right)$$

BEST POSSIBLE USING GRADIENT APPROACH

Still no luck, but …

IT TAKES n STEPS FOR CHARGE TO TRAVEL ACROSS



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BEST POSSIBLE USING GRADIENT APPROACH

Combining Representation and Iteration

Gaussian elimination and gradient methods seem **complementary**

- Gaussian elimination is nearly-linear on **paths** (and trees),
  but slow on **expanders**.

- Gradient methods are nearly-linear time on **expanders**,
  but slow on **paths**.

MAIN APPROACH TO FAST SOLVERS:

Combine Gaussian elimination and gradient methods
to obtain **best of both worlds**

Combining Representation and Iteration: Combinatorial Preconditioning

$$Lv = \chi$$

Gradient methods fail when condition number is large.

IDEA: modify system to improve condition number

$$L_H^+ Lv = L_H^+ \chi$$

where H is a **preconditioner** graph.

DESIRED PROPERTIES OF H:

1. New matrix is well conditioned: $L_H^+ L$

2. Linear systems in L_H can be solved quickly by Gaussian elimination

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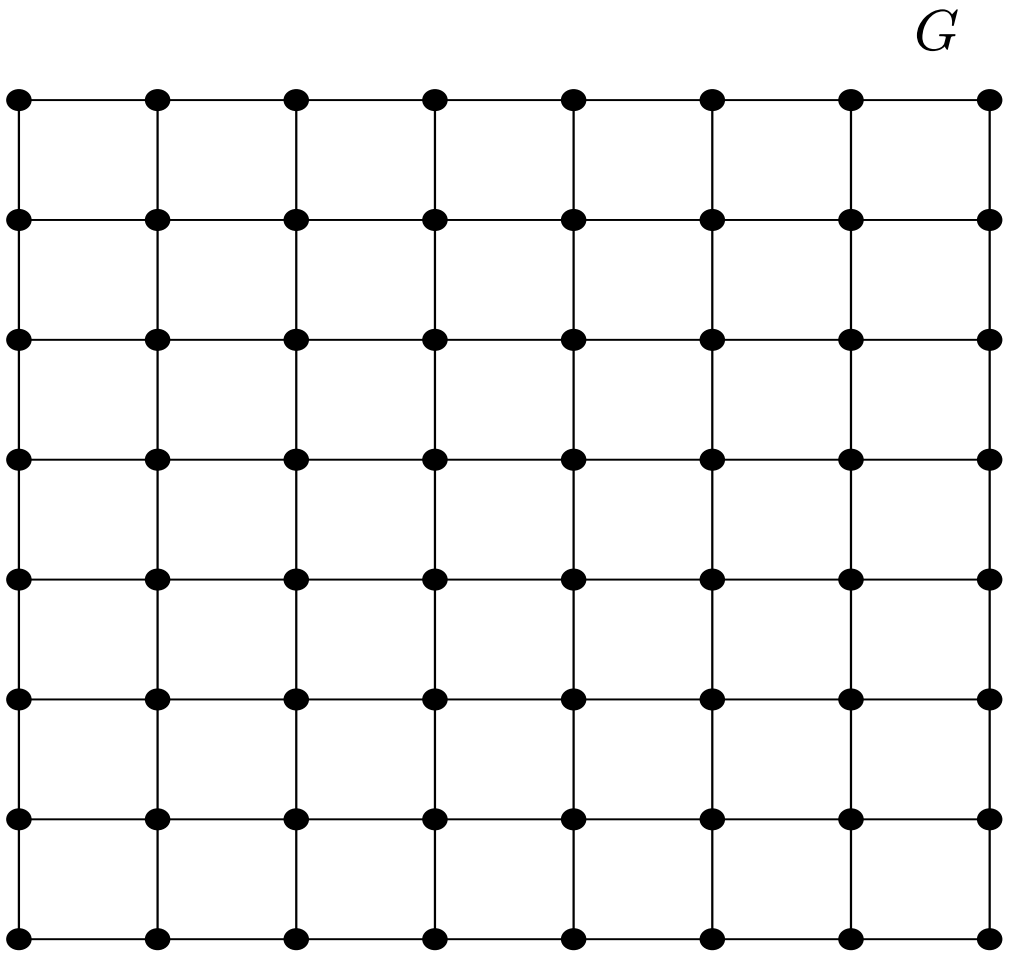
KEEPS ITERATIONS LINEAR TIME

Preconditioning: Low-Stretch Trees



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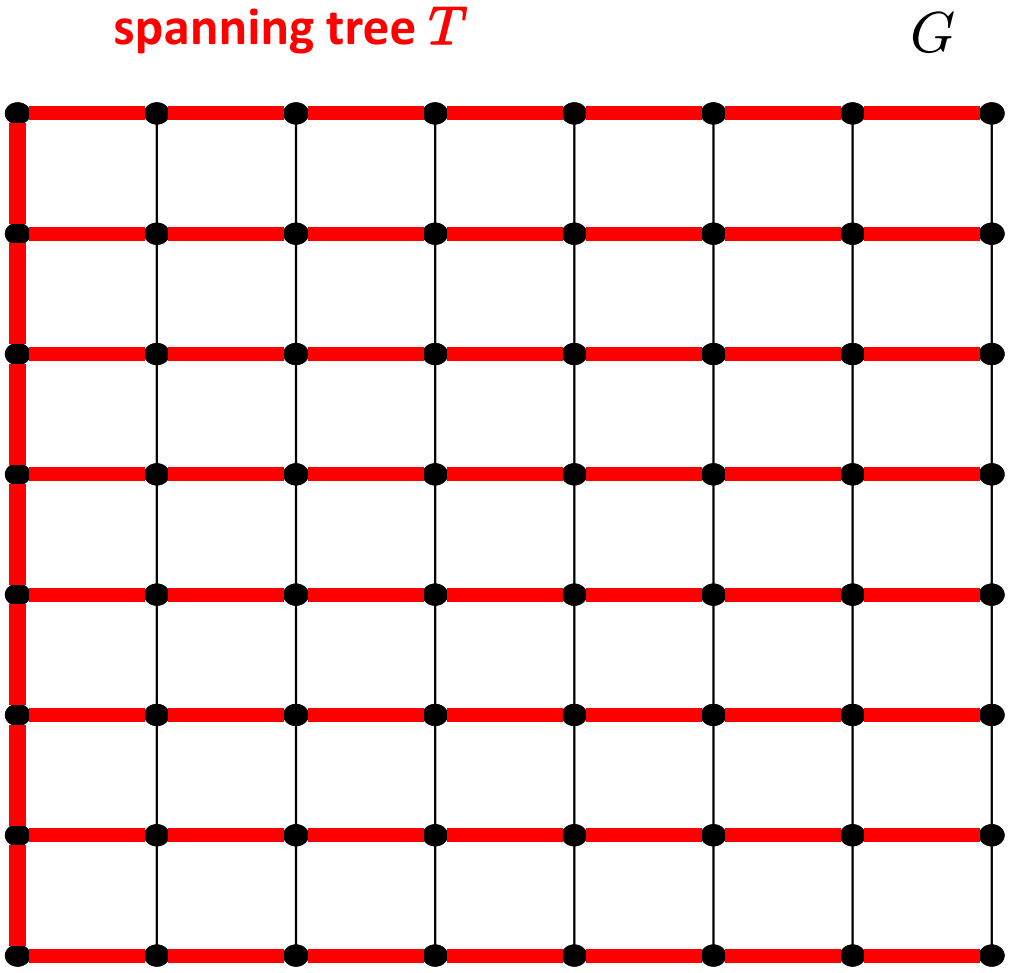

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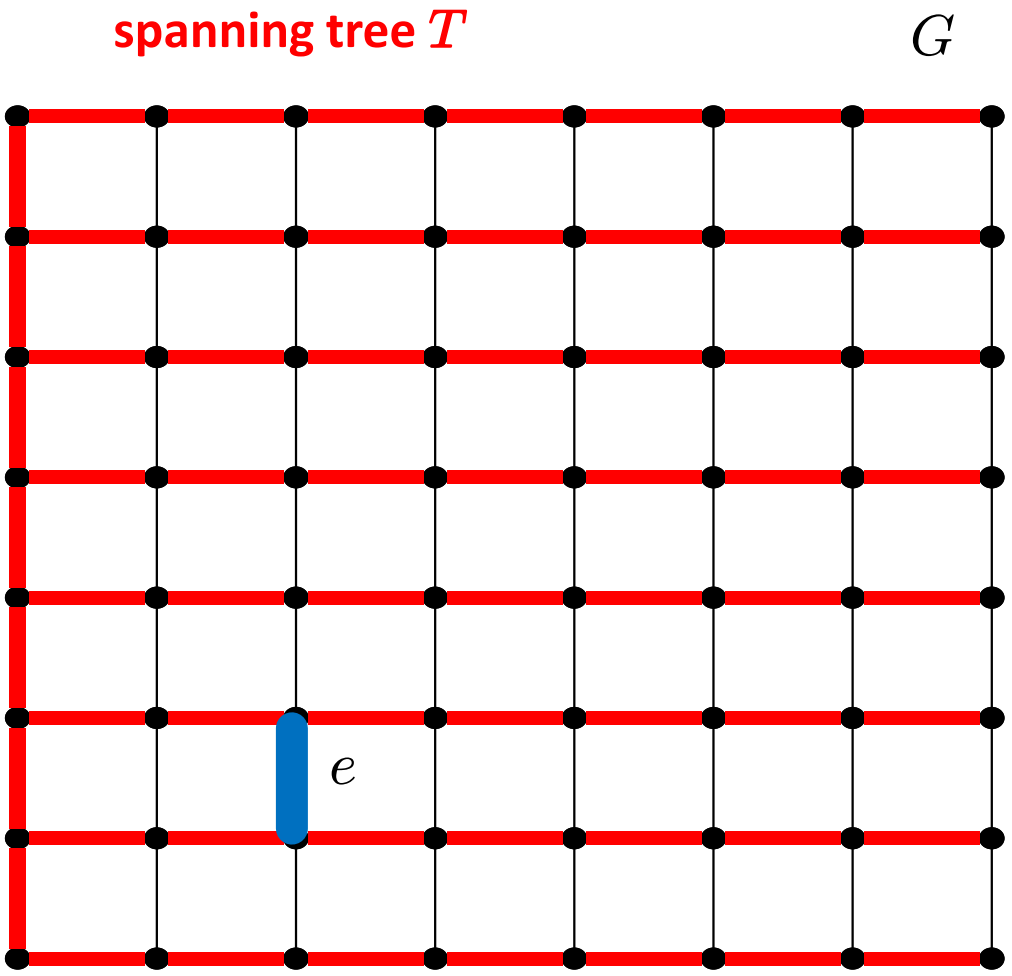
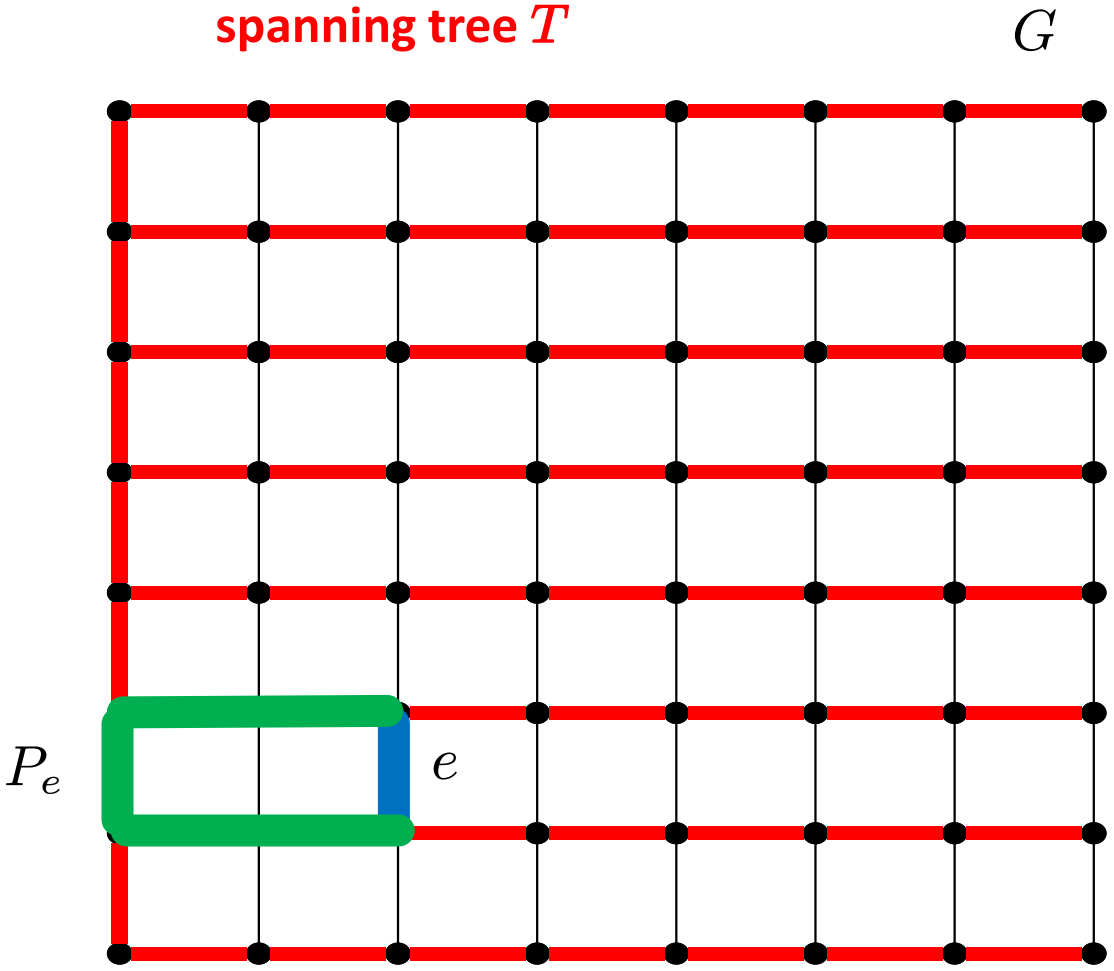


Preconditioning: Low-Stretch Trees

$$\mathrm{st}(e) = \frac{1}{r_e} \sum_{e' \in P_e} r_{e'}$$

Stretch of e



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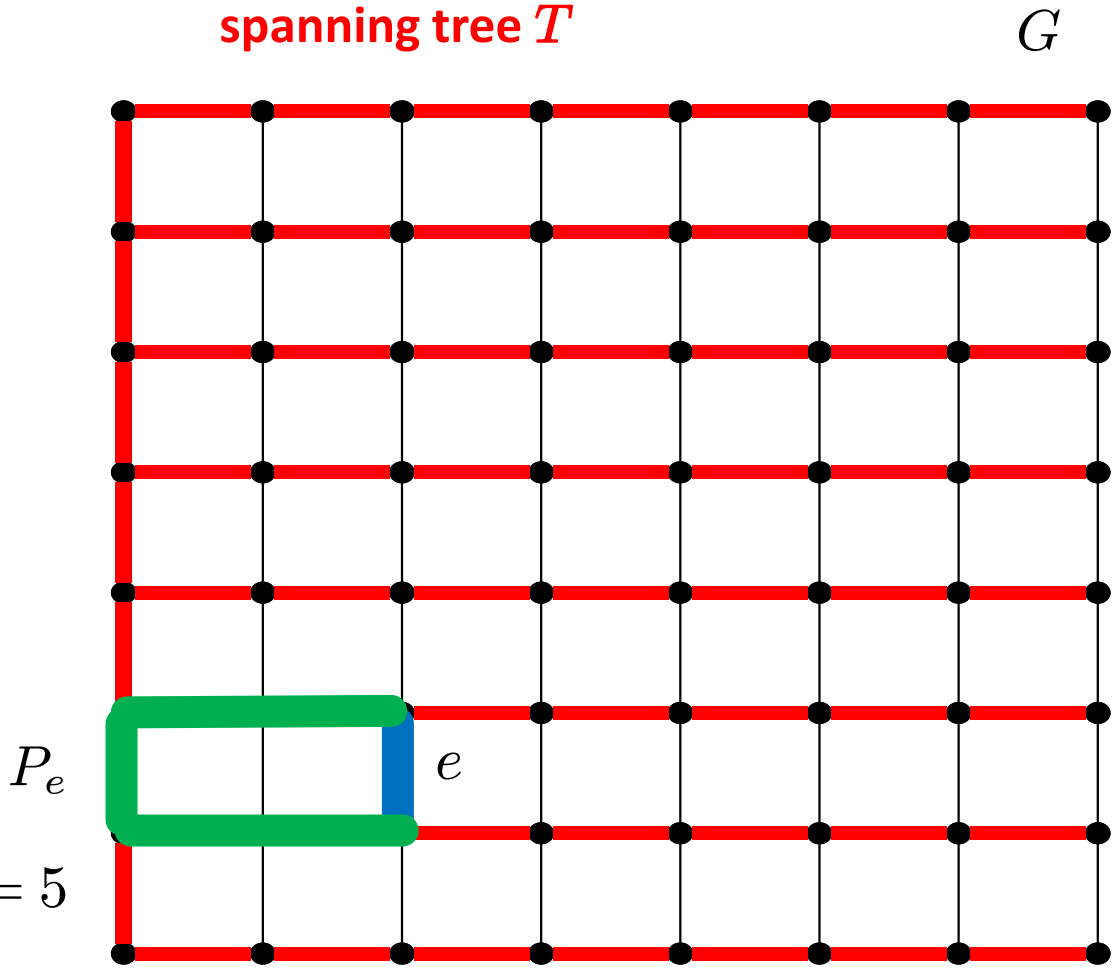

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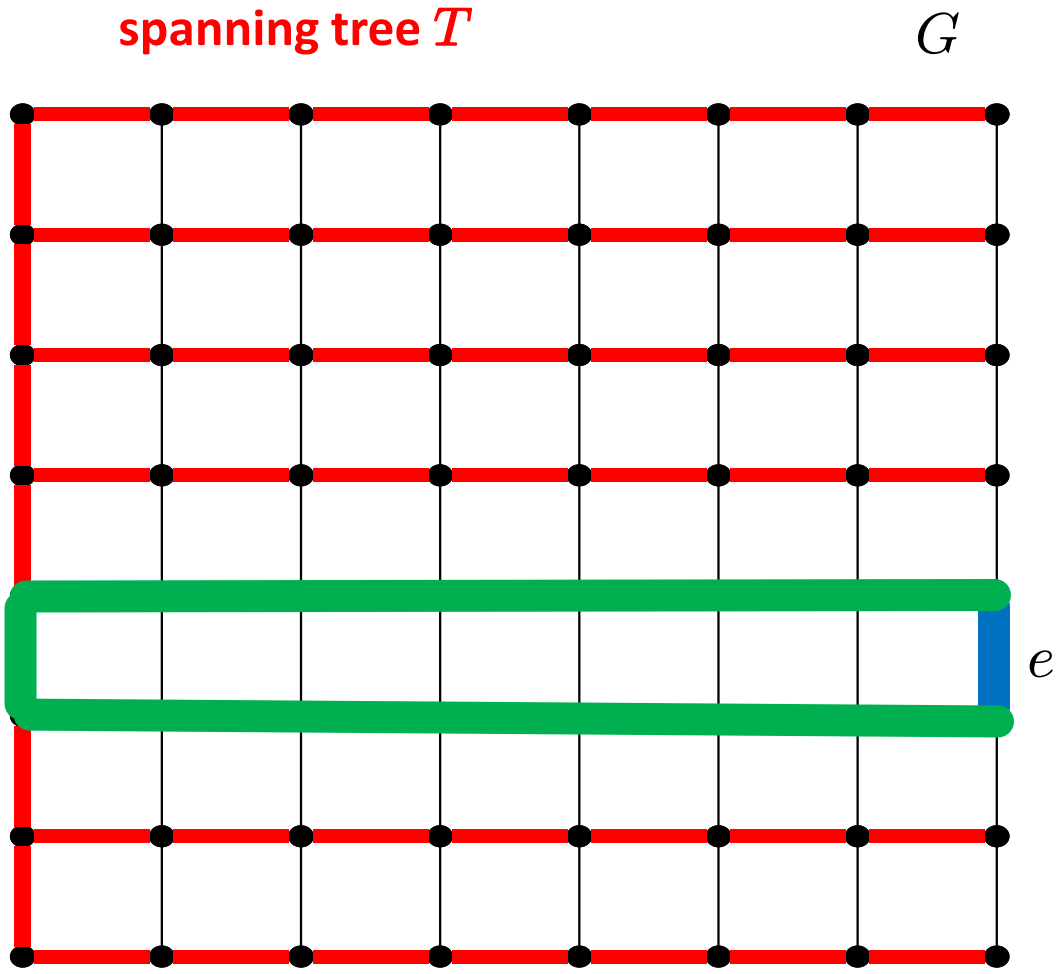

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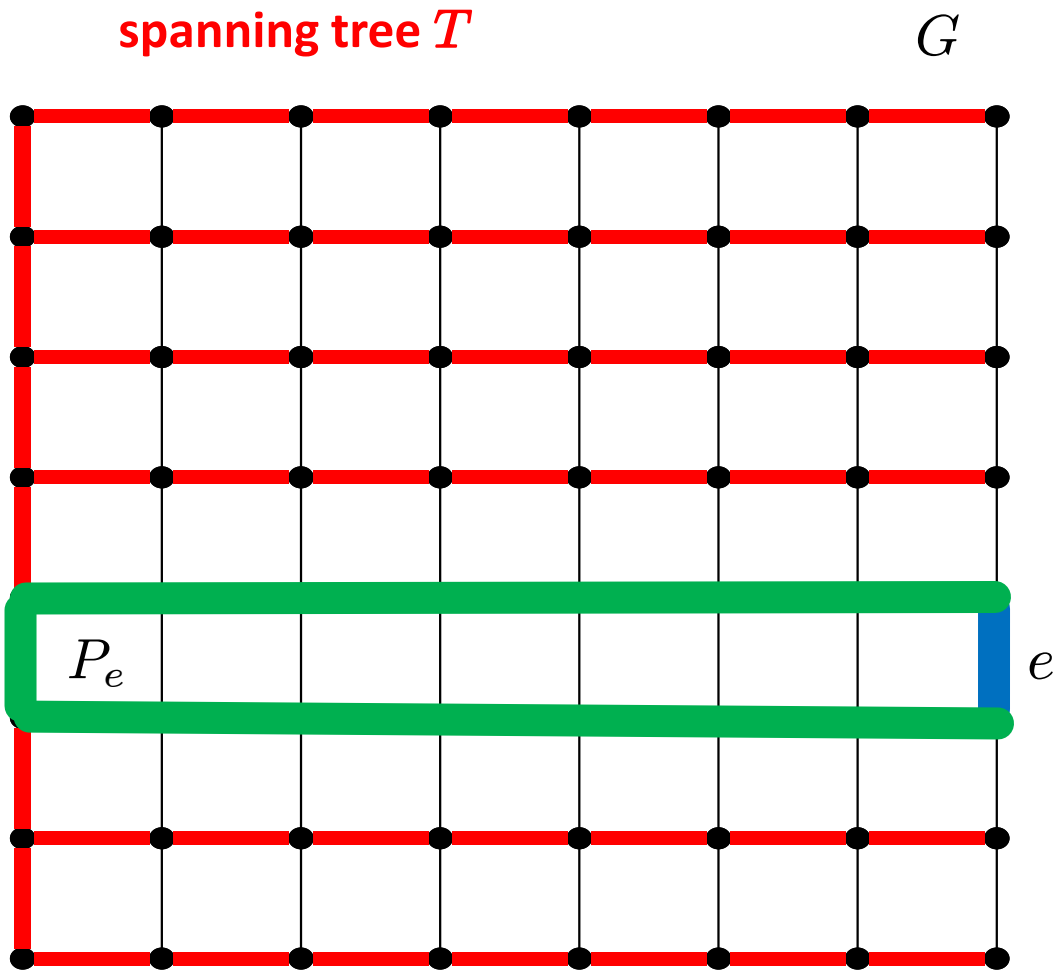
Total Stretch of e



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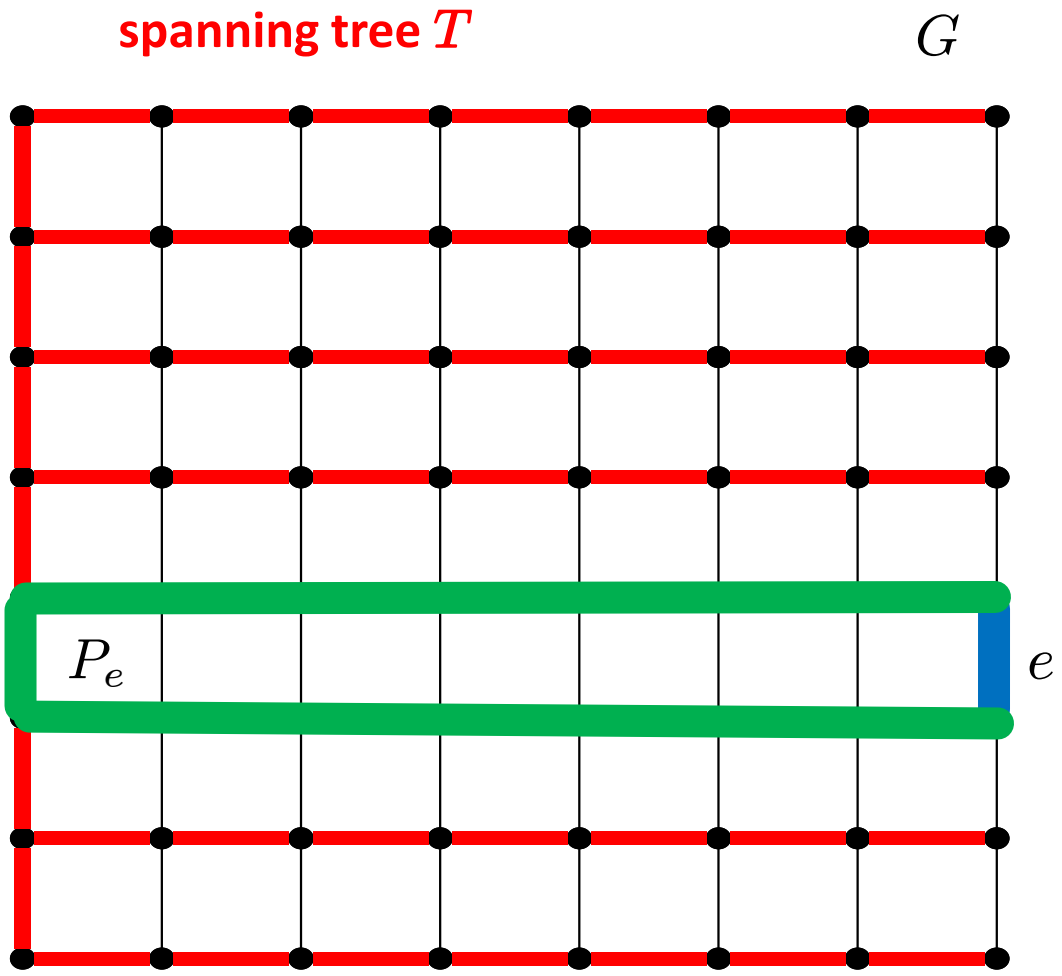


$$\mathrm{st}(T) = O(n^{1.5})$$

Preconditioning: Low-Stretch Trees

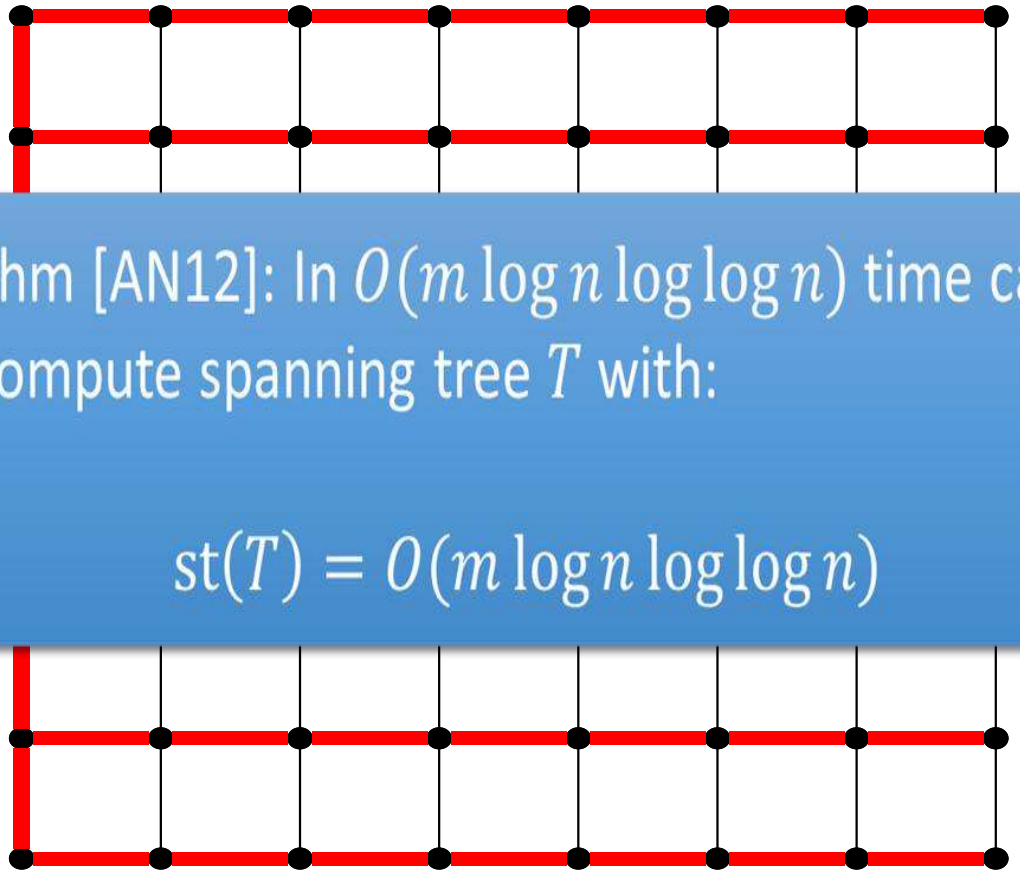
$$\mathrm{st}(e) = \frac{1}{r_e} \sum_{e' \in P_e} r_{e'}$$

Stretch of e

$$\mathrm{st}(T) = \sum_{e \in E} \mathrm{st}(e)$$

Total Stretch of e

spanning tree T G



Thm [AN12]: In $O(m \log n \log\log n)$ time can compute spanning tree T with:

$$\mathrm{st}(T) = O(m \log n \log\log n)$$

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Q: How does this help?

A: Can help to bound condition number of system preconditioned by T

$$1 \cdot I \preceq L_T^{+} L_G \preceq \mathrm{st}(T) \cdot I$$

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Spanning tree property

Low-stretch-tree property

Preconditioning: Low-Stretch Trees

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EASY PROOF:

$$\lambda_{\max}\left(L_T^+ L_G\right) \cdot \ \mathrm{Tr}\left(L_T^+ L_G\right) = \sum_{e \in E} \chi_e^{\mathrm{T}} L_T^+ \chi_e = \mathrm{st}(T)$$

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Trace bounds eigenvalue

Preconditioning: Low-Stretch Trees

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Tree resistance is path length

Preconditioning: Partial Results

1. Preconditioning by low-stretch spanning trees[BH01]

$$\text{RunTime} = \tilde{O}(\underbrace{\sqrt{m \log n} \log\left(\tfrac{n}{\epsilon}\right)}) \cdot [O(m) + O(n)]$$

Condition number

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Condition number

Multiplication by $L_T^+ L$

Preconditioning: Partial Results

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$$\text{RunTime} = \tilde{O}(\underline{\sqrt{m \log n}} \log\left(\tfrac{n}{\epsilon}\right)) \cdot \underline{[O(m) + O(n)]}$$

Condition number — Multiplication by $L_T^+ L$

2. Improved analysis using Conjugate Gradient and Trace bound [SW09]

$$\text{RunTime} = \tilde{O}(m^{4/3} \text{polylog } n)$$

NB: Low-stretch is a trace bound, stronger than eigenvalue bound!

Recursive Preconditioning: Spielman-Teng and Koutis-Miller-Peng

Nearly-linear-time algorithms at last: [ST04], [KMP10],[KMP11]

IDEA: Low-stretch trees do not provide good enough condition number.
Use better preconditioner graphs

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$$L_{G_1}^{+} L v = L_{G_1}^{+} \chi$$

PROBLEM: Hard to solve system for G_1.

SOLUTION: Solve recursively.

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$$L_{G_1}^{+} L v = L_{G_1}^{+} \chi$$

PROBLEM: Hard to solve system for G_1.

SOLUTION: Solve recursively.

$$L_{G_2}^{+} L_{G_1} x = L_{G_2}^{+} b$$

MAIN IDEA: At every recursive level, $G_i$becomes smaller as some low-degree vertices are eliminated via **Gaussian elimination**.

Our Algorithm

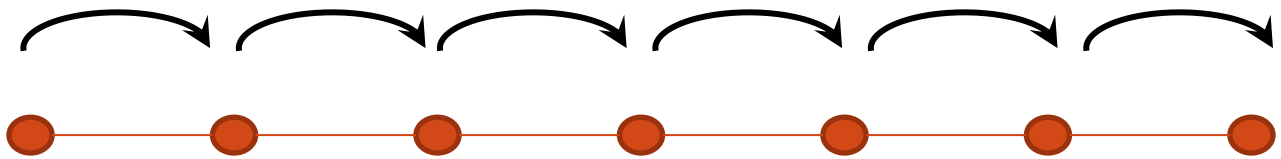
Our Algorithm
(at last)

Choosing the Right Representation
Solve for the Flow

All algorithms discussed so far aim to solve voltage problem .

$$\min_{x \perp \vec{1}} \quad \tfrac{1}{2} \cdot v^{\mathrm{T}} L v - v^{\mathrm{T}} \chi$$

This is particularly problematic for gradient-based methods:



CURSE OF LONG PATHS

Choosing the Right Representation
Solve for the Flow

All algorithms discussed so far aim to solve voltage problem .

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Our algorithm targets the minimum-energy flow problem:

$$\begin{aligned} \min \quad & f^T R f \\ \text{s.t} \quad & B^T f = \chi \end{aligned}$$

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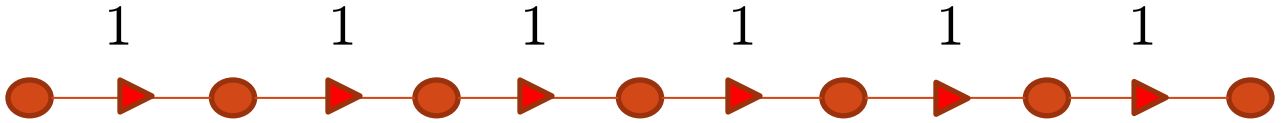
This choice opens up different representational questions:



Could this be a flow path in our basic representation?

Optimality Conditions for Flow Problem

1. Ohm's Law:

$$\exists v: \qquad f = R^{-1}Bv$$

2. Kirchoff's Conservation Law:

$$B^T f = \chi$$

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$$\sum_{e \in C} r_e f_e = 0$$

Flow-induced voltage drop along cycle is 0.

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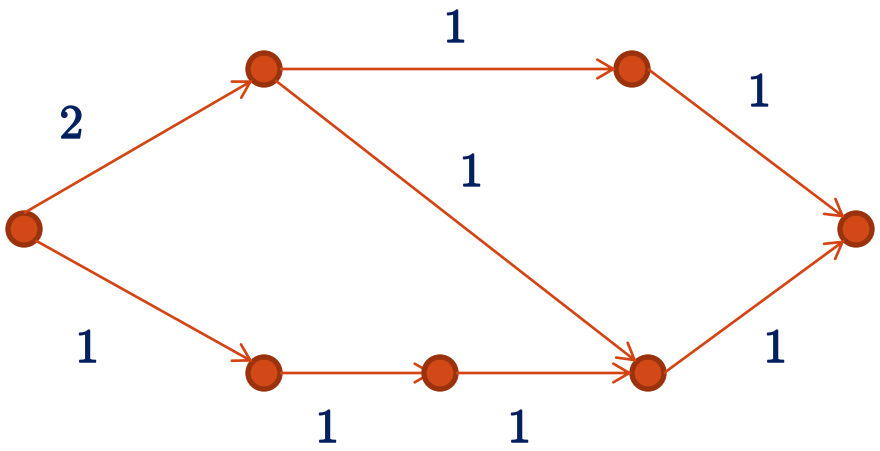
Flow values
Unit resistances



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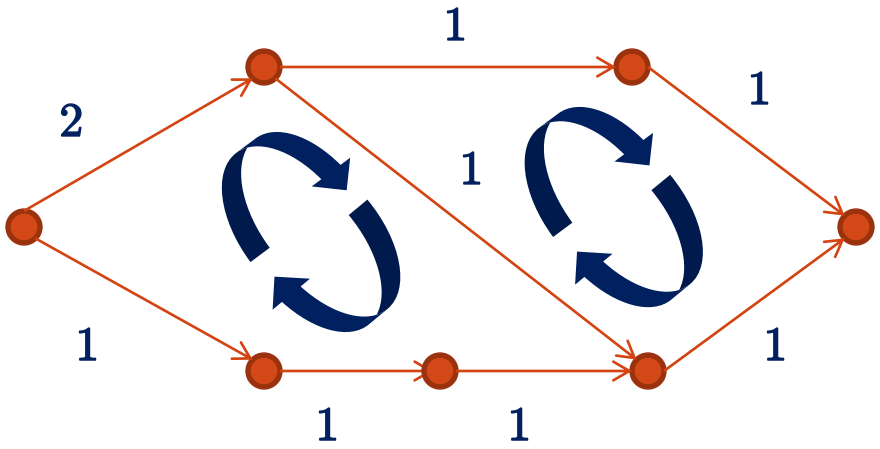

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Flow values
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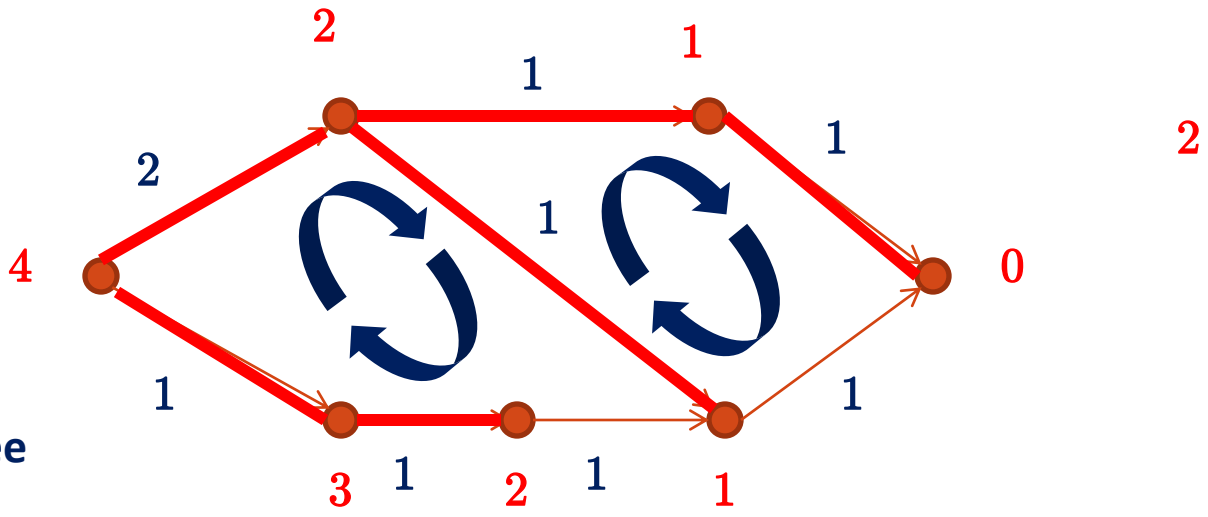
Set voltages using spanning tree



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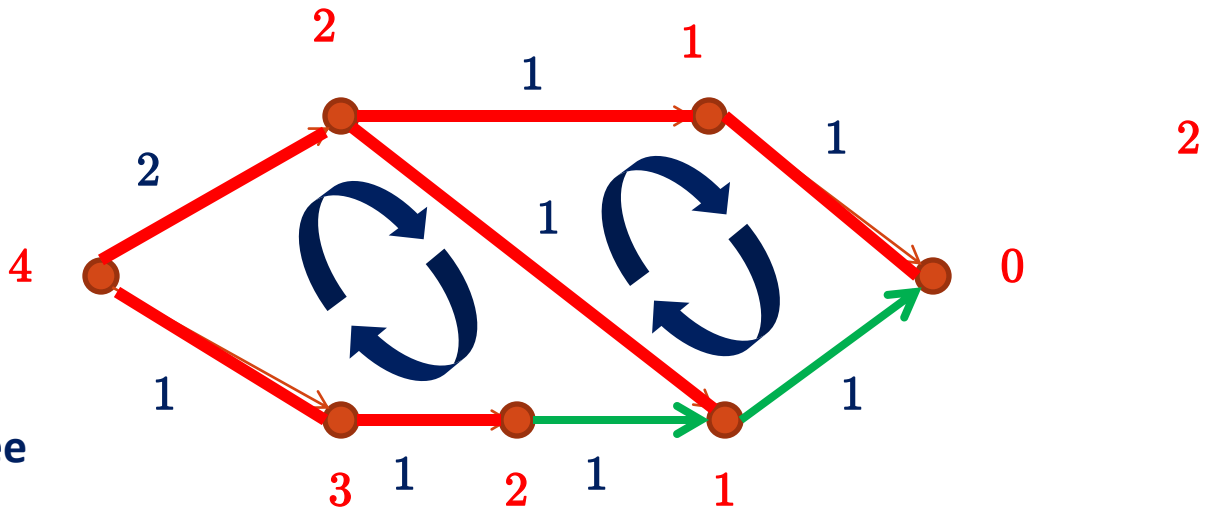


KCL ensures that off-tree edges respect Ohm's Law

Algorithmic Approach

1. Kirchoff's Cycle Law: For any cycle C in G, the optimal electrical flow has

$$\sum_{e \in C} r_e f_e = 0$$

ITERATIVELY FIX BY ADDING/REMOVING FLOW ALONG CYCLES

2. Kirchoff's Conservation Law:

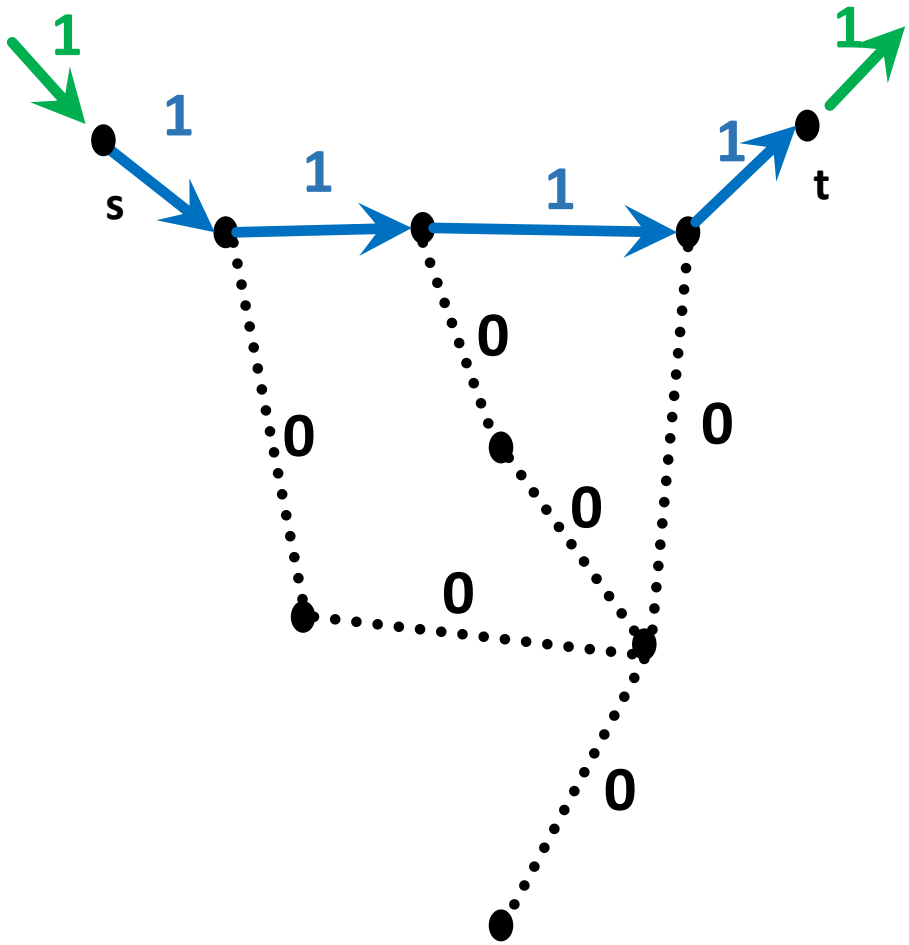
$$B^T f = \chi$$

MAINTAIN SATISFIED

Algorithmic Approach

Initialize: Get a crude feasible $\vec{f}$

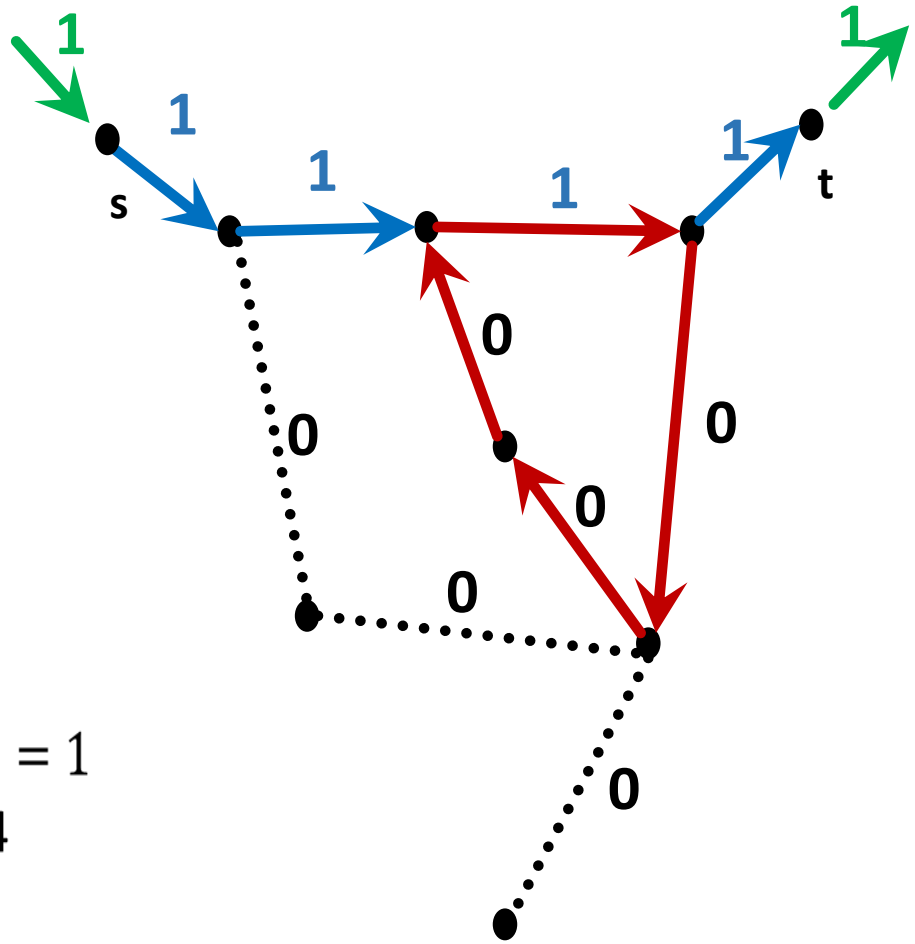
Improve:
- Pick a cycle



Algorithmic Approach

Initialize: Get a crude feasible $\vec{f}$

Improve:
- Pick a cycle
- Fix it



- Potential Drop: $\Delta = \sum_{e \in C} r_e \vec{f}(e) = 1$
- Cycle Resistance: $R = \sum_{e \in C} r_e = 4$

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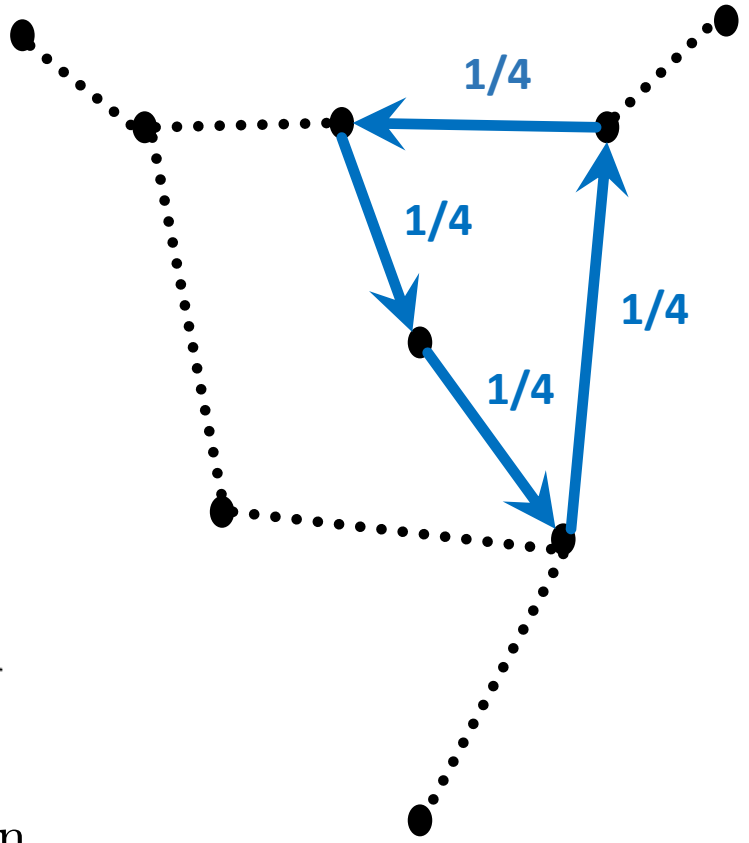

- Potential Drop: $\Delta = \sum_{e \in C} r_e \vec{f}(e) = 1$
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Send $\frac{\Delta}{R}$ flow in opposite direction

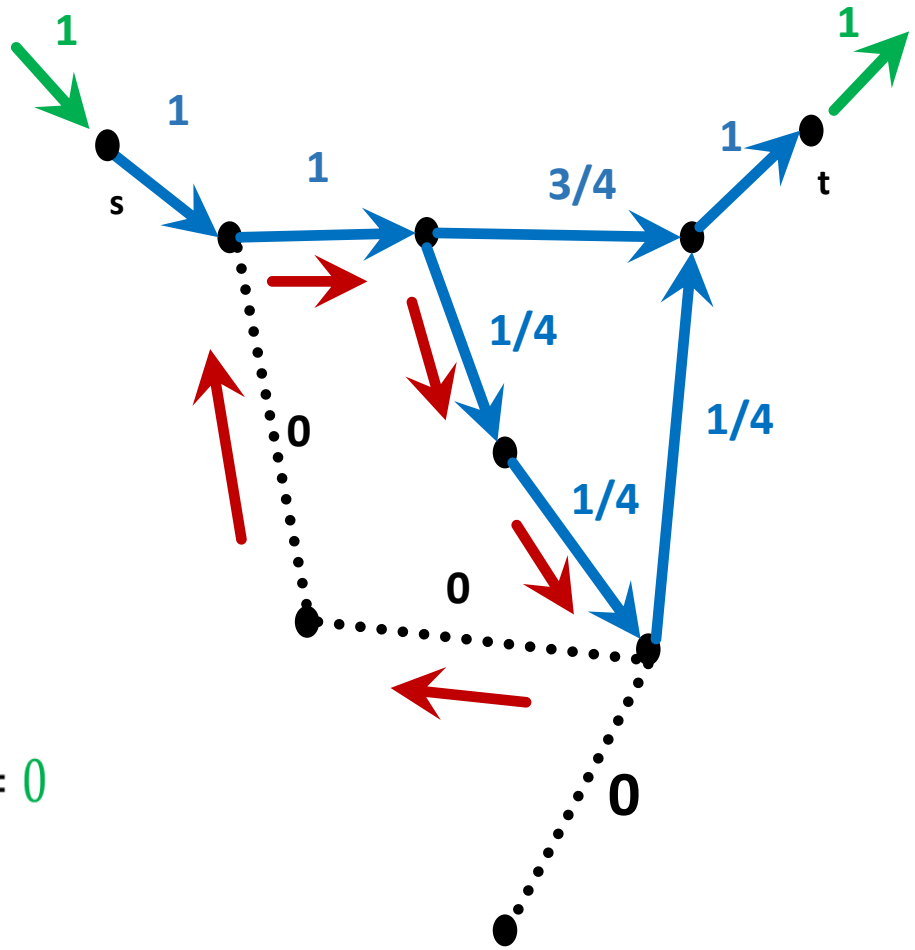
Algorithmic Approach

Initialize:

Get a crude feasible $\vec{f}$

Improve:

- Pick a cycle
- Fix it
- Repeat



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Algorithmic Approach

Initialize: Get a crude feasible $\vec{f}$

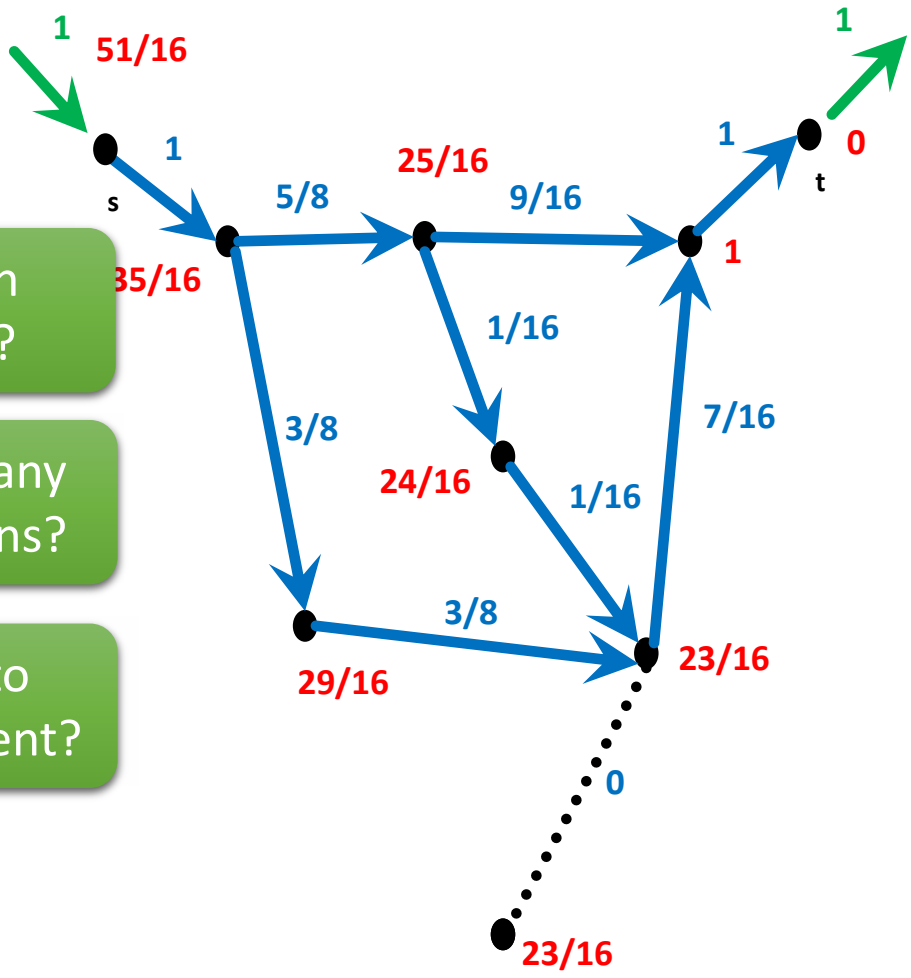
Improve:
- Pick a cycle
- Fix it
- Repeat

Which cycle?

How many iterations?

How to implement?

Output: Round to voltages v



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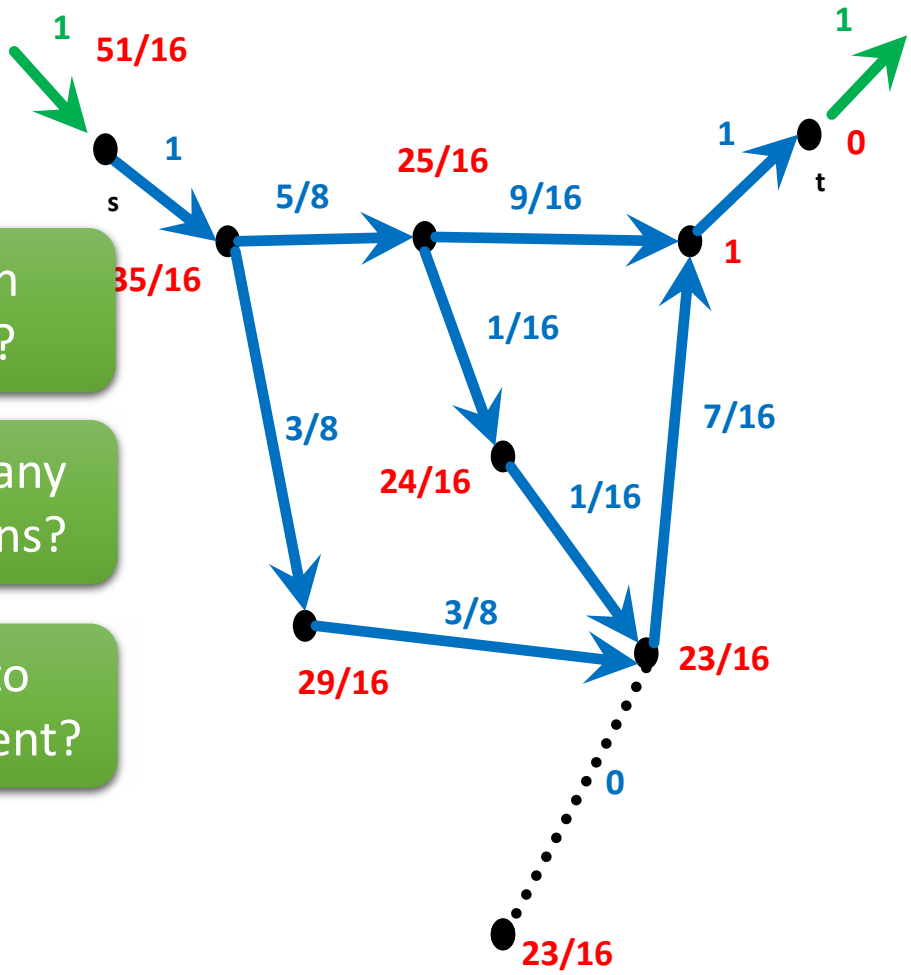
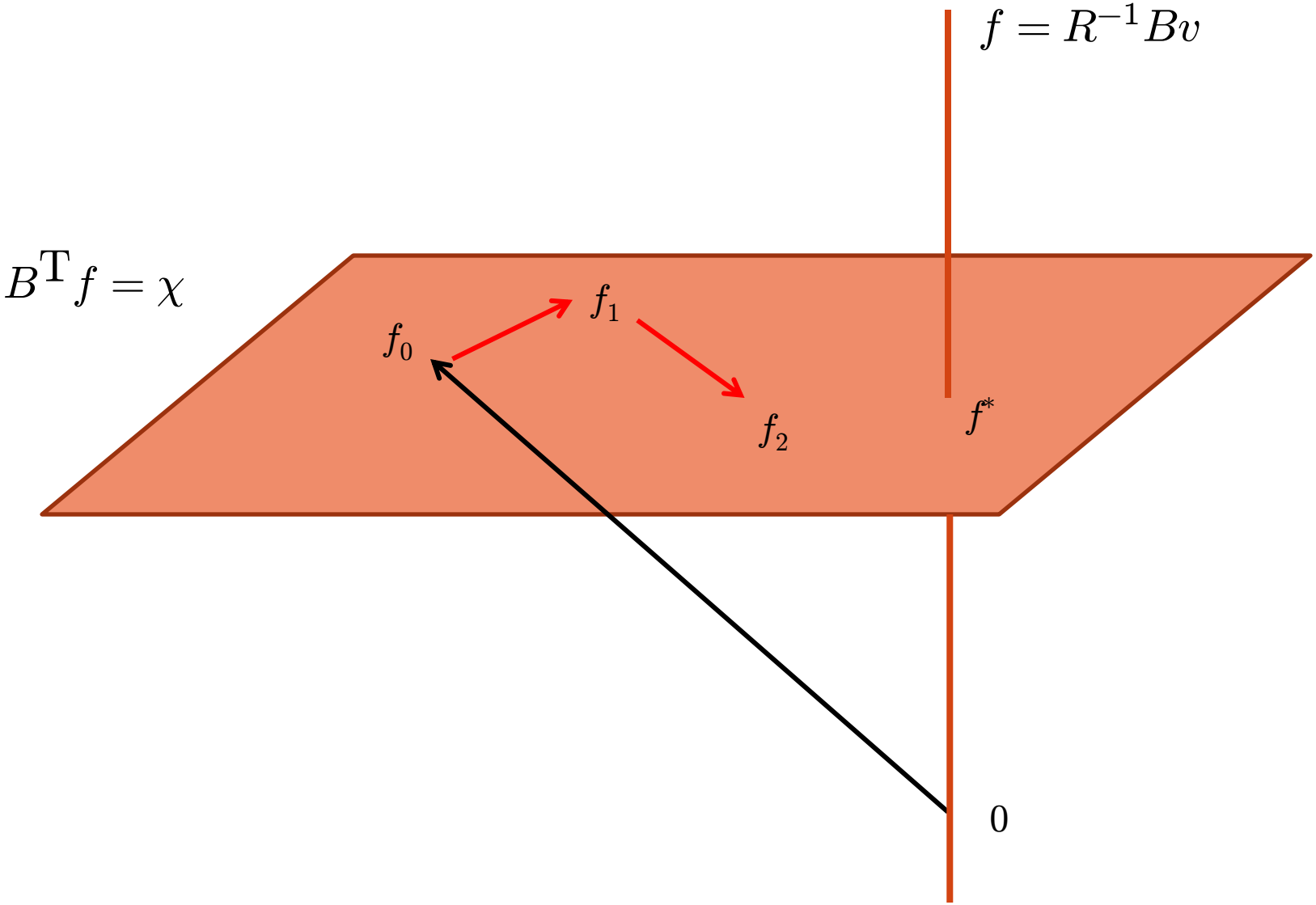


IDEA: A nearly-linear number of cheap (i.e. $\mathrm{O}(\log n)$ iterations!

Choosing the Right Representation: Cycle Space



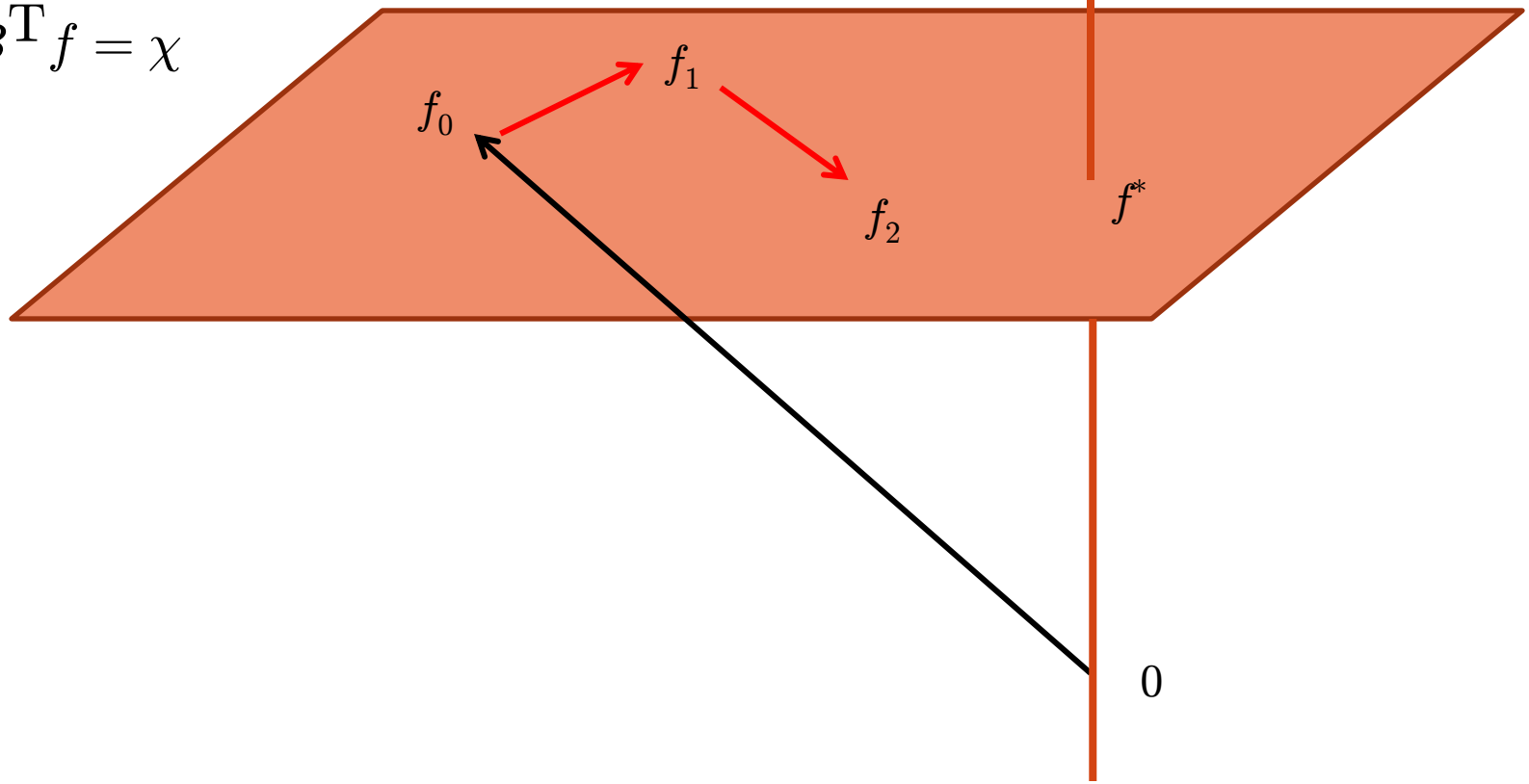
Choosing the Right Representation: Cycle Space

An affine translation of span of cycles (cycle space C_G)

$$C_G = \{c : B^{\mathrm{T}} c = 0\}$$

$$B^{\mathrm{T}} f = \chi$$

$$f = R^{-1} B v$$



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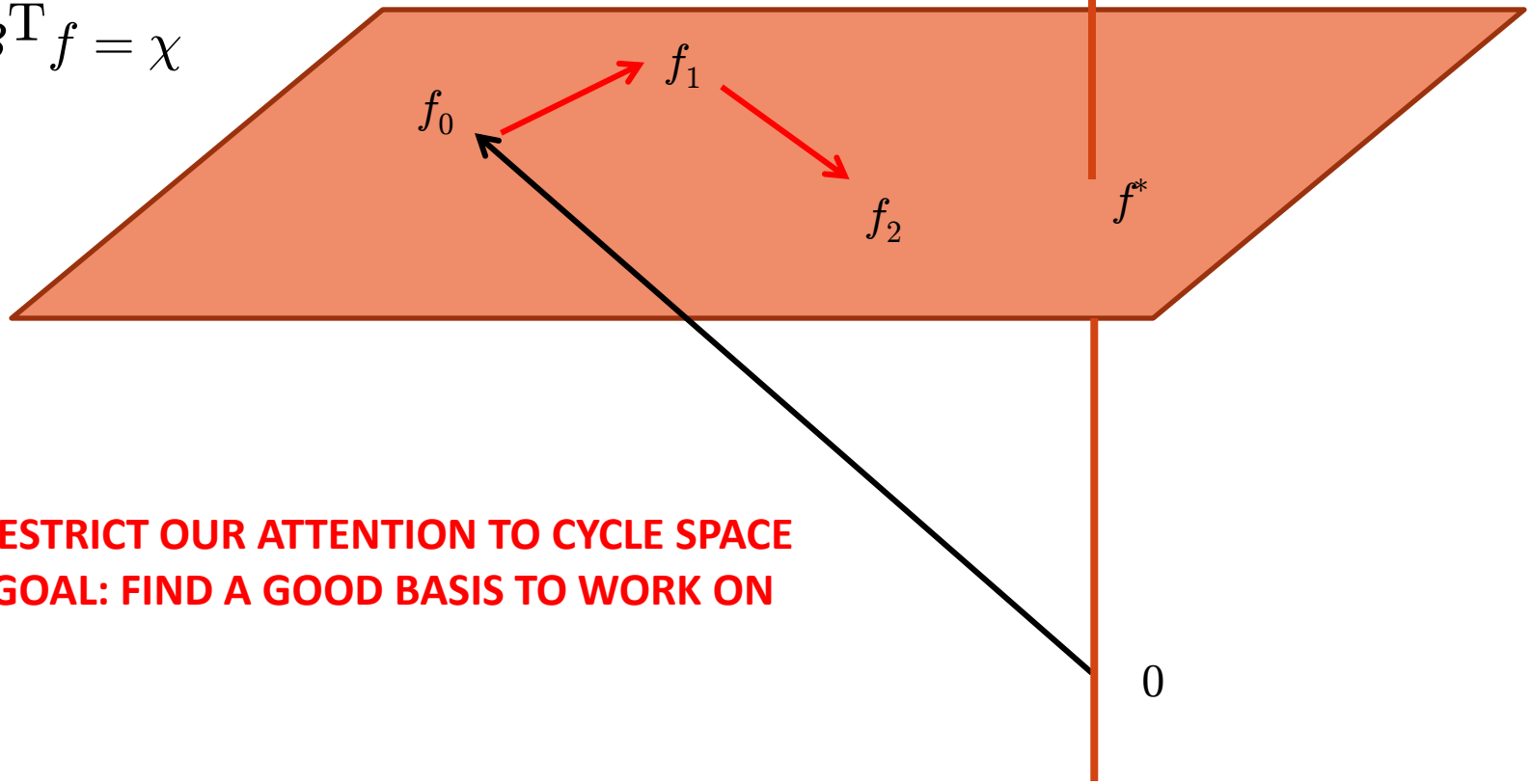
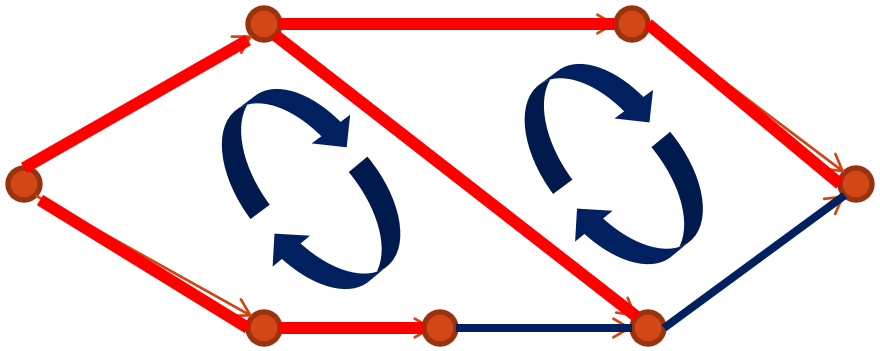


RESTRICT OUR ATTENTION TO CYCLE SPACE
GOAL: FIND A GOOD BASIS TO WORK ON

Choosing the Right Representation: Basis of Cycle Space

Fact:
Fundamental cycles of spanning tree give a basis of cycle space



Fix spanning tree **T**

Choosing the Right Representation: Basis of Cycle Space

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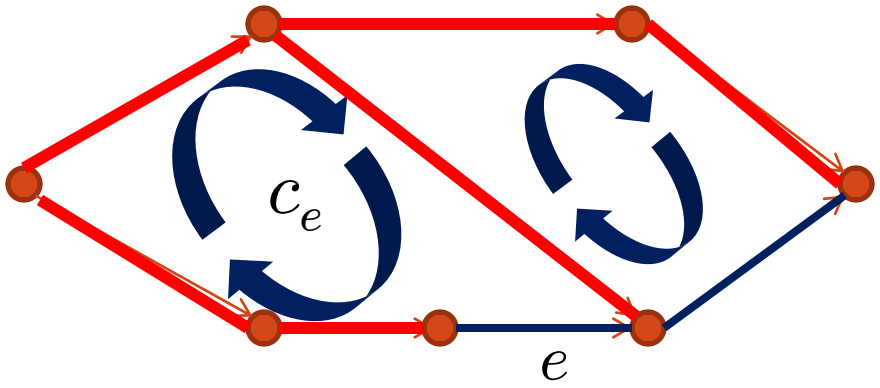

Fix spanning tree **T**

Our New Representation

ORIGINAl PROBLEM

$$\forall c \in C_G : \quad c^T R(f_0 + f) = 0$$

$$B^T f = 0$$

Use spanning tree T
basis for C_G

$$\forall e \in E \setminus T : \quad c_e^T R(f_0 + f) = 0$$

$$f = \sum_{e \in E \setminus T} f_e c_e$$

Our New Representation

ORIGINAl PROBLEM

$$\forall c \in C_G : \quad c^T R(f_0 + f) = 0$$
$$B^T f = 0$$

Use spanning tree T basis for C_G

$$\forall e \in E \setminus T : \quad c_e^T R(f_0 + f) = 0 \qquad m - n + 1 \text{ constraints}$$
$$f = \sum_{e \in E \setminus T} f_e c_e \qquad m - n + 1 \text{ variables}$$

Our New Representation

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Iteratively fix fundamental cycles of T

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WHAT ITERATIVE METHOD IS THIS? Iteratively fix fundamental cycles of T

Algorithmic Approach

Initialize: Get a crude feasible $\vec{f}$

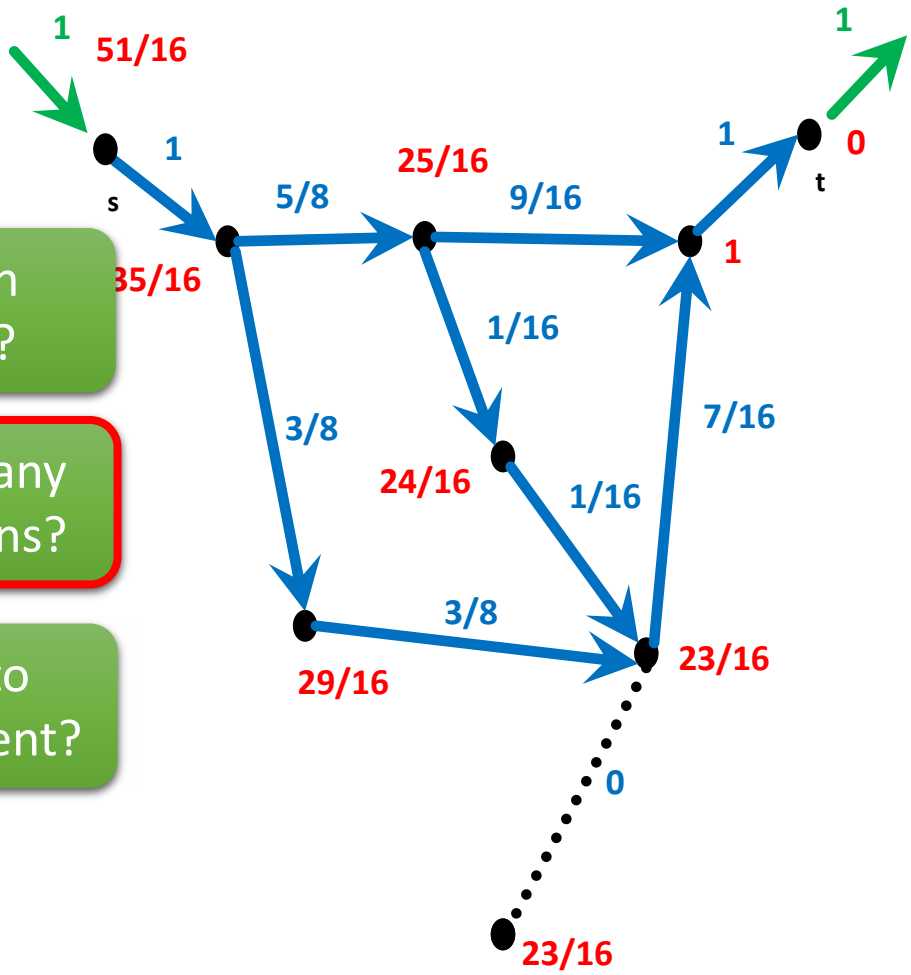
Improve:
- Pick a cycle
- Fix it
- Repeat

Which cycle?

How many iterations?

How to implement?

Output: Round to voltages v



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Choosing the Right Iteration Gradient Descent?

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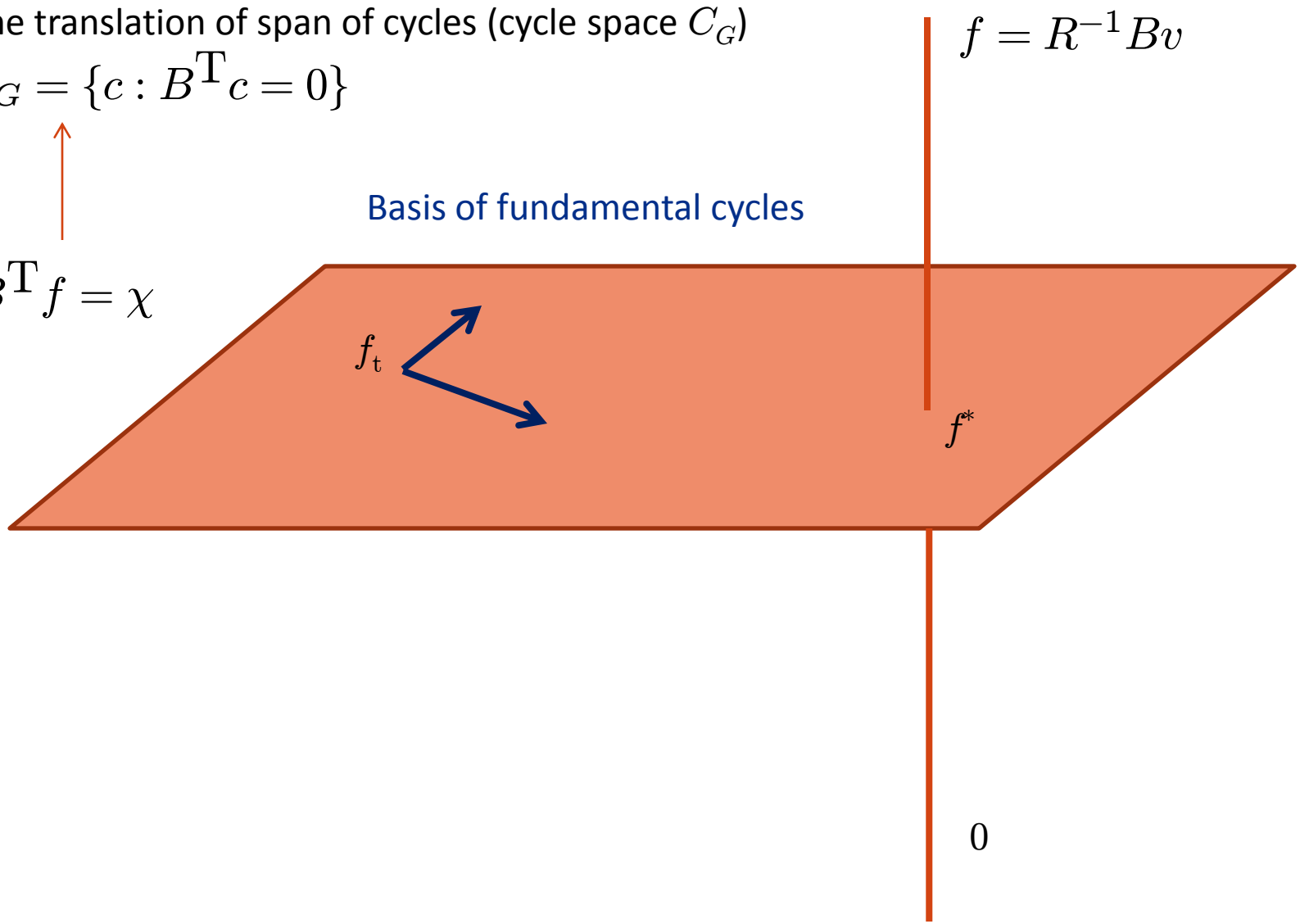
$$B^{\mathrm{T}} f = \chi$$

Basis of fundamental cycles



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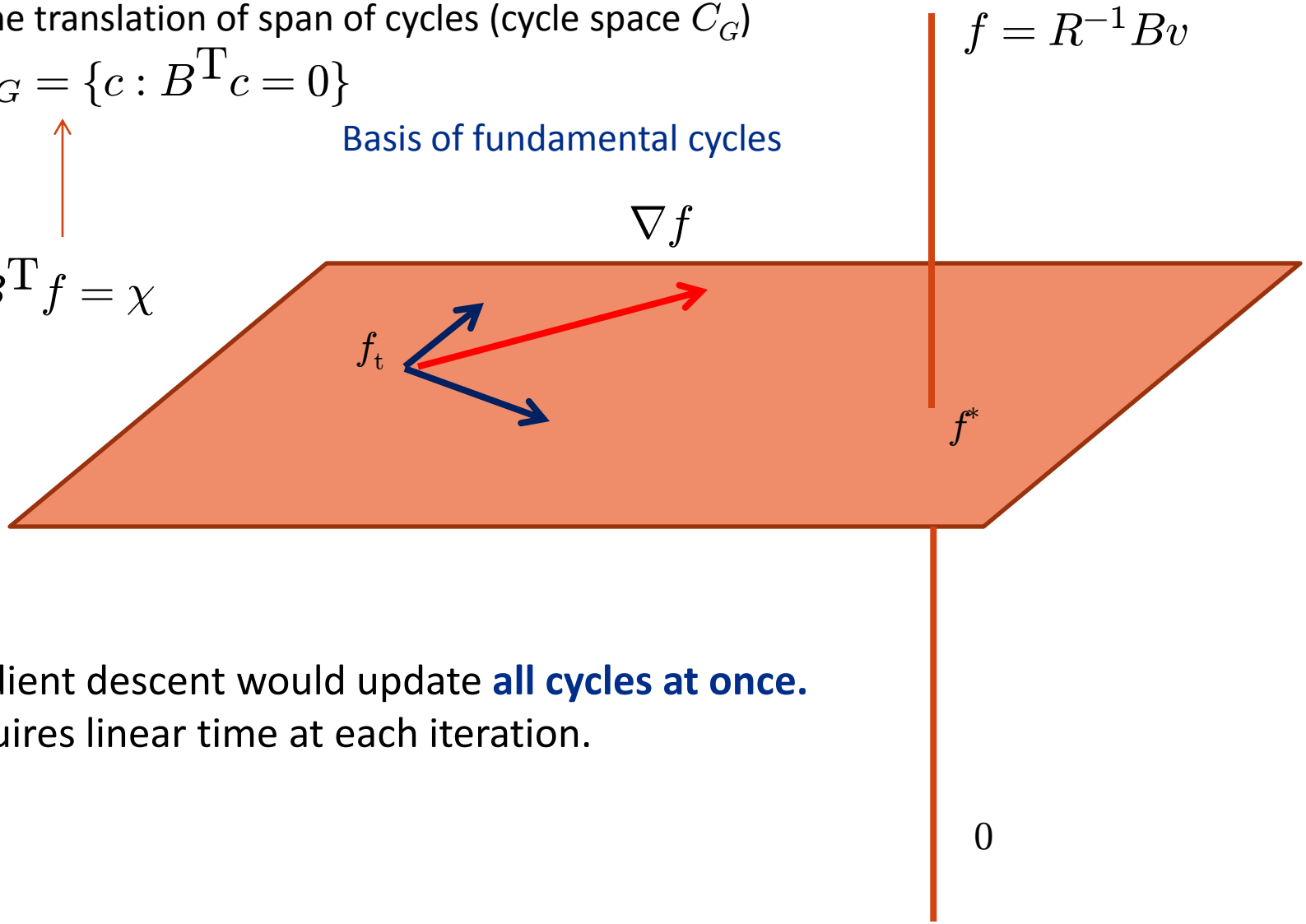

Gradient descent would update **all cycles at once.**
Requires linear time at each iteration.

Choosing the Right Iteration
Randomized Cordinate Descent!

An affine translation of span of cycles (cycle space C_G)

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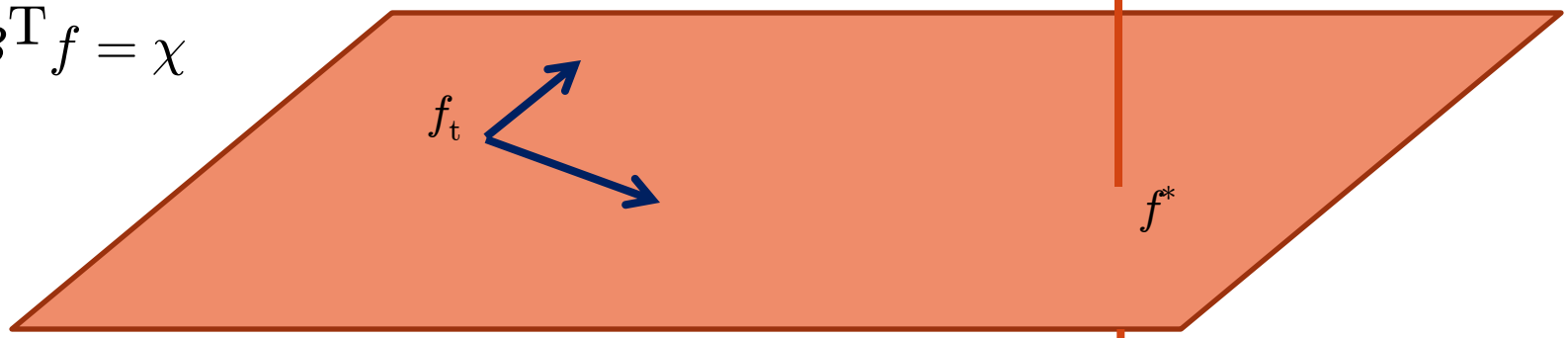
Basis of fundamental cycles

$$B^{\mathrm{T}} f = \chi$$

$$f = R^{-1} B v$$



Move along one coordinate direction at a time.
Choose basis vector c_e w.p. proportional to

$$\frac{c_e^T R c_e}{r_e} = \mathrm{st}(e) + 1$$

[SV09]

0

Convergence analysis?

Coordinate Descent: Convergence

$$\min_{x \perp \vec{1}} \quad \frac{1}{2} \cdot x^{\mathrm{T}} A x - b^{\mathrm{T}} x$$

$$\forall e \in E \setminus T: \quad c_e^T R(f_0 + f) = 0$$

$$f = \sum_{e \in E \setminus T} f_e c_e$$

Iteration count for gradient descent:

$$T = O\left(\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \log\left(\frac{n}{\epsilon}\right)\right)$$

Iteration count for randomized coordinate descent:

$$T = O\left(\frac{\mathrm{Tr}(A)}{\lambda_{\min}(A)} \log\left(\frac{n}{\epsilon}\right)\right)$$

Coordinate Descent: Convergence

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Worse by a factor of $m - n + 1$ in our case

Convergence

$$\min_{x \perp \vec{1}} \quad \frac{1}{2} \cdot x^{\mathrm{T}} A x - b^{\mathrm{T}} x$$

$$\forall e \in E \setminus T: \quad c_e^T R(f_0 + f) = 0$$

$$f = \sum_{e \in E \setminus T} f_e c_e$$

$$\lambda_{\min}(A) = 1 \qquad \lambda_{\max}(A) \cdot \mathrm{Tr}(A) = \mathrm{st}(T) + O(n)$$

Tree stretch

Same calculation as for voltage preconditioning.

Iteration count for randomized coordinate descent:

$$T = O\left(\frac{\mathrm{Tr}(A)}{\lambda_{\min}(A)} \log\left(\frac{n}{\epsilon}\right)\right)$$

New Condition Numbers

$$\min_{x \perp \vec{1}} \quad \frac{1}{2} \cdot x^{\mathrm{T}} A x - b^{\mathrm{T}} x$$

$$\forall e \in E \setminus T: \quad c_e^T R(f_0 + f) = 0$$

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~~Worse by a factor of $m - n + 1$ in our case~~

We start with a trace guarantee,
Not an eigenvalue one

Final Convergence

$$\min_{x \perp \vec{1}} \quad \frac{1}{2} \cdot x^{\mathrm{T}} A x - b^{\mathrm{T}} x$$

$$\forall e \in E \setminus T: \quad c_e^T R(f_0 + f) = 0$$

$$f = \sum_{e \in E \setminus T} f_e c_e$$

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Tree stretch

Same calculation as for voltage preconditioning.

Pick spanning tree T to be low stretch.

Iteration count for randomized coordinate descent:

$$T = O\left(\frac{\mathrm{Tr}(A)}{\lambda_{\min}(A)} \log\left(\frac{n}{\epsilon}\right)\right) = \tilde{O}\left(m \log n \log\left(\frac{n}{\epsilon}\right)\right)$$

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NEARLY-LINEAR

Algorithmic Approach

Initialize: Get a crude feasible $\vec{f}$

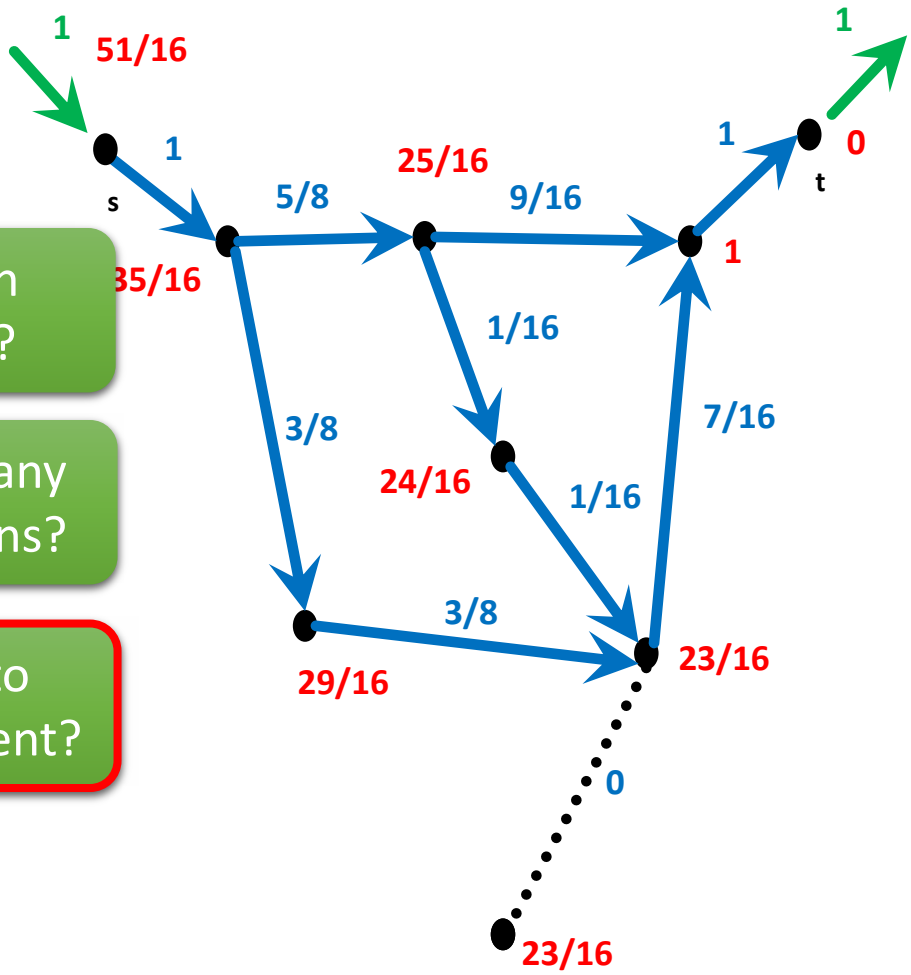
Improve:
- Pick a cycle
- Fix it
- Repeat

Which cycle?

How many iterations?

How to implement?

Output: Round to voltages v



IDEA: A nearly-linear number of cheap (i.e. $\mathrm{O}(\log n)$ iterations!

Implement updates efficiently

- Need to compute the potential drop along a tree cycle C_e

$$\sum_{e' \in C_e} f_e r_e$$

- Need to add a constant α to a tree cycle C_e

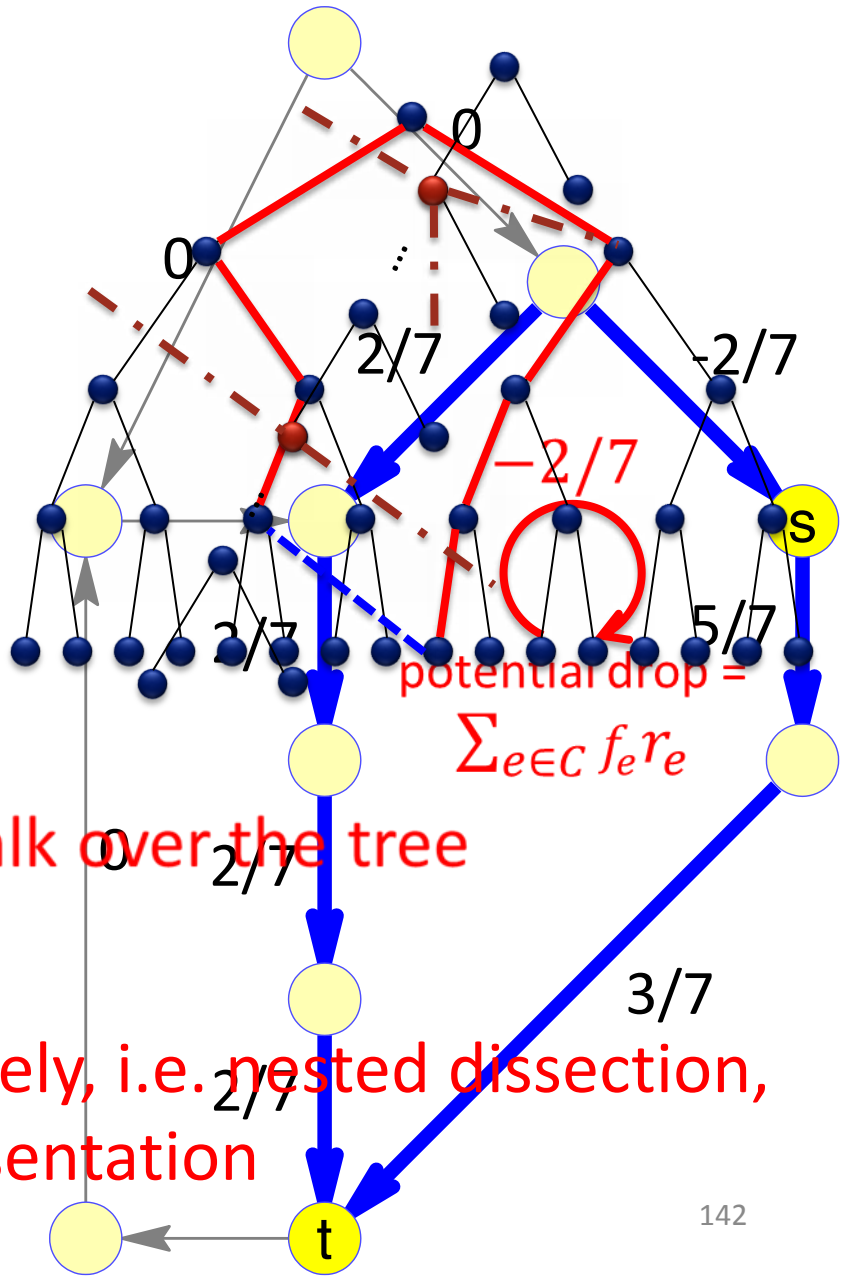
$$\forall e' \in C_e, \qquad f_e \leftarrow f_e + \alpha;$$

- When the tree is balanced

  requires $O(\log n)$ time to walk over the tree

- Otherwise

  decompose the tree recursively, i.e. nested dissection, gives $\mathrm{O}(\log\ n)$ sparse representation



Resulting Algorithm

Which cycle?

Fundamental cycle from low-stretch spanning tree T, sampled with probability proprtional to stretch.

- Pick a cycle
- Fix it
- Repeat

How many iterations?

How to implement?

Resulting Algorithm

Which cycle?

Fundamental cycle from low-stretch spanning tree T, sampled with probability proprtional to stretch.

- Pick a cycle
- Fix it
- Repeat

How many iterations?

$$\tilde{O}\left(m \log n \log\left(\frac{n}{\epsilon}\right)\right)$$

Randomized coordinate descent analysis

How to implement?

Resulting Algorithm

- Pick a cycle
- Fix it
- Repeat

Which cycle?

Fundamental cycle from low-stretch spanning tree T, sampled with probability proprtional to stretch.

How many iterations?

$$\tilde{O}\left(m \log n \log\left(\frac{n}{\epsilon}\right)\right)$$

Randomized coordinate descent analysis

How to implement?

$$O(\log n)$$

Exploiting separators of size 1 in tree

Resulting Algorithm

Which cycle?

Fundamental cycle from low-stretch spanning tree T, sampled with probability proprtional to stretch.

- Pick a cycle
- Fix it
- Repeat

How many iterations?

$$\tilde{O}\left(m \log n \log\left(\frac{n}{\epsilon}\right)\right)$$

Randomized coordinate descent analysis

How to implement?

$$O(\log n)$$

Exploiting separators of size 1 in tree

TOTAL RUNNING TIME: $\tilde{O}\left(m \log^2 n \log\left(\frac{n}{\epsilon}\right)\right)$

Running Time Improvements

$$\tilde{O}\left(m \log^2 n \log\left(\tfrac{n}{\epsilon}\right)\right)$$

Multiple phases with restarts

$$\tilde{O}\left(m \log^2 n \log\left(\tfrac{1}{\epsilon}\right)\right)$$

Lee, Sidford at FOCS'13

Accelerated Coordinate Descent

$$\tilde{O}\left(m \log^{1.5} n \log\left(\tfrac{1}{\epsilon}\right)\right)$$

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$$\tilde{O}\left(m \log^{1.5} n \log\left(\tfrac{1}{\epsilon}\right)\right)$$

Open Questions

RELATED TO THIS ALGORITHM:

- Can this algorithm be parallelized? **Practically important**.

- Can it be made deterministic? Random order works on any RHS with high probability.

MORE GENERAL:

- Other application of idea about **many cheap iterations**?
  E.g. cycle update view of undirected maximum flow result?

- Other applications of underlying idea: **right representation + right iterative method**
  Almost-linear-time undirected maximum flow falls in this framework
  Can we tackle **other important graph problems**?
  e.g. directed maximum flow, better multicommodity flows, …