Open access • Journal Article • DOI:10.1016/S0764-4442(99)80288-2

# A simple definition for locally compact quantum groups — Source link

Johan Kustermans, Stefaan Vaes

**Institutions:** University College Cork, Katholieke Universiteit Leuven

**Published on:** 15 May 1999 - Comptes Rendus De L Academie Des Sciences Serie I-mathematique (Elsevier Masson)

**Topics:** Locally compact quantum group, Locally compact group, Pontryagin duality, Quantum t-design and Haar measure

# A simple definition for locally compact quantum groups

*C.R. Acad. Sci., Paris, Sér. I* **328** (10) (1999), 871–876.

Johan Kustermans, Department of Mathematics, University College Cork, Western Road, Cork, Ireland, e-mail : johank@ucc.ie

Stefaan Vaes[1], Department of Mathematics, KU Leuven, Celestijnenlaan 200B, Leuven, Belgium, e-mail : Stefaan.Vaes@wis.kuleuven.ac.be

**Abstract**

In this Note we propose a simple definition of a locally compact quantum group in reduced form. By the word 'reduced' we mean that we suppose the Haar weight to be faithful, and hence we define in fact arbitrary locally compact quantum groups represented on the $L^2$-space of the Haar weight. We construct the multiplicative unitary associated with our quantum group. We construct the antipode with its polar decomposition, and the modular element. We prove the unicity of the Haar weights, define the dual and prove a Pontryagin duality theorem.

## 1 Introduction

Following the common paradigm that C$^*$-algebras are quantized locally compact spaces, Woronowicz proposed to use the C$^*$-algebra framework to define locally compact quantum groups ([14]). The most general definition of a locally compact quantum group available up to now is formulated in the von Neumann algebra framework by Masuda and Nakagami ([7]) and a C$^*$-version of this was announced by Masuda, Nakagami and Woronowicz in some lectures ([8]). There is however an objection to their definition: they propose a long list of axioms and hence they suppose in fact a priori a lot of the nice properties one would like to prove from a simpler definition. Nevertheless they are able to obtain the dual and to prove it satisfies again all the axioms. That is their major result.

In this Note we will present a simple definition of a reduced C$^*$-algebraic quantum group and we show that all the axioms of [7] hold in this case. Following [3] we also construct the dual and prove a Pontryagin duality theorem. The fact that we assume the Haar weight to be faithful – that is what 'reduced' means – is not essential. All the theories of Kac algebras over multiplicative unitaries to the Woronowicz algebras of [7] are reduced as well. The reduced case is just a form in which an arbitrary locally compact quantum group comes to us when represented on the $L^2$-space of its Haar weight.

Because we think an existence proof of a Haar weight is still far away, our definition seems to be the simplest one can hope for at the moment.

In this Note we give an overview of our results, for more details and proofs we refer to [5].

## 2 Conventions

Our results will depend heavily on the theory of weights on C$^*$-algebras. For the usual conventions we refer to [2]. By a proper weight we will mean a lower-semicontinuous, densely defined weight which is non-zero. All weights in this Note will be proper. Let $\varphi$ be a proper weight on a C$^*$-algebra $A$. Recall the following notation introduced by Combes:

$$\mathcal{G}_\varphi = \{\, \alpha\,\omega \mid \alpha \in ]0,1[, \omega \in A^*_+, \omega \leq \varphi \,\}.$$

[1]Research Assistant of the Fund for Scientific Research – Flanders (Belgium) (F.W.O.)

Then $\mathcal{G}_\varphi$ is directed, when equiped with the usual order on $A^*_+$. We have $\varphi(x) = \lim(\omega(x))_{\omega \in \mathcal{G}_\varphi}$ for all $x \in A^+$. When $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ is a GNS-construction for $\varphi$ we can define for every $\omega \in \mathcal{G}_\varphi$ a unique element $\tilde{\omega} \in \pi_\varphi(A)''_*$ such that $\tilde{\omega}\,\pi_\varphi = \omega$. Then we can define for $x \in (\pi_\varphi(A)'')^+$

$$\tilde{\varphi}(x) = \lim\big(\tilde{\omega}(x)\big)_{\omega \in \mathcal{G}_\varphi}.$$

So $\tilde{\varphi}$ is a normal and semi-finite weight on the von Neumann algebra $\pi_\varphi(A)''$. We will call our proper weight $\varphi$ approximately KMS when $\tilde{\varphi}$ is faithful. We will call $\varphi$ KMS when there exists a norm continuous one-parameter group $\sigma$ of automorphisms of $A$ such that

- $\varphi\,\sigma_t = \varphi$ for all $t \in \mathbb{R}$.
- $\varphi(a^*a) = \varphi\big(\sigma_{\frac{i}{2}}(a)\sigma_{\frac{i}{2}}(a)^*\big)$ for all $a \in D(\sigma_{\frac{i}{2}})$,

where $D(\sigma_{\frac{i}{2}})$ denotes the domain of the analytic continuation of $\sigma$ in $\frac{i}{2}$. This is not the usual definition of the KMS-property, but equivalent with it, as the first author proves in [4]. We will call $\sigma$ a modular group for $\varphi$ and this is uniquely determined when $\varphi$ is faithful. Note that a KMS weight is approximately KMS.

For a proper weight $\varphi$ it is possible to define the slice map $\iota \otimes \varphi$ in such a way that it becomes a useful object. We refer to [6] and [5]. When $\varphi$ is an approximate KMS weight with GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ it is possible to define a canonical GNS-construction $(H_\varphi \otimes H_\varphi, \pi_\varphi \otimes \pi_\varphi, \Lambda_\varphi \otimes \Lambda_\varphi)$ for the tensor product weight $\varphi \otimes \varphi$.

We will denote with $M(A)$ the multiplier algebra of a C$^*$-algebra $A$. When $\Delta : A \rightarrow M(A \otimes A)$ is a non-degenerate $^*$-homomorphism we will call a proper weight $\varphi$ on $(A, \Delta)$ left invariant (resp. right invariant) when

$$\varphi((\omega \otimes \iota)\Delta(a)) = \omega(1)\varphi(a) \qquad \text{resp.} \quad \varphi((\iota \otimes \omega)\Delta(a)) = \omega(1)\varphi(a)$$

for all $a \in \mathcal{M}^+_\varphi$ and $\omega \in A^*_+$.

## 3 The definition of a reduced C$^*$-algebraic quantum group

We will now give our main definition.

**Definition 3.1** *Consider a C$^*$-algebra $A$ and a non-degenerate $^*$-homomorphism $\Delta : A \rightarrow M(A \otimes A)$ such that*

- *$\Delta$ is coassociative, meaning that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.*
- *We have the following density conditions : the closed linear spans of*

  $$\{\,(\omega \otimes \iota)\Delta(a) \mid \omega \in A^*, a \in A\,\} \qquad \text{and of} \qquad \{\,(\iota \otimes \omega)\Delta(a) \mid \omega \in A^*, a \in A\,\}$$

  *are both equal to $A$.*

*Assume moreover the existence of*

- *a faithful left invariant approximate KMS weight $\varphi$ on $(A, \Delta)$,*
- *a right invariant approximate KMS weight $\psi$ on $(A, \Delta)$.*

*Then we will call the pair $(A, \Delta)$ a reduced C$^*$-algebraic quantum group.*

**Remark :** The last axiom, concerning the right invariant weight, is a bit unusual. But we could as well suppose the existence of a $^*$-antiautomorphism $\theta$ of $A$ such that $\Delta\,\theta = \chi\,(\theta \otimes \theta)\Delta$ where $\chi$ denotes the flip map on $A \otimes A$. This would give us a definition which looks like the definition of a Kac algebra, but which is even simpler. In this case $\varphi\,\theta$ would give a right invariant approximate KMS weight. Later on we will show that the existence of such a $\theta$ follows from the definition we give. It will be the unitary antipode. Hence both kinds of definitions are equivalent.

From now on we will fix weights $\varphi$ and $\psi$ on $(A, \Delta)$ satisfying the conditions of the definition. We can show that in this case $\psi$ is automatically faithful as well. We will also fix a GNS-construction $(H, \pi, \Lambda)$ for $\varphi$.

The next theorem is crucial, but the proof is involved.

**Theorem 3.2** *The weights $\varphi$ and $\psi$ are KMS.*

# 4 The multiplicative unitary

**Definition 4.1** *Define the operator $W \in B(H \otimes H)$ such that*

$$W^* \Lambda(a) \otimes \Lambda(b) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1))$$

*for all $a, b \in \mathcal{N}_\varphi$.*

It is easy to check that $W^*$ is well defined and isometric. We want to prove that $W$ is unitary, and then $W$ will be a multiplicative unitary. For the proof of the unitarity of $W$ we will walk along different paths as [3] or [7] and give an inversion formula for $W^*$.

**Proposition 4.2** *Let $a, b \in \mathcal{N}_\psi$ and $c, d \in \mathcal{N}_\varphi$. Put $x = (\psi \otimes \iota \otimes \iota)\big(\Delta_{13}(a^* c)\Delta_{12}(b)\big)$. Then $x(d \otimes 1) \in \mathcal{N}_{\varphi \otimes \varphi}$ and*

$$W^*(\Lambda \otimes \Lambda)(x(d \otimes 1)) = \Lambda(d) \otimes \Lambda\big((\psi \otimes \iota)(\Delta(a^* c)(b \otimes 1))\big).$$

Notice that we use the 'leg-numbering' notation. For instance $\Delta_{12}(b) = \Delta(b) \otimes 1$.

Once we have this formula at our disposal it is not so difficult any more – although a bit subtle – to prove that the range of $W^*$ equals $H \otimes H$. So we have

**Theorem 4.3** *The operator $W$ is unitary and satisfies the pentagonal equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$.*

Again we used the leg-numbering : $W_{12}(\xi_1 \otimes \xi_2 \otimes \xi_3) = W(\xi_1 \otimes \xi_2) \otimes \xi_3$ for instance.

# 5 The antipode and its polar decomposition

We still have to define the antipode of our reduced C$^*$-algebraic quantum group in this section, but we would like that $S$ is an antihomomorphism and that it satisfies the relation

$$S\big((\psi \otimes \iota)((x^* \otimes 1)\Delta(y))\big) = (\psi \otimes \iota)(\Delta(x^*)(y \otimes 1)).$$

When working with Kac algebras one would call this relation the 'strong right invariance'. We will define $S$ through its polar decomposition and hence it is not surprising that it would be nice to define a (possibly unbounded) operator $G$ in $H$ such that $G\,\Lambda(x) = \Lambda(S(x^*))$, in order to make the polar decomposition of $G$ afterwards. Taking into account these remarks the first step lies in the possibility of the next definition.

**Definition 5.1** *There exists a unique closed densily defined anti-linear operator $G$ in $H$ such that the elements*

$$\{ \Lambda\big((\psi \otimes \iota)(\Delta(x^*)(y \otimes 1))\big) \mid x, y \in \mathcal{N}_\varphi^* \mathcal{N}_\psi \}$$

*span a core for $G$ and such that*

$$G\,\Lambda\big((\psi \otimes \iota)(\Delta(x^*)(y \otimes 1))\big) = \Lambda\big((\psi \otimes \iota)(\Delta(y^*)(x \otimes 1))\big)$$

*for all $x, y \in \mathcal{N}_\varphi^* \mathcal{N}_\psi$.*

Remark that we immediately get that $G$ is involutive. By making the polar decomposition of $G$ we obtain operators which potentially induce the polar decomposition of the antipode.

**Definition 5.2**

- *We define $N = G^* G$, so $N$ is a strictly positive operator in $H$.*
- *We define the anti-unitary $I$ on $H$ such that $G = I N^{\frac{1}{2}}$.*

Because $G$ is involutive we get $I^2 = 1$, $I^* = I$ and $IN = N^{-1} I$. Because we expect the mapping $x \mapsto S(x^*)$ to be multiplicative the following proposition is not so surprising. Its proof however is technical and not completely straightforward.

**Proposition 5.3** *Consider $a, b \in \mathcal{N}_\psi$. Then we have*

$$\pi\big((\psi \otimes \iota)(\Delta(b^*)(a \otimes 1))\big)\, G \subseteq G\, \pi\big((\psi \otimes \iota)(\Delta(a^*)(b \otimes 1))\big).$$

Recall that the weights $\varphi$ and $\psi$ are KMS and faithful. We will denote their (unique) modular groups by $\sigma$ and $\sigma'$ respectively. For technical reasons we need a Tomita algebra:

$$\mathcal{T}_\psi = \{ a \in A \mid a \text{ is analytic w.r.t. } \sigma' \text{ and } \sigma'_z(a) \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \text{ for all } z \in \mathbb{C} \}.$$

By making, in a certain sense, the polar decomposition of the commutation in the previous proposition we obtain the following.

**Proposition 5.4** *Consider $a, b \in \mathcal{T}_\psi$. Then we have*

$$\begin{aligned} N^{-it}\, \pi\big((\psi \otimes \iota)((b^* \otimes 1)\Delta(a))\big)\, N^{it} &= \pi\big((\psi \otimes \iota)((\sigma'_{-t}(b^*) \otimes 1)\Delta(\sigma'_{-t}(a)))\big) \quad \textit{and} \\ I\, \pi\big((\psi \otimes \iota)((b^* \otimes 1)\Delta(a))\big)^* I &= \pi\big((\psi \otimes \iota)(\Delta(\sigma'_{-\frac{i}{2}}(b^*))(\sigma'_{-\frac{i}{2}}(a) \otimes 1))\big). \end{aligned}$$

Taking into account that $\pi$ is faithful we can now define the antipode with its polar decomposition.

**Definition 5.5**

- *There exists a unique norm continuous one-parameter group $\tau$ of automorphisms of $A$ such that $\pi(\tau_t(x)) = N^{-it} \pi(x) N^{it}$ for all $t \in \mathbb{R}$ and $x \in A$.*
- *There exists a unique ${}^*$-antiautomorphism $R$ of $A$ such that $\pi(R(x)) = I \pi(x)^* I$ for all $x \in A$.*
- *We define $S = R\, \tau_{-\frac{i}{2}}$ where $\tau_{-\frac{i}{2}}$ denotes the analytic continuation of $\tau$ in $-\frac{i}{2}$.*

Before stating some properties of $R, \tau$ and $S$ we want to stress that these objects still depend on the particular choice of $\varphi$ and $\psi$ we made in the beginning of the story. Later on however we will show that they are really independent of $\varphi$ and $\psi$. We will denote by $D(S)$ the domain of $S$.

**Proposition 5.6**

- *We have the following commutations*

$$\begin{aligned} \Delta\, \sigma'_t &= (\sigma'_t \otimes \tau_{-t})\Delta & \qquad \Delta\, \sigma_t &= (\tau_t \otimes \sigma_t)\Delta \\ \Delta\, \tau_t &= (\tau_t \otimes \tau_t)\Delta \end{aligned}$$

- *For all $x \in D(S)$ we have $S(x)^* \in D(S)$ and $S(S(x)^*)^* = x$.*
- *We have $\chi\,(R \otimes R)\Delta = \Delta\,R$, where $\chi$ denotes the flip map on $A \otimes A$.*

**Proposition 5.7** *For all $a, b \in \mathcal{N}_\psi$ we have $(\psi \otimes \iota)((a^* \otimes 1)\Delta(b)) \in D(S)$ and*

$$S\big((\psi \otimes \iota)((a^* \otimes 1)\Delta(b))\big) = (\psi \otimes \iota)(\Delta(a^*)(b \otimes 1)).$$

*The linear span of such elements $(\psi \otimes \iota)((a^* \otimes 1)\Delta(b))$ is a core for $S$.*

*For all $a, b \in \mathcal{N}_\varphi$ we have $(\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b)) \in D(S)$ and*

$$S\big((\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b))\big) = (\iota \otimes \varphi)((1 \otimes a^*)\Delta(b)).$$

*The linear span of such elements $(\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b))$ also gives a core for $S$.*

We can now argue that $R, \tau$ and $S$ do not depend on the particular choice of $\varphi$ and $\psi$. Because of the previous proposition we get that $S$ does not depend on $\psi$, and neither on $\varphi$. But from proposition 5.6 we get that

$$\tau_t\big((\iota \otimes \omega)\Delta(a)\big) = (\iota \otimes \omega\sigma_{-t})\Delta(\sigma_t(a))$$

for all $\omega \in A^*$, $a \in A$ and $t \in \mathbb{R}$ and hence the density condition in definition 3.1 gives that $\tau$ does not depend on $\psi$. Analogously it does not depend on $\varphi$. Because $S = R\,\tau_{-\frac{i}{2}}$ we see that $R$ does not depend on $\varphi$ or $\psi$. So we can call $(S, R, \tau)$ the antipodal triple of the reduced C$^*$-algebraic quantum group $(A, \Delta)$. We call $R$ the unitary antipode and $\tau$ the scaling group of $(A, \Delta)$.

From now on we will fix a faithful left invariant approximate KMS weight $\varphi$ on $(A, \Delta)$ and we will make a particular choice for $\psi$, namely $\psi = \varphi\,R$. Because of proposition 5.6 $\psi$ is right invariant, and it is obviously approximately KMS.

# 6 Relative invariance properties

Before we can define, in the next section, the modular element as the Radon-Nikodym derivative of the right Haar weight with respect to the left Haar weight, we will need a certain commutation between the Haar weights. To prove this we need a unicity result for our Haar weights. This is not the strongest result we can prove (cfr. section 8) but we first need this version.

The proof of the next proposition can be given by lifting everything to the same von Neumann algebra, using Radon-Nikodym there, and mimicking the proof of [3]. However also a more elementary proof is possible, see [5].

**Proposition 6.1** *Consider a left invariant proper weight $\eta$ on $(A, \Delta)$ such that there exists a number $\lambda > 0$ satisfying $\eta\,\sigma_t = \lambda^t \eta$ for all $t \in \mathbb{R}$. Then there exists a number $r > 0$ such that $\eta = r\,\varphi$.*

This proposition can be used to prove the following.

**Proposition 6.2**
- *We have the commutation $\Delta\,\tau_t = (\sigma_t \otimes \sigma'_{-t})\Delta$ for all $t \in \mathbb{R}$.*
- *The automorphism groups $\sigma, \sigma'$ and $\tau$ commute pairwise.*
- *There exists a number $\nu > 0$ such that*

$$\begin{aligned} \varphi\,\sigma'_t &= \nu^t\,\varphi & \qquad \psi\,\sigma_t &= \nu^{-t}\psi \\ \psi\,\tau_t &= \nu^{-t}\,\psi & \qquad \varphi\,\tau_t &= \nu^{-t}\,\varphi \end{aligned}$$

We call the number $\nu > 0$ the scaling constant of $(A, \Delta)$. This terminology will be justified once we will have established the uniqueness of the Haar weights in section 8. In order to satisfy the axioms of [8] one needs this scaling constant to be one. We still do not have an example in which $\nu \neq 1$, but we think the appearance of $\nu$ is very natural in this theory.

Because of the previous proposition we can define a strictly positive operator $P$ in $H$ such that $P^{it}\,\Lambda(a) = \nu^{\frac{t}{2}}\,\Lambda(\tau_t(a))$ for all $t \in \mathbb{R}$ and $a \in \mathcal{N}_\varphi$. Then one can verify that $P^{\frac{1}{2}}$ 'manages' the multiplicative unitary $W$, and hence we obtain, refering to definition 1.2 of [12], the following.

**Proposition 6.3** *The multiplicative unitary $W$ is manageable.*

# 7 The modular element

Let $(H_\psi, \pi_\psi, \Lambda_\psi)$ be a GNS-construction for $\psi$. Then we prove that there is a unique $^*$-isomorphism $\theta : \pi_\psi(A)'' \rightarrow \pi(A)''$ such that $\theta\,\pi_\psi = \pi$. Using this and the remarks in section 2 we can canonically lift $\psi$ to a weight $\tilde{\psi}$ on the von Neumann algebra $\tilde{A} = \pi(A)''$. We can also lift $\varphi$ to $\tilde{\varphi}$. From proposition 6.2 it follows that $\tilde{\varphi}\,\tilde{\sigma}'_t = \nu^t\,\tilde{\varphi}$ for all $t \in \mathbb{R}$ where $\tilde{\sigma}'$ denotes the modular group of the normal semi-finite faithful weight $\tilde{\psi}$. By the Radon-Nikodym theorem of [10], which is a generalization of [9], we obtain the existence of a strictly positive operator $\tilde{\delta}$ affiliated with $\tilde{A}$ such that $\tilde{\sigma}_t(\tilde{\delta}) = \nu^t\,\tilde{\delta}$ and $\tilde{\psi}(x) = \tilde{\varphi}(\tilde{\delta}^{\frac{1}{2}}\,x\,\tilde{\delta}^{\frac{1}{2}})$ for $x \in \tilde{A}^+$ formally. It is easy to check that

$$W^*(1 \otimes \pi(x))W = (\pi \otimes \pi)\Delta(x)$$

for all $x \in A$. Hence we can extend the comultiplication $\Delta$ on $A$ to a comultiplication $\tilde{\Delta}$ on the von Neumann algebra $\tilde{A}$ by defining $\tilde{\Delta}(y) = W^*(1 \otimes y)W$ for all $y \in \tilde{A}$. In this section we want to prove that $\tilde{\Delta}(\tilde{\delta}) = \tilde{\delta} \otimes \tilde{\delta}$. Before we can do so we need the following lemma, which is interesting on itself because it gives a serious meaning to the formula $(\iota \otimes \psi)\Delta(a) = \psi(a)\,\delta^{-1}$ which exists on the algebraic level (see [11]).

**Lemma 7.1** *Consider $a \in \mathcal{M}_{\tilde{\psi}}$ and $v, w \in D(\tilde{\delta}^{-\frac{1}{2}})$. Then $(\omega_{v,w}\bar{\otimes}\iota)\tilde{\Delta}(a) \in \mathcal{M}_{\tilde{\psi}}$ and*

$$\tilde{\psi}\big((\omega_{v,w}\bar{\otimes}\iota)\tilde{\Delta}(a)\big) = \tilde{\psi}(a)\,\langle \tilde{\delta}^{-\frac{1}{2}}v\,,\,\tilde{\delta}^{-\frac{1}{2}}w\rangle.$$

Remark that $\omega_{v,w}$ denotes the vector functional defined by $\omega_{v,w}(x) = \langle xv, w\rangle$. Using this lemma it is not so difficult to prove the following lemma. We use the notation $\tilde{\Delta}^{(2)} = (\tilde{\Delta}\bar{\otimes}\iota)\tilde{\Delta} = (\iota\bar{\otimes}\tilde{\Delta})\tilde{\Delta}$.

**Lemma 7.2** *Consider $a \in \mathcal{M}^+_{\tilde{\psi}}$ and $v \in D(\tilde{\delta}^{-\frac{1}{2}} \otimes \tilde{\delta}^{-\frac{1}{2}}) \cap D(\tilde{\Delta}(\tilde{\delta}^{-\frac{1}{2}}))$. Then*

$$\tilde{\psi}(a)\,\|(\tilde{\delta}^{-\frac{1}{2}} \otimes \tilde{\delta}^{-\frac{1}{2}})v\|^2 = \tilde{\psi}\big((\omega_{v,v}\bar{\otimes}\iota)\tilde{\Delta}^{(2)}(a)\big) = \tilde{\psi}(a)\,\|\tilde{\Delta}(\tilde{\delta}^{-\frac{1}{2}})v\|^2.$$

Remembering that $\tilde{\sigma}'_t(\tilde{\sigma}_{-t}(x)) = \tilde{\delta}^{it}\,x\,\tilde{\delta}^{-it}$ for all $x \in \tilde{A}$ and $t \in \mathbb{R}$ and using the commutation rules of propositions 5.6 and 6.2 it is easy to prove that $\tilde{\Delta}(\tilde{\delta})$ and $\tilde{\delta} \otimes \tilde{\delta}$ commute. The previous lemma gives then the following.

**Proposition 7.3** *We have $\tilde{\Delta}(\tilde{\delta}) = \tilde{\delta} \otimes \tilde{\delta}$.*

A standard trick now allows to prove that $\tilde{\delta}$ is affiliated with $\pi(A)$ in the C$^*$-algebra sense ([1],[13]). So we can define the modular element of our reduced C$^*$-algebraic quantum group as the strictly positive element affiliated with the C$^*$-algebra $A$ such that $\pi(\delta) = \tilde{\delta}$. We can then pull down the properties of $\tilde{\delta}$ and obtain that $\Delta(\delta) = \delta \otimes \delta$, $\tau_t(\delta) = \delta$ for all $t \in \mathbb{R}$ and $R(\delta) = \delta^{-1}$. The fact that $\delta$ is affiliated with $A$ is a quantum version of the fact that the modular function of a locally compact group is continuous.

# 8 Unicity of the Haar weights

Using the modular element defined in the previous section we can prove that every left invariant proper weight on $(A, \Delta)$ automatically satisfies the extra condition of proposition 6.1. Hence we obtain

**Theorem 8.1** *Consider a left invariant proper weight $\eta$ on $(A, \Delta)$. Then there exists a number $r > 0$ such that $\eta = r\,\varphi$.*

By applying the unitary antipode $R$ and using proposition 5.6 we can immediately deduce

**Theorem 8.2** *Consider a right invariant proper weight $\eta$ on $(A, \Delta)$. Then there exists a number $r > 0$ such that $\eta = r\,\psi$.*

# 9 The reduced dual of a reduced C*-algebraic quantum group

In this section we construct the dual of a reduced C*-algebraic quantum group following chapter 3 of [3] and relying on [12].

**Definition 9.1** *We define*

- *The set $\hat{A}$ as the norm closure of $\{ (\omega \otimes \iota)(W) \mid \omega \in B(H)_* \}$.*
- *The injective linear map $\hat{\Delta} : \hat{A} \to B(H \otimes H)$ such that $\hat{\Delta}(x) = \Sigma\, W(x \otimes 1)W^*\, \Sigma$ for all $x \in \hat{A}$.*

We use the flip map $\Sigma$ on $H \otimes H$ to guarantee that the dual weight constructed from $\varphi$ will again be left invariant, rather than right invariant. In [3] and [7] it is done in the same way, but on the algebraic level of [11] the flips are left out.

Because we already mentioned the manageability of $W$ we now get the following theorem for free, thanks to theorem 1.5 and proposition 5.1 of [12].

**Theorem 9.2** *The set $\hat{A}$ is a non-degenerate sub-C*-algebra of $B(H)$ and the mapping $\hat{\Delta}$ is a non-degenerate $^*$-homomorphism from $\hat{A}$ into $M(\hat{A} \otimes \hat{A})$ such that*

- *$(\hat{\Delta} \otimes \iota)\hat{\Delta} = (\iota \otimes \hat{\Delta})\hat{\Delta}$.*
- *$\hat{\Delta}(\hat{A})(\hat{A} \otimes 1)$ and $\hat{\Delta}(\hat{A})(1 \otimes \hat{A})$ are dense subsets of $\hat{A} \otimes \hat{A}$.*

We will now introduce a notation which strengthens the analogy with the classical group case. We define $L^1(A)$ to be the closed linear span of $\{ a\,\varphi\, b^* \mid a, b \in \mathcal{N}_\varphi \}$ in $A^*$. We use the notation $(a\,\varphi\, b^*)(x) = \varphi(b^* x a)$. For every $\omega \in L^1(A)$ there is a unique $\tilde{\omega} \in \tilde{A}_*$ such that $\tilde{\omega}\,\pi = \omega$ and hence we can define the contractive linear mapping $\hat{\pi} : L^1(A) \to \hat{A}$ such that $\hat{\pi}(\omega) = (\tilde{\omega} \otimes \iota)(W)$. In the group case this is the left regular representation of $L^1(G)$ on $L^2(G)$ given by convolution.

We also mention that the expression $\omega\,\mu = (\omega \otimes \mu)\Delta$ turns $L^1(A)$ into a Banach algebra, and that $\hat{\pi}$ becomes multiplicative this way. Now we want to define a left invariant weight on $(\hat{A}, \hat{\Delta})$. We start off the same way as in [3] and copy their definition 2.1.6.

**Definition 9.3** *We define the subset $\mathcal{I}$ of $L^1(A)$ as follows:*

$$\mathcal{I} = \{ \omega \in L^1(A) \mid \textit{There exists a number } M \geq 0 \textit{ s.t. } |\omega(x^*)| \leq M\, \|\Lambda(x)\| \textit{ for all } x \in \mathcal{N}_\varphi \} .$$

It is clear that $\mathcal{I}$ is a subspace of $L^1(A)$. By Riesz' theorem for Hilbert spaces, there exists for every $\omega \in \mathcal{I}$ a unique element $v(\omega) \in H$ such that $\omega(x^*) = \langle v(\omega), \Lambda(x) \rangle$ for $x \in \mathcal{N}_\varphi$. We should think of $\omega \mapsto v(\omega)$ as a part of the GNS-map of the still to be constructed dual weight $\hat{\varphi}$. Hence we want $\hat{\varphi}(\hat{\pi}(\omega)^* \hat{\pi}(\omega)) = \|v(\omega)\|^2$ for all $\omega \in \mathcal{I}$. We will construct $\hat{\varphi}$ with the inverse GNS-construction of [4]. So we have to define the modular group of $\hat{\varphi}$.

**Proposition 9.4** *The expression $\rho_t(\omega)(x) = \omega(\delta^{-it}\,\tau_{-t}(x))$, for $\omega \in L^1(A)$, $x \in A$ and $t \in \mathbb{R}$, defines a norm continuous one-parameter representation $\rho$ of $\mathbb{R}$ on $L^1(A)$. There exists a unique norm continuous one-parameter group $\hat{\sigma}$ on $\hat{A}$ such that $\hat{\sigma}_t(\hat{\pi}(\omega)) = \hat{\pi}(\rho_t(\omega))$ for all $t \in \mathbb{R}$ and $\omega \in L^1(A)$.*

Using this one-parameter group $\hat{\sigma}$ we can now execute the techniques of [4] and hence we get the following.

**Proposition 9.5** *There exists a unique closed densely defined linear map $\hat{\Lambda}$ from $D(\hat{\Lambda}) \subseteq \hat{A}$ into $H$ such that $\hat{\pi}(\mathcal{I})$ is a core for $\hat{\Lambda}$ and $\hat{\Lambda}(\hat{\pi}(\omega)) = v(\omega)$ for all $\omega \in \mathcal{I}$.*

*There exists a unique KMS weight $\hat{\varphi}$ on $\hat{A}$ such that $(H, \iota, \hat{\Lambda})$ is a GNS-construction for $\hat{\varphi}$. We have moreover that $\hat{\varphi}$ is faithful and that $\hat{\sigma}$ is its modular group.*

We have a remarkably easy proof for the following fundamental proposition.

**Proposition 9.6** *The weight $\hat{\varphi}$ on $(\hat{A}, \hat{\Delta})$ is left invariant.*

We now want to show that $(\hat{A}, \hat{\Delta})$ is actually a reduced C$^*$-algebraic quantum group. So we must produce a right invariant approximate KMS weight on $(\hat{A}, \hat{\Delta})$. The most easy way to do this is the classical way (see [3]), by producing the unitary antipode of $(\hat{A}, \hat{\Delta})$.

**Proposition 9.7** *There exists a unique $^*$-antiautomorphism $\hat{R}$ on $\hat{A}$ such that $\hat{R}(\hat{\pi}(\omega)) = \hat{\pi}(\omega\, R)$ for all $\omega \in L^1(A)$. This $^*$-antiautomorphism satisfies the relation $\chi\,(\hat{R} \otimes \hat{R})\hat{\Delta} = \hat{\Delta}\,\hat{R}$.*

So we get that $\hat{\varphi}\,\hat{R}$ is a right invariant KMS weight on $(\hat{A}, \hat{\Delta})$ and hence we can conclude the following.

**Theorem 9.8** *The pair $(\hat{A}, \hat{\Delta})$ is a reduced C$^*$-algebraic quantum group.*

We call the pair $(\hat{A}, \hat{\Delta})$ the reduced dual of $(A, \Delta)$. In the next proposition we identify the antipodal triple of $(\hat{A}, \hat{\Delta})$.

**Proposition 9.9**
- *The linear space $\{\, (\omega \otimes \iota)(W^*) \mid \omega \in B(H)_* \,\}$ is a core for the antipode $\hat{S}$ of $(\hat{A}, \hat{\Delta})$ and $\hat{S}\big((\omega \otimes \iota)(W^*)\big) = (\omega \otimes \iota)(W)$ for all $\omega \in B(H)_*$.*
- *The $^*$-antiautomorphism $\hat{R}$ is the unitary antipode of $(\hat{A}, \hat{\Delta})$.*
- *There exists a unique norm continuous one-parameter group $\hat{\tau}$ on $\hat{A}$ such that $\hat{\tau}_t(\hat{\pi}(\omega)) = \hat{\pi}(\omega\,\tau_{-t})$ for all $\omega \in L^1(A)$. This one-parameter group $\hat{\tau}$ is the scaling group of $(\hat{A}, \hat{\Delta})$.*

Finally we will give the multiplicative unitary of $(\hat{A}, \hat{\Delta})$ and from this we will be able to conclude a Pontryagin duality theorem.

**Proposition 9.10** *We have for all $x, y \in \mathcal{N}_{\hat{\varphi}}$ that*

$$(\Sigma W \Sigma)(\hat{\Lambda}(x) \otimes \hat{\Lambda}(y)) = (\hat{\Lambda} \otimes \hat{\Lambda})(\hat{\Delta}(y)(x \otimes 1))$$

Hence we have that $\Sigma W^* \Sigma$ is the multiplicative unitary of $(\hat{A}, \hat{\Delta})$ in the GNS-construction $(H, \iota, \hat{\Lambda})$ for $\hat{\varphi}$. From definition 9.1 now follows the following fundamental theorem.

**Theorem 9.11** *The reduced C$^*$-algebraic quantum groups $(A, \Delta)$ and $(\hat{\hat{A}}, \hat{\hat{\Delta}})$ are isomorphic. More specific, the mapping $\pi : A \rightarrow \hat{\hat{A}}$ is a $^*$-isomorphism such that $(\pi \otimes \pi)\Delta = \hat{\hat{\Delta}}\,\pi$.*

As a concluding remark we mention that $\hat{\hat{\varphi}}\,\pi = \varphi$, although a priori we only know that $\hat{\hat{\varphi}}\,\pi$ and $\varphi$ are proportional.

## References


[1] S. Baaj, Multiplicateurs non bornés. *Thèse 3ème cycle, Université Paris 6* (1980).

[2] F. Combes, Poids sur une C*-algèbre. *J. Math. pures et appl.* **47** (1968), 57–100.

[3] M. Enock & J.-M. Schwartz, Kac Algebras and Duality of Locally Compact Groups. *Springer-Verlag, Berlin* (1992).

[4] J. Kustermans, KMS weights on C*-algebras. *Preprint Odense Universitet* (1997). #funct-an/9704008

[5] J. Kustermans & S. Vaes, Locally compact quantum groups. *Ann. Sc. Ec. Norm. Sup.* **33 (6)** (2000), 837–934.

[6] J. Kustermans & S. Vaes, Weight theory for C*-algebraic quantum groups. *Preprint University College Cork, KU Leuven* (1999). #math/9901063

[7] T. Masuda & Y. Nakagami, A von Neumann algebra framework for the duality of the quantum groups. *Publ. RIMS, Kyoto University* **30** (1994), 799–850.

[8] T. Masuda, Y. Nakagami & S. L. Woronowicz, Lectures at the Fields Institute and at the University of Warsaw, 1995.

[9] G.K. Pedersen & M. Takesaki, The Radon-Nikodym theorem for von Neumann algebras. *Acta Math.* **130** (1973), 53–87.

[10] S. Vaes, A Radon-Nikodym theorem for von Neumann algebras. *J. Operator Theory* **46** (3) (2001), 477–489.

[11] A. Van Daele, An algebraic framework for group duality. *Adv. in Math.* **140** (1998), 323–366.

[12] S.L. Woronowicz, From multiplicative unitaries to quantum groups. *Int. J. Math.* **Vol. 7, No. 1** (1996), 127–149.

[13] S.L. Woronowicz, Unbounded elements affiliated with C*-algebras and non-compact quantum groups. *Commun. Math. Phys.* **136** (1991), 399–432.

[14] S.L. Woronowicz, Pseudospaces, pseudogroups and Pontrjagin duality. Proceedings of the International Conference on Mathematical Physics, Lausanne, 1979. *Lecture Notes in Physics* **116** (1980), 407–412.