A simple cosmology: General relativity not required

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A pedagogical cosmology illustrates general relativity concepts, without requiring general or special relativity. Topics examined are the existence of a global time scale, proper vs coordinate variables, the variation of light speed in an expanding universe, the look-back paradox, the horizon, the red shift, the age of the universe, and the dynamics of the universe. An Appendix is devoted to space and time in general relativity, but can be skipped by readers unfamiliar with general relativity. © *2001 American Association of Physics Teachers.*
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I. INTRODUCTION

This pedagogical article examines a simple, one-dimensional model expanding universe.[1,2] It is readily understandable to physics students who have no background in general relativity (GR), yet it emphasizes, compares, and contrasts many general relativistic features of cosmology.

The model universe of this article is an expanding rubber band marked with a uniform string of dot galaxies. The one spatial expanding distance is the radial direction in an isotropic, homogeneous universe. The dots represent comoving distances, as well as galaxies. The comoving system is a spatially expanding system with a common clock, which is useful in solving the GR field equation. In GR, one such universe, which we will be simulating here, is the flat Robertson–Walker (FRW) universe, which begins with a singularity at time zero, and expands forever with a continually decreasing expansion rate. The usual measure of distance to a point on this system, the proper distance, equals the comoving distance multiplied by the expansion factor for the universe as a whole. In the model presented here, a fixed, stationary ruler adjacent to the expanding rubber band measures this distance. The left end of the rubber band is attached to the left end of the ruler, and the right end of the rubber band is stretched. The spatial coordinates on the rubber band grid coincide with the comoving system on the ruler at time t_0, called the epoch, the time that we now live in.

In this epoch t_0, we can observe light that was emitted early in the universe, even though the galactic material that emitted that light is now far from us. Light, initially emitted close to us, moved away from us early in the expansion, carried by the expansion itself. As the expansion slowed, the light reversed direction and came toward us. Eventually it increased its speed of approach to c, the normal velocity of light, when it reached us. That light's spectrum is shifted along with the expansion of the grid of the universe. We discover both how far out radially we can see in this universe, and the classical dynamics of the model universe. Some calculations are performed in FRW coordinates, although the speed of light is not constant in FRW coordinates. Other calculations are done using the comoving system, where a comoving time is defined so that the comoving speed of light is a constant. An Appendix discussing the GR concepts of proper time and distance is included for readers familiar with GR.

II. THE BASICS OF THE MODEL

Let x represent proper distance as measured by the fixed ruler. The ruler is laid out on a stretched rubber band at some time, t_0, on which the comoving distance, x_c, is marked on the rubber band. When the rubber band is stretched more or contracted, the distances between those ruler marks stretch or contract with it, so that a mark on the rubber band is always located at the same comoving distance. If the end of the rubber band is stretched at a constant rate, then, as time passes, a specific dot on the rubber band moves farther away from the origin, according to[3]

$$x(t) = \frac{t}{t_0} \cdot x_0 = \frac{t}{t_0} \cdot x_C,$$ (2.1)

where $x(t)$ is the dot's proper location, x_0 is the dot's location along the ruler in this epoch, and where x_C, the comoving value of that specific dot on the rubber band, never changes as time proceeds. Proper distance, x, and comoving distance, x_C, coincide in this epoch, t_0. The time t will turn out to be a universal time, just as most people visualize it. It is also proper time in FRW coordinates. GR readers see the Appendix. The non-GR reader is advised that GR deals with clocks whose reading depends on gravity and velocity. However, there is no need to deal with such subtleties in this paper, because the time t does turn out to be a universal time, as already mentioned in Sec. I.

One basic assumption must be built into the model. We will assume that light propagates through the rubber band grid at constant physical speed c, whether or not the grid is expanding, so that the proper speed of light, relative to the ruler, is

$$\left(\frac{dx}{dt}\right)_{\text{proper}} = \left(\frac{dx}{dt}\right)_{\text{proper, relative to grid}}$$
$$+ \left(\frac{dx}{dt}\right)_{\text{proper, of grid relative to ruler}}.$$ (2.2)

The velocity addition expressed by Eq. (2.2) is a nonrelativistic assumption, which should be familiar from everyday experience, but which will still give many of the GR features. Substituting $(dx/dt)_{\text{proper, relative to grid}} = c$ and $(dx/dt)_{\text{proper, of grid relative to ruler}} = x/t$ from the derivative of Eq. (2.1) gives[4]

$$\frac{dx}{dt} = \pm c + \frac{x}{t} \tag{2.3}$$

as the basic equation for radial light propagation. In Eq. (2.3) the positive sign corresponds to outward traveling light, and the negative sign corresponds to inward traveling light. Here we have a basic result of GR—a variable speed of light, understood quantitatively without the aid of GR or even special relativity.

In Eq. (2.1), (t/t_0) is the factor by which the universe has expanded since the present epoch. This factor can easily be generalized to a more realistic GR model by replacing the (t/t_0) scale factor[5] by $(a(t)/a_0)$, where $a(t)$ is the cosmic scale factor for a realistic GR solution, and $a_0 = a(t_0)$. For a FRW universe,[6] $(a(t)/a_0) = (t/t_0)^{2/3}$. Equations (2.1) and (2.3) are replaced by the more realistic

$$\text{(Galaxies)} \quad x(t) = \left(\frac{t}{t_0}\right)^{2/3} \cdot x_0 = \left(\frac{t}{t_0}\right)^{2/3} \cdot x_c \tag{2.4}$$

and[7]

$$\text{(Light)} \quad \frac{dx}{dt} = \pm c + \frac{2x}{3t}. \tag{2.5}$$

The general solution of Eq. (2.5) is

$$\text{(Light)} \quad x = A(ct)^{2/3} \pm 3ct, \tag{2.6}$$

where A is a constant of integration to be evaluated depending on the particular initial conditions.

The proper time t is a global time, useful not only for the fixed ruler, but also for the expanding grid in the sense that all galaxies on the grid can agree on the same time, and that the travel time of light is independent of the direction of light travel. To understand this happening, suppose that we at the origin emit a light signal at time t_0, and that it propagates radially outward with variable light proper speed as determined by Eq. (2.5). A dot galaxy, which, at this emission time t_0, is at $x = x_0$, receives this signal somewhat later. When is the signal received, and where is the galaxy then located? Evaluating Eq. (2.6) at emission time gives $A = -3(ct_0)^{1/3}$. This light is received by the galaxy when $x_{\text{Galaxy}} = x_0 \cdot (t/t_0)^{2/3}$ is equal to $x_{\text{Light}} = 3ct[1 - (t_0/t)^{1/3}]$, giving

$$t_{\text{receive}} = t_0 \cdot \left(1 + \frac{x_0}{3ct_0}\right)^3, \quad x_{\text{receive}} = x_0 \cdot \left(1 + \frac{x_0}{3ct_0}\right)^2. \tag{2.7}$$

On the other hand, suppose the distant galaxy located at x_0 at time t_0 emits a light signal at the same time t_0. Equation (2.6), with A evaluated to $A = (x_0 + 3ct_0)/(ct_0)^{2/3}$, shows that we, at the origin, receive this signal at the same time as the time in Eq. (2.7) when our signal was received by the distant galaxy, and that the distant galaxy was at the same location, Eq. (2.7), as when it received our signal. Therefore, a universal time t can unambiguously be assigned to every point on the expanding grid, so that, if t is used, the elapsed time is independent of the direction in which the light traveled. Hence, events can be universally described on the same universal time scale, with no problem of lack of simultaneity, such as that encountered in special relativity. This universal time is a key feature of the full GR treatment of comoving coordinate systems as described in the FRW metric. However, although proper distance and proper time are associated together in this article, this does not imply that proper time is

what is measured on clocks on the fixed ruler measuring proper distance, but actually is measured on clocks fixed to the rubber band. See the Appendix.

A useful comoving time, t_C, can be assigned to accompany the comoving distance defined in Eq. (2.4). This comoving time is a mathematical convenience to make this model simulate FRW, but not the time on clocks attached to the moving rubber band, where proper time is still t. See the Appendix. Substituting x from Eq. (2.4) into Eq. (2.5) gives

$$\left(\frac{t}{t_0}\right)^{2/3} \frac{dx_C}{dt} = \pm c. \tag{2.8}$$

The comoving time,[8] t_c, can now be defined, so that Eq. (2.8) becomes the constant light speed equation, $dx_C/dt_C = \pm c$, in terms of comoving time and comoving distance. The comoving time, t_C, is found from the chain rule to be

$$t_C = 3t_0 \left[\left(\frac{t}{t_0}\right)^{1/3} - 1\right] + t_0, \tag{2.9}$$

where the times have been evaluated so that both proper and comoving times agree at t_0. The comoving light equation is then

$$\frac{dx_C}{dt_C} = \pm c, \tag{2.10}$$

so that light travels at *constant* speed using pure comoving independent variables.

Although either set of independent variables can be used for the same calculation, one set is often much more efficient than the other. Using comoving variables (x_C, t_C) it is immediately obvious that communication times are independent of the direction of light travel, since the speed of light is constant, and hence a global comoving time exists. This same result was obtained earlier using (x, t) variables, but it required two calculations of relative travel time, each being a relative speed problem with nonconstant speeds.

III. LOOK BACK, RED SHIFT, AND HORIZON

Our ordinary Earth experience might lead us to conclude that we can see no farther than halfway back to the time of the singularity.[9,10] Consider matter speeding outward from the origin at the fastest possible speed, nearly the speed of light, emitting radiation at a time that is half the present age of the universe. That light requires essentially the same time to travel back to us at the origin. We see that light as it was at half the age of the universe. Of course, light reaching us now from slower moving matter (e.g., a nearby galaxy) would have been emitted more recently. Therefore, it may seem that we should be able to see no farther back in time than half the way back to the singularity.

Using our rubber band model universe, we get a different result, which is confirmed by an accurate GR treatment[11] using universal time. Our model explains that different result as follows. Consider the event of light emission at proper location and time $(x_{\text{emit}}, t_{\text{emit}})$, so early in the history of the universe that the outward expansion proper speed, $2x_{\text{emit}}/3t_{\text{emit}}$, of the grid was greater than the inward proper speed, c, of the light traveling through the grid, or $2x_{\text{emit}}/3t_{\text{emit}} > c$. That inward traveling light is actually carried outward (proper distance) for a while by the faster moving grid. As time proceeds, the light travels inward along the

grid to regions of slower grid expansion.[12] There comes some time, t_{max}, when this light travels inward through the grid exactly as fast as the grid expands outward. At this moment, the inward light actually stands still (proper distance and time). It is at its maximum proper distance, x_{max}, from us at the origin. Subsequently the outward proper speed of the grid becomes less than the constant inward proper speed of the light through the grid, and the light begins to travel (proper speed) toward us. Finally, the light reaches us at the origin at our present proper time, t_0, in this epoch. Applying standard max–min techniques to Eq. (2.6) yields

$$t_{max} = \frac{8}{27}\left(1 + \frac{x_{emit}}{3ct_{emit}}\right)^3 t_{emit}, \quad x_{max} = \frac{4}{9}\left(1 + \frac{x_{emit}}{3ct_{emit}}\right)^3 ct_{emit}. \tag{3.1}$$

Equations (3.1) are valid only if the inward light is moving physically outward at the time of emission, which means that

$$t_{emit} < \frac{2x_{emit}}{3c} \tag{3.2}$$

from Eq. (2.5). Inequality (3.2) is equivalent to the reasonable initial condition that $t_{emit} < t_{max}$, when t_{max} is given by Eq. (3.1).

As a numerical example, consider radiation emitted at $t_{emit} = t_0/1000$, and suppose that this ancient radiation reaches us at the origin now, at epoch t_0. Equation (2.6) yields

$$x_{emit} = 3ct_0\left[\frac{(t_0/t_{emit})^{1/3} - 1}{(t_0/t_{emit})}\right] = 0.027ct_0 \tag{3.3}$$

as the proper distance at the emission time $t_0/1000$. Equation (3.1) yields

$$t_{max} = \tfrac{8}{27}t_0 = 0.30t_0, \quad x_{max} = \tfrac{4}{9}ct_0 = 0.44ct_0. \tag{3.4}$$

This radiation emitted at proper time $0.001t_0$ had proper distance of only $0.024ct_0$ at the time of emission. The inward traveling light was actually dragged outward (proper distance) until it slowed to an instantaneous stop at a time of 30% the present proper age of the universe at a proper distance of $0.44ct_0$. It required the last 70% of the age of the universe to plow its way inward, through the ever-slowing, outward-expanding grid, to reach us now at epoch t_0. The matter that emitted the early light is now at a proper distance of

$$x_0 = 3ct_0 \cdot \left[1 - \left(\frac{t_{emit}}{t_0}\right)^{1/3}\right] = 2.7ct_0. \tag{3.5}$$

Equations (2.4) and (3.3) were combined to solve for x_0 above. According to Eq. (3.5), the galaxy is now so far away that it had to travel at an average speed of nearly three times the speed of light. This result, impossible in special relativity, is understandable here in terms of our model universe.

To determine the red shift of the ancient light reaching us now at epoch t_0, it is advantageous to use comoving time and distance because (a) the comoving speed of light is constant, (b) the emitting galaxy is motionless, and (c) we at the origin are motionless. Therefore there is no comoving red shift, so

$$dt_{C,receive} = dt_{C,emit}, \tag{3.6}$$

where $dt_{C,emit}$ and $dt_{C,receive}$ are the comoving periods. Differentiation of Eq. (2.9) converts comoving periods to proper periods to give

$$dt_{Proper,receive} = \left(\frac{t_{Proper,receive}}{t_{Proper,emit}}\right)^{2/3} dt_{Proper,emit}. \tag{3.7}$$

Since the local speed of light is c both at the emitter at emission time and at the receiver at reception time, multiplying Eq. (3.7) by c gives the wavelengths $\lambda = cdt$,

$$\lambda_{Proper,receive} = \lambda_{Proper,emit}\left(\frac{t_{Proper,receive}}{t_{Proper,emit}}\right)^{2/3}. \tag{3.8}$$

Comparing Eq. (3.8) with Eq. (2.4) shows that a proper light wave expands along with the expansion of the universe as a whole. This is the same result that is obtained from the full GR treatment of FRW.

If the proper light horizon is defined as the greatest proper distance of light, emitted at the origin since the singularity, then the proper horizon is $3ct_0$, from Eq. (2.6) with the constant A evaluated so that $x=0$ when $t = t_{emit} \rightarrow 0$.

Suppose one defines the comoving light horizon as the comoving distance through which light has traveled since the beginning of the universe at proper time $t=0$. From Eq. (2.9), the comoving time, $t_c = -2t_0$, corresponds to $t=0$ proper time. This light expanding out to proper time $+t_0$ corresponds to comoving time $+t_0$, since proper and comoving times are defined to be identical in the present epoch. Hence, elapsed comoving time is $3t_0$. Light has traveled at a constant comoving speed c for a comoving time $3t_0$ since the singularity. The comoving horizon is $3ct_0$.

The proper particle horizon is defined as the present distance of the objects that emitted the oldest light that we can presently observe.[13] Equation (3.5), in the $(t_{emit}/t_0) \rightarrow 0$ limit, gives a proper particle horizon of $3ct_0$. The comoving particle horizon would also be $3ct_0$, since comoving and proper distances coincide at t_0.

IV. HUBBLE DISTANCES AND MODEL DYNAMICS

Cosmological distances are measured using the Hubble relation, stating that the distance to a galaxy is proportional to its red shift, as long as the distances are not too great. The Hubble parameter, H, is defined to be the ratio[14]

$$H(t) = \frac{\dot{a}(t)}{a_0}, \tag{4.1}$$

where $a(t)$ is the expansion parameter of the universe, given by Eq. (2.4) for our model universe. Thus, for our spatially flat universe,

$$H(t) = \frac{2}{3t}, \tag{4.2}$$

a well-known result[15] for the FRW universe in GR. There can actually be several different Hubble parameters. $H_0 = 2/3t_0$ in our epoch, although it was larger at earlier epochs. Recalling that a galaxy's speed is $\nu = 2x/3t$,

$$x_0 = \left(\frac{1}{H_0}\right)\nu_0 \tag{4.3}$$

would give the distance to a galaxy in this epoch, if its speed were known in this epoch. V_0 is determined from the red shift, Z, defined[16] by

$$Z = \frac{\lambda_0 - \lambda_{\text{emit}}}{\lambda_{\text{emit}}}, \tag{4.4}$$

giving

$$x \approx \frac{c}{H_0} \cdot Z \tag{4.5}$$

for relatively nearby galaxies. λ_0 is the normal wavelength in our epoch. For more distant galaxies, the travel time of light is a significant fraction of the age of the universe. Those more distant galaxies, whether the Hubble relation gives the galactic distances and velocities as they were when they emitted their light, or as they are now in the present epoch, can be resolved by deriving the exact Hubble relation for the rubber band model universe without recourse to GR or special relativity. Combining Eqs. (2.4) and (3.3) gives

$$\left(\frac{t_{\text{emit}}}{t_0}\right)^{1/3} = 1 - \frac{x_0}{3ct_0}. \tag{4.6}$$

This equation, along with Eqs. (4.4) and (3.8), yields

$$x_0 = 3ct_0\left(1 - \frac{1}{\sqrt{1+Z}}\right) \tag{4.7}$$

as the location, x_0, in this epoch, t_0, of the matter that emitted the earlier light, whose light we receive now at time t_0. Using Eqs. (2.4) and (4.6) in Eq. (4.7) gives

$$x_{\text{emit}} = \frac{3ct_0}{(1+Z)} \cdot \left(1 - \frac{1}{\sqrt{1+Z}}\right) \tag{4.8}$$

as the location of the emitting matter when it emitted its radiation. Eliminating x_0 between Eq. (4.6) and Eq. (4.7) gives

$$t_{\text{emit}} = \frac{t_0}{(1+Z)^{3/2}} \tag{4.9}$$

as the time at which the radiation was emitted.

As in Eq. (2.5), the proper speed of a dot galaxy on the expanding grid is

$$v = \frac{2x}{3t}, \tag{4.10}$$

so

$$H_{\text{emit}} = \frac{v_{\text{emit}}}{x_{\text{emit}}} = \frac{2}{3t_{\text{emit}}} \tag{4.11}$$

is the Hubble parameter for the universe when the galaxy emitted the light that we observe now, at epoch t_0. Using Eqs. (4.1), (4.9), and (4.11) gives

$$H_{\text{emitted}} = H_0 \cdot (1 + Z_{\text{measured now}})^{3/2} = \frac{2(1 + Z_{\text{measured now}})^{3/2}}{3t_0}. \tag{4.12}$$

The advantage offered by Eq. (4.12) is that it can be used in conjunction with Eq. (4.11) to yield a value for H_0 that is the same for all galaxies, whereas Eq. (4.11) yields different values for H_{emit}, as each galaxy has its own emission time

that allows its light to arrive at our location in the present epoch. Note that it is

$$H_0 = \frac{H_{\text{emit}}}{(1 + Z_{\text{measured now}})^{3/2}} \tag{4.13}$$

that is the Hubble constant for our epoch, although it is the various values of the individual and different H_{emit}'s that are actually measured using the galactic red shifts.

The following two numerical examples illustrate the process. First, consider a red shift of $Z = 99$, illustrative of the background radiation. Then $t_{\text{emit}} = t_0/1000$ as in our earlier example. Then $x_0 = 3ct_0(1 - 1/\sqrt{1+99}) = 2.7ct_0$, $x_{\text{emit}} = x_0/(1+99) = 0.027ct_0$, $v_0 = 2x_0/3t_0 = 1.8c$, $v_{\text{emit}} = v_0(t_0/t_{\text{emit}})^{1/3} = 18c$, $H_{\text{emit}} = v_{\text{emit}}/x_{\text{emit}} = 667/t_0$, and $H_0 = H_{\text{emit}}/(1+99)^{3/2} = 0.667/t_0$. As a somewhat realistic example of a quasar, begin with a measured red shift of $Z = 3$. Then $t_{\text{emit}} = t_0/(1+3)^{3/2} = 0.125t_0$, $x_0 = 3ct_0(1 - 1/\sqrt{1+3}) = 1.5ct_0$, $x_{\text{emit}} = x_0/(1+3) = 0.375ct_0$, $v_0 = 2x_0/3t_0 = c$, $v_{\text{emit}} = v_0(t_0/t_{\text{emit}})^{1/3} = 2c$, $H_{\text{emit}} = v_{\text{emit}}/x_{\text{emit}} = 5.33/t_0$, and $H_0 = H_{\text{emit}}/(1+3)^{3/2} = 0.667/t_0$. Note that H_0 is the same in either case, although the two values for H_{emit} are quite different. Also note that the quasar was traveling at a proper speed of twice the proper speed of light when it emitted its radiation that we see now, and that it has slowed down to its present speed of exactly the speed of light. These are astounding statements for students accustomed to special relativity, but are simple results from the rubber band model universe. Note also that the quasar is now at a proper distance 50% greater than a beam of light could travel at constant speed c during the entire evolution of the universe. Such scenarios occur in the rubber band model and in the actual general relativistic FRW universe, because the proper speed of light changes during the evolution of the universe.

The dynamic of this rubber band model and the FRW universe, which it imitates, is realized by evaluating the total mechanical energy of the expanding rubber band dot galaxies. Let ΔM be the mass of a comoving expanding spherical shell and ρ be the uniform density of matter within this shell at some time during the expansion, in the Newtonian sense that mass is independent of speed. The classical total mechanical energy, ΔE, of the expanding shell is the sum of the classical (Newtonian) gravitational potential energy of the shell due to the matter within it plus the classical (Newtonian) kinetic energy of this expanding shell. Classically, the total mass within the shell is constant, and is $(4/3)\pi x_0^3 \rho_0$, where x_0 is the proper radius of the shell in the present epoch. Then, using Eqs. (2.5) and (4.10), ΔE at any time can be written in terms of parameters in the present epoch as

$$(\Delta E)_t = (\Delta E)_0 \cdot \left(\frac{t_0}{t}\right)^{4/3},$$

$$(\Delta E)_0 = \Delta M \cdot \left\{\frac{1}{2}v_0^2 - \frac{4\pi G\rho_0 x_0^2}{3}\right\}. \tag{4.14}$$

Conservation of total classical mechanical energy during the expansion requires that

$$(\Delta E)_t = (\Delta E)_0 = 0, \tag{4.15}$$

giving

$$\rho_0 = \frac{3}{8\pi G} \cdot H_0^2, \quad \rho(t) = \rho_0 \cdot \left(\frac{t_0}{t}\right)^2 \tag{4.16}$$

as the mass density of galaxies in the universe in the present epoch. Since the change in ρ is entirely due to the expansion of the universe,

$$\rho(t) \cdot x^3(t) = \rho(t_0) \cdot x^3(t_0). \quad (4.17)$$

Therefore, the rubber band model universe has the dynamics of a classical, isotropic, homogeneous universe, expanding so that its total mass and energy are separately conserved, with total energy always zero. It is only this zero mechanical energy universe that is imitated by the rubber band model. Nonzero mechanical energy universes evolve differently. Those with positive total mechanical energy expand toward infinity faster than our model. Those with negative total mechanical energy expand slower, and eventually reach a maximum size [in the sense that the expansion factor $a(t)$ for those universes reaches a maximum for some value of t] and then collapse upon themselves.[17,18]

V. SUMMARY

The rubber band model universe is used to explain, compare, and contrast many GR features of a FRW expanding universe, without ever resorting to GR itself. The model gives a speed of light that is variable, even reversing in direction, because of the expansion of the rubber band grid. It defines proper time, proper distance, and comoving distance to be the same as in the FRW universe. It defines a comoving time so that the comoving speed of light is constant. The existence of a global proper time, in the sense that transmission of light signals is independent of the direction of the signal, is verified using both proper and comoving variables. The look-back paradox is explained and resolved with a numerical example. Light speeds much greater than c are encountered and understood in the model universe. The red shift formula shows that light expands along with the grid of the model universe. The light horizon and the particle horizons, both proper and comoving, are derived. The Hubble parameter is derived in terms of redshift, and the Hubble parameter at emission is distinguished from the Hubble parameter at reception. Two numerical examples are presented. It is shown that this model universe corresponds to both a classical isotropic, homogeneous universe expanding with zero total energy and constant mass, and to the FRW universe.

APPENDIX: TIME AND DISTANCE IN GR

This section reviews some of the GR details relating to the concepts of time and distance used in the simplified model of this article. This Appendix may be skipped by those readers not acquainted with GR.

1. Definitions

The definition of proper distance is based on the concept that the proper distance between two events is measured in a coordinate frame in which those two events occur simultaneously. The definition of proper time is based on the concept that the proper time between two events is the time measured in a coordinate frame in which those two events occur at the same location. Expressing these concepts mathematically, proper distance and proper time are defined in terms of the metric expression for the squared differential interval, $(ds)^2$, as follows. The differential proper distance, dx_p, satisfies[19] $(ds)^2 = -0 + (dx_p)^2$ at constant time. The differential proper time, dt_p, satisfies[20] $(ds)^2 = -c^2(dt_p)^2 + 0$ at constant location.

2. Comoving coordinates

The FRW metric for a flat expanding universe is

$$(ds)^2 = -c^2(dt)^2 + \left(\frac{a(t)}{a_0}\right)^2 [(dr)^2 + r^2(d\theta)^2$$

$$+ r^2 \sin^2 \theta (d\varphi)^2] \rightarrow -c^2(dt)^2 + \left(\frac{a(t)}{a_0}\right)^2 (dr)^2,$$

$$(A1)$$

where the arrow recognizes that this article deals only with radial motion. In this metric, r is the comoving distance, x_C, of the rubber band model, and t is the time t of the rubber band model. Setting $dt=0$ and $(ds)^2 = (dx_p)^2$ in Eq. (A1) gives the differential proper distance, dx_p, as $dx_p = (a(t)/a_0) \cdot dr \rightarrow (t/t_0)^{2/3} \cdot dx_C$, which, for constant t, integrates[21] to x in Eq. (2.4). Setting $dr=0$ and $(ds)^2 = -c^2(dt_p)^2$ in Eq. (A1) shows that $t_p = t$, to within an additive constant here taken to be zero. Thus, using comoving coordinates, proper distance and proper time are, respectively, the x and t of this article.

3. Fixed ruler coordinates

Clocks attached to the fixed ruler are at a constant $R = a(t) \cdot r$. Following Fletcher[22] and Gautreau,[23] for such a clock, differentiating $R = a(t) \cdot r$ gives $0 = a(t)dr + \dot{a}(t)rdt$, where the dot indicates differentiation with respect to t. The metric for constant R becomes $(ds)^2 = -c^2[1 - (RH(t)/c)^2](dt)^2$, not simply $(ds)^2 = -c^2(dt)^2$, as it does for fixed r in comoving coordinates. Here $H(t)$ is the Hubble variable, which for FRW is $H(t) = \dot{a}(t)/a(t) = 2/3t$. A change of variables from (R,t) to (R,t_p) is required to bring the metric into the form $(ds)^2 = -c^2(dt_p)^2$ for constant R. Integration gives $t_p(R,t) = t \cdot \sqrt{1 - (2R/3ct)^2} + 2R \cdot \sin^{-1}(2R/3ct)/3c + f(R)$, where $f(R)$ is a function of only R, and where $\partial t_p(R,T)/\partial R$ is chosen to eliminate a cross product metric term. Both Fletcher and Gautreau discuss the complete transformation, but such is beyond the scope of this article. This t_p is the proper time measured in fixed ruler coordinates in terms of R and t. Note that only at $R=0$ is $t_p = t$ [if $f(0)$ is properly chosen], but at all other values of R, the time t_p depends on both R and t in a nontrivial way. Hence, t works well to measure time at $R=0$, but t is neither a proper time nor a universal time for clocks attached to the fixed ruler, where distance is measured by R.

Holding proper time constant, the metric becomes $(ds)^2 = (dR)^2/[1 - (RH/c)^2]$, showing that R is not the proper distance when using proper time, t_p, as the measure of time. However, when using the time t as defined in the rubber

band model, constant t yields $(ds)^2 = (dR)^2$ in Eq. (A1), so that R is proper distance when using (R,t) to measure distance and time.

4. Comoving time

The FRW metric of Eq. (A1) can be manipulated into a form having a constant speed of light by factoring out $(t/t_0)^{4/3}$, giving $(ds)^2 = (t/t_0)^{4/3} \cdot [(t_0/t)^{4/3} c^2 (dt)^2 + (dr)^2]$, and making the change of time variable from time t to comoving time t_C defined by Eq. (2.9), resulting in a FRW transformed metric,

$$(ds)^2 = \left[\frac{1}{3}\left(\frac{t_C}{t_0} + 2\right)\right]^4 \cdot [-c^2(dt_C)^2 + (dx_C)^2], \qquad (A2)$$

expressed in terms of the article's comoving distance and time. With the metric in this form, it is obvious that light propagation, defined by $ds = 0$, is constant speed, as expressed in Eq. (2.10). It is also obvious from this metric form that (x_C, t_C) do not constitute proper distance and time, even though these coordinates may be useful for calculations, since the speed of light is constant in terms of them.

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[1]The three spatial dimensional model for constant expansion is called the Milne model. See Abraham N. Silverman, ''Resolution of a cosmological paradox using concepts from general relativity,'' Am. J. Phys. **54**, 1092–1096 (1986), especially p. 1095.

[2]See W. M. Stuckey, ''Kinematics between comoving photon exchangers in a closed matter-dominated universe,'' Am. J. Phys. **60**, 554–560 (1992), where Stuckey emphasizes and clearly illustrates the concept of proper distance.

[3]One arrives at Eq. (2.1) by noting that, for a dot galaxy traveling at constant speed V, $x = Vt$ for any time t and, at time t_0 in particular, $x_0 = Vt_0$. Eliminating V between these two distance equations gives Eq. (2.1).

[4]The one spacial dimension FRW metric is given by $(ds)^2 = -c^2(dt)^2 + [a(t)/a_0]^2 (dx_c)^2$, where $a(t)/a_0 = t/t_0$ is used so far in this article, although $a(t)/a_0$ will shortly be replaced by the actual FRW $(t/t_0)^{2/3}$. Hence light travel is given by $dx_c/dt = \pm c/(a(t)/a_0)$. Substituting $x_c = x/(a(t)/a_0)$ from Eq. (2.1) gives $dx/dt = \pm c + x/t$ for light, which is Eq. (2.3), arrived at in this article in a different, non-GR way.

[5]Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, *Gravitation* (W. H. Freeman, New York, 1973), pp. 721, 722.

[6]Steven Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972), p. 483, Eq. (15.3.14); Joseph Silk, *The Big Bang* (W. H. Freeman, San Francisco, 1980), p. 335.

[7]The grid speed that is added to c in Eq. (2.5) is found by differentiating the logarithm of Eq. (2.4) to give $(dx/dt)_{\text{proper, grid relative to ruler}} = 2x/3t$.

[8]For emphasis, the comoving time is not the time on clocks on the rubber band, but merely a useful mathematical time to accompany the comoving distance.

[9]See Robert C. Fletcher, ''Light exchange in an expanding universe in fixed coordinates,'' Am. J. Phys. **62**, 648–656 (1994). Fletcher presents an elegant and detailed calculation for this paradox for a matter-dominated universe of three spatial dimensions.

[10]Jayant V. Narlikar, ''Spectral shifts in general relativity,'' Am. J. Phys. **62**, 903–907 (1994), especially Eq. (1).

[11]Robert C. Fletcher, ''Light exchange in an expanding universe in fixed coordinates,'' Am. J. Phys. **62**, 648–656 (1994).

[12]The decrease in the overall rate of grid expansion as time proceeds hastens the slowdown of light effect.

[13]Steven Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972), p. 489.

[14]Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, *Gravitation* (W. H. Freeman, New York, 1973), p. 709, Eq. (10).

[15]Steven Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972), p. 483, Eq. 15.3.15; Silk, *ibid.*, p. 335.

[16]G. Börner, *The Early Universe* (Springer-Verlag, Berlin, 1988), p. 7.

[17]In GR, Eqs. (4.13)–(4.15) arise from a solution of the GR tensor field equations, where the only mass-energy for the mass-energy tensor is that of the mass of the galaxies. Gravitational potential energy is not a part of this tensor setup; it is incorporated in the metric one uses for GR. The limitations of Eqs. (4.12)–(4.15) occur in GR as the representation of a flat RW universe, often chosen by assigning a value of $k = 0$ to the more general structure of the RW interval,

$(ds)^2 = -c^2(dt)^2 + [a(t)/a(t_0)]^2 \cdot \{(dr)^2/1 - k \cdot r^2 + r^2[(d\theta)^2 + \sin^2 \theta(d\phi)^2]\}$

for all universe densities, as we have done in this paper. See Stephen Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972), p. 412, Eq. (14.2.1).

[18]The RW universe with $k = -1$ corresponds to our model universe with positive total mechanical energy, while the RW universe with $k = +1$ corresponds to our model universe of negative total mechanical energy.

[19]Bernard F. Schultz, *A First Course in General Relativity* (Cambridge University Press, Cambridge, 1985), p. 22, Fig. 1.14.

[20]See Ronald J. Adler, Maurice J. Bazin, and Menahem Schiffer, *Introduction to General Relativity*, 2nd ed. (McGraw-Hill, New York, 1975), pp. 122–124.

[21]Constant t must be used in the integration because $dt = 0$, according to the definition of proper distance.

[22]Robert C. Fletcher, ''Light exchange in an expanding universe in fixed coordinates,'' Am. J. Phys. **62**, 648–656 (1994).

[23]Ronald Gautreau, ''Curvature coordinates in cosmology,'' Phys. Rev. D **29** (2), 186–197 (1984); ''Newton's absolute time and space in general relativity,'' Am. J. Phys. **68**, 350–366 (2000).

STRING THEORY

String theory is an example of science being driven by fashion. And I have mixed feelings about it. Some of the mathematical notions which people associate with string theory are very appealing. But just because they are appealing doesn't mean that they are right.

Roger Penrose, New York Times, 19 January 1999, p. D3.