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A Simple Derivation of the Gauss-Bonet Theorem

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Abstract

A simple derivation of the Gauss-Bonet theorem is presented based on the representation of spherical polygons by Euler angles and Rodrigues transposition theorem. This leads to a derivation of the theorem which avoids completely the explicit evaluation of rotation matrices.

Introduction

Mukherjee and Pukrushpan [1] have presented a new derivation of the Gauss-Bonet Theorem [1, 2] using explicit expressions for the rotation matrix and the relations of spherical trigonometry. In fact, a much simpler and more transparent derivation is possible using the general properties of the rotation matrix [3] and the results of Junkins and Shuster [4], who showed a strong interplay between spherical trigonometry and the Euler angles. This new derivation, which is the subject of the present work, does not require the calculation of individual elements of the rotation matrix as in [1]. The techniques employed in [4] make easier not only the proof for spherical triangles but also the generalization of the Gauss-Bonet theorem for arbitrary spherical polygons.

According to the Gauss-Bonet Theorem [1, 2], if a spherical triangle is inscribed on a unit sphere and that sphere rolls without slipping on a plane so that the point of contact of the plane and the sphere traces successively the three arcs of the spherical triangle, then the net result of this motion is a translation of the sphere together with a rotation of the sphere about the direction perpendicular to the plane. The angle of rotation is equal to the excess of the spherical triangle.

The simplifications in the derivation of the Gauss-Bonet theorem offered here come from the use of formal relationships satisfied by the rotation matrix. In order to use these relationships unambiguously we will be careful to distinguish clearly between active and passive descriptions of rotation and space-referenced and body-referenced Euler angles. These are explained in [3].

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In our approach to the Gauss-Bonet theorem we consider the rolling of a sphere initially from the perspective of an observer fixed on the surface on which it is rolling. Thus, initially our perspective is of a sequence of active rotations about inertially-fixed axes. The perspective of [4] and of most engineers dealing with spacecraft attitude is of rotations observed from the perspective of the rotating object and the rotations are with respect to axes fixed in the object. Hence, the perspective in this case is passive and body-referenced. The connection between the two descriptions is exceedingly simple, but one must be cautious in setting up the problem, else inconsistencies are inevitable.

Proof of the Gauss-Bonet Theorem

Consider the spherical triangle as shown in Fig. 1.[3] Following [1], the net rotation of the sphere is written as a sequence of three rotations

$$\mathcal{R} = \mathcal{R}_3 \circ \mathcal{R}_2 \circ \mathcal{R}_1 = \mathcal{R}(\hat{\boldsymbol{n}}_3, c) \circ \mathcal{R}(\hat{\boldsymbol{n}}_2, b) \circ \mathcal{R}(\hat{\boldsymbol{n}}_1, a) \tag{1}$$

Here $\mathcal{R}$ denotes an abstract rotation and "$\circ$" denotes the abstract composition rule for abstract rotations. Bold italic $\hat{\boldsymbol{n}}$ denotes an abstract axis of rotation (a unit vector). By abstract we mean without reference to a coordinate system. By representation we mean the elements of these quantities with respect to a coordinate system as 3×3 and 3×1 arrays. Representations of matrices and vectors will be represented by the corresponding non-bold italic symbols.

The three abstract rotations carry one orthonormal triad of vectors into another. To make our notation specific, we write

$$\mathcal{R}(\hat{\boldsymbol{n}}_1, a) : \boldsymbol{\mathcal{E}}_1 \rightarrow \boldsymbol{\mathcal{E}}_2, \qquad \mathcal{R}(\hat{\boldsymbol{n}}_2, b) : \boldsymbol{\mathcal{E}}_2 \rightarrow \boldsymbol{\mathcal{E}}_3, \qquad \mathcal{R}(\hat{\boldsymbol{n}}_3, c) : \boldsymbol{\mathcal{E}}_3 \rightarrow \boldsymbol{\mathcal{E}}_4 \tag{2}$$

Consider now the spherical triangle of Fig. 1. The spherical triangle is positioned so that initially the z-axis pierces the triangle at the vertex A. As the sphere rolls on the surface with the point of contact on the spherical triangle, the vertical will trace the spherical triangle. By convention we write $\boldsymbol{\mathcal{E}}_1 = \{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$. The unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are parallel to the plane of the surface, while $\hat{\boldsymbol{k}}$ is perpendicular to that plane. The first rotation, by convention, is about the direction $\hat{\boldsymbol{\imath}}$. Hence the first rotation can be written as

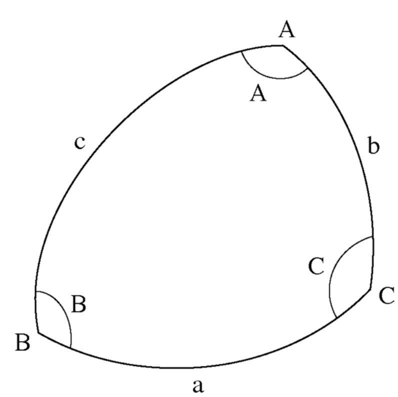

FIG. 1. A Spherical Triangle.

[3]We differ here from the conventions of [1] in that we have labeled the vertices of the spherical triangle by the included dihedral angles, while [1] has assigned an independent sequence of letters for the vertices.

$$\mathcal{R}_1 = \mathcal{R}(\hat{\boldsymbol{n}}_1, a) \tag{3}$$

The second rotation will be about an axis

$$\begin{aligned} \hat{\boldsymbol{n}}_2 &= \cos(\pi - C)\,\hat{\boldsymbol{\imath}} + \sin(\pi - C)\,\hat{\boldsymbol{\jmath}} \\ &= -\cos C\,\hat{\boldsymbol{\imath}} + \sin C\,\hat{\boldsymbol{\jmath}} \end{aligned} \tag{4}$$

The reason for this particular choice of coefficients can be seen from Fig. 2. The reason that $(\pi - C)$ rather than C appears is that the angle from the first axis of rotation to the second axis of rotation is not the dihedral angle of the spherical triangle but its complement. Here the trace is that of the sphere-fixed z-axis as the sphere rolls over the plane.

In a similar fashion, the axis for the third axis of rotation is

$$\begin{aligned} \hat{\boldsymbol{n}}_3 &= \cos(2\pi - A - C)\,\hat{\boldsymbol{\imath}} + \sin(2\pi - A - C)\,\hat{\boldsymbol{\jmath}} \\ &= \cos(A + C)\,\hat{\boldsymbol{\imath}} - \sin(A + C)\,\hat{\boldsymbol{\jmath}} \end{aligned} \tag{5}$$

If we are to evaluate the result of the three rotations of equation (1), we must deal with representations, that is rotation matrices, rather than abstract rotations. We note first that with regard to space axes (the reference frame $\mathcal{E}_1$), the representation of the three axes of rotation are

$$(\hat{\mathbf{n}}_1)_{\mathcal{E}_1} = \hat{\mathbf{1}}, \tag{6a}$$

$$(\hat{\mathbf{n}}_2)_{\mathcal{E}_1} = -\cos C\,\hat{\mathbf{1}} + \sin C\,\hat{\mathbf{2}} \tag{6b}$$

$$(\hat{\mathbf{n}}_3)_{\mathcal{E}_1} = \cos(A + C)\,\hat{\mathbf{1}} - \sin\,(A + C)\,\hat{\mathbf{2}} \tag{6c}$$

The arrays $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$, and $\hat{\mathbf{3}}$ denote the arrays

$$\hat{\mathbf{1}} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{2}} \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{3}} \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{7}$$

and the individual rotation matrices can be written using [3]

$$R(\hat{\mathbf{n}}, \theta) = I + \frac{\sin\theta}{\theta}[[\hat{\mathbf{n}}]] + \frac{1 - \cos\theta}{\theta^2}[[\hat{\mathbf{n}}]]^2 \tag{8}$$

where I is the identity matrix, and

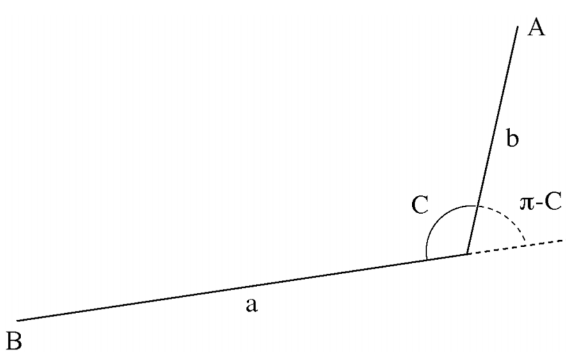

FIG. 2. Action of Tracing Two Successive Sides of a Spherical Triangle.

$$[[\mathbf{u}]] = \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix} \tag{9}$$

Some workers prefer to use the cross-product matrix $[\mathbf{u}\times] \equiv -[[\mathbf{u}]]$.

In practice, we are more accustomed to writing a sequence of rotations (Euler angles, for example) as a sequence of rotations with the axes of rotations referenced to body axes. Thus, we are accustomed to writing for the rotation matrix of the complete sequence in equation (1)

$$R = R\big((\hat{\mathbf{n}}_3)_{\mathcal{E}_3}, c\big)\, R\big((\hat{\mathbf{n}}_2)_{\mathcal{E}_2}, b\big)\, R\big((\hat{\mathbf{n}}_1)_{\mathcal{E}_1}, a\big) \tag{10}$$

However, it can be shown (equation (117) of [3]) that this can be written equivalently as

$$R = R\big((\hat{\mathbf{n}}_1)_{\mathcal{E}_1}, a\big)\, R\big((\hat{\mathbf{n}}_2)_{\mathcal{E}_1}, b\big)\, R\big((\hat{\mathbf{n}}_3)_{\mathcal{E}_1}, c\big) \tag{11}$$

The representations are now all with respect to $\mathcal{E}_1$ as given in equation (6), but the order of matrices is reversed. This result is known as Rodrigues' Transposition Theorem.

We note further that the representations of equation (6) can be rewritten as

$$(\hat{\mathbf{n}}_1)_{\mathcal{E}_1} = \hat{\mathbf{1}} \tag{12a}$$

$$(\hat{\mathbf{n}}_2)_{\mathcal{E}_1} = -R(\hat{\mathbf{3}}, C)\,\hat{\mathbf{1}} \tag{12b}$$

$$(\hat{\mathbf{n}}_3)_{\mathcal{E}_1} = R(\hat{\mathbf{3}}, A + C)\,\hat{\mathbf{1}} \tag{12c}$$

We may now apply the general result

$$R(R_0\hat{\mathbf{n}}, \theta) = R_0\, R(\hat{\mathbf{n}}, \theta)\, R_0^{\mathrm{T}} \tag{13}$$

to write equation (11) equivalently

$$R = R(\hat{\mathbf{1}}, a)\;\; R(\hat{\mathbf{3}}, C)\, R(-\hat{\mathbf{1}}, b)\, R^{\mathrm{T}}(\hat{\mathbf{3}}, C)\;\; R(\hat{\mathbf{3}}, A + C)\, R(\hat{\mathbf{1}}, c)\, R^{\mathrm{T}}(\hat{\mathbf{3}}, A + C) \tag{14}$$

and we have added extra spaces to make the origin of the factors more apparent.

We now note that

$$-\hat{\mathbf{1}} = R(\hat{\mathbf{3}}, \pi)\,\hat{\mathbf{1}} \tag{15}$$

and

$$R(\hat{\mathbf{3}}, \pi) = R(\hat{\mathbf{3}}, -\pi) \tag{16}$$

Thus, equation (14) becomes

$$R = R(\hat{\mathbf{1}}, a)\, R(\hat{\mathbf{3}}, C)\, R(\hat{\mathbf{3}}, \pi)\, R(\hat{\mathbf{1}}, b)\, R(\hat{\mathbf{3}}, \pi)\, R(\hat{\mathbf{3}}, A)\, R(\hat{\mathbf{1}}, c)\, R(\hat{\mathbf{3}}, -A - C) \tag{17}$$

and collecting factors

$$\begin{aligned} R = R(\hat{\mathbf{1}}, a)\, R(\hat{\mathbf{3}}, \pi + C)\, R(\hat{\mathbf{1}}, b)\, R(\hat{\mathbf{3}}, \pi + A)\, R(\hat{\mathbf{1}}, c)\, R(\hat{\mathbf{3}}, \pi + B) \\ \cdots R(\hat{\mathbf{3}}, \pi - A - B - C) \end{aligned} \tag{18}$$

We now note that the arc lengths a, b, c, and the dihedral angles $-A$, $-B$, and $-C$ constitute a spherical triangle. (The minus sign arise from the fact that we have originally defined angles as being for active rotations rather than the passive rotations of [4]. As a result, it follows that the leftmost six matrices must satisfy [4]

$$R(\hat{\mathbf{1}}, a)\, R(\hat{\mathbf{3}}, \pi + C)\, R(\hat{\mathbf{1}}, b)\, R(\hat{\mathbf{3}}, \pi + A)\, R(\hat{\mathbf{1}}, c)\, R(\hat{\mathbf{3}}, \pi + B) = I \tag{19}$$

Therefore,

$$R = R(\hat{\mathbf{3}}, -E) \tag{20}$$

where

$$E = A + B + C - \pi \tag{21}$$

is the spherical excess. Again, the minus sign results from the fact that equation (20) is written from the perspective of a passive rotation rather than active, so that the rotations are in the opposite directions. Thus, the net rotation is that of a rotation about the vertical axis by the spherical excess. Note also that because we have expressed our spherical triangle in terms of Euler angles, the arc lengths are actually rotations about an axis, hence dihedral angles, which are defined on the interval $[0, \pi)$. In geometrical proofs of the Gauss-Bonet theorem, naturally, signs do not enter. Q.E.D.

Note that in the proof above, not a single rotation matrix has been evaluated explicitly.

The General Gauss-Bonet Theorem for Spherical Polygons

The methods employed above for a spherical triangle may be applied to derive the theorem for an arbitrary spherical polygon. Consider an arbitrary spherical polygon as shown in Fig. 3. The ground trace is shown in Fig. 4.

We may write the net rotation of the sphere as

$$\mathcal{R} = \mathcal{R}_n \circ \cdots \circ \mathcal{R}_2 \circ \mathcal{R}_1 \tag{22}$$

where $\mathcal{R}_l$ is the rotation about the physical axis $\hat{\boldsymbol{n}}_l$ through an angle a_l, $l = 1, 2, \ldots, n$, with

$$\hat{\boldsymbol{n}}_l = \cos\phi_l\, \hat{\boldsymbol{\imath}} + \sin\phi_l\, \hat{\boldsymbol{\jmath}} \tag{23}$$

and

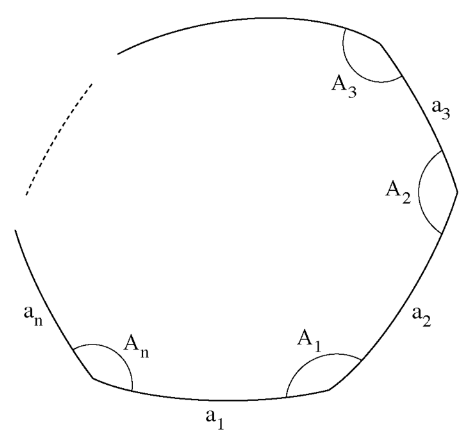

FIG. 3. An Arbitrary Spherical Polygon.

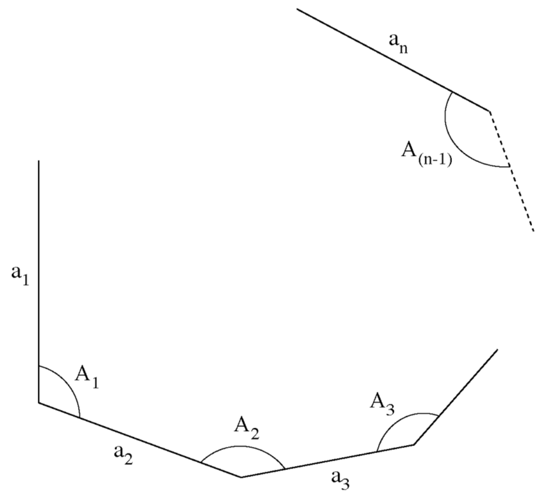

FIG. 4. Ground Trace for Spherical Polygon in Fig. 3.

$$\phi_l = \begin{cases} 0, & \text{for } l = 1 \\ (l-1)\pi - \sum_{m=1}^{l-1} A_m, & \text{for } l = 2, 3, \ldots, n \end{cases} \tag{24}$$

As before

$$\begin{aligned} (\hat{\mathbf{n}}_l)_{\mathcal{E}_1} &= \cos\phi_l\, \hat{\mathbf{1}} + \sin\phi_l\, \hat{\mathbf{2}} \\ &= R(\hat{\mathbf{3}}, -\phi_l)\, \hat{\mathbf{1}} \end{aligned} \tag{25}$$

Note that we have made a small change in the conventions for the arc lengths to simplify the notation.

It follows that

$$\begin{aligned} R = {} & R(\hat{\mathbf{3}}, -\phi_1)\, R(\hat{\mathbf{1}}, a_1)\, R^{\mathrm{T}}(\hat{\mathbf{3}}, -\phi_1)\, R(\hat{\mathbf{3}}, -\phi_2)\, R(\hat{\mathbf{1}}, a_2)\, R^{\mathrm{T}}(\hat{\mathbf{3}}, -\phi_2) \\ & \cdots R(\hat{\mathbf{3}}, -\phi_n)\, R(\hat{\mathbf{1}}, a_n)\, R^{\mathrm{T}}(\hat{\mathbf{3}}, -\phi_n) \end{aligned} \tag{26}$$

and combining matrices

$$\begin{aligned} R = {} & R(\hat{\mathbf{3}}, -\phi_1)\, R(\hat{\mathbf{1}}, a_1)\, R(\hat{\mathbf{3}}, \phi_1 - \phi_2)\, R(\hat{\mathbf{1}}, a_2)\, R(\hat{\mathbf{3}}, \phi_2 - \phi_3) \\ & \cdots R(\hat{\mathbf{3}}, \phi_{n-1} - \phi_n)\, R(\hat{\mathbf{1}}, a_n)\, R(\hat{\mathbf{3}}, \phi_n) \end{aligned} \tag{27}$$

Noting that

$$\phi_l - \phi_{l+1} = -\pi + A_l \tag{28}$$

and

$$R(\hat{\mathbf{3}}, -\pi + A_l) = R(\hat{\mathbf{3}}, \pi + A_l) \tag{29}$$

it follows that

$$\begin{aligned} R = {} & R(\hat{\mathbf{1}}, a_1)\, R(\hat{\mathbf{3}}, \pi + A_1)\, R(\hat{\mathbf{1}}, a_2)\, R(\hat{\mathbf{3}}, \pi + A_2) \\ & \cdots R(\hat{\mathbf{3}}, \pi + A_{n-1})\, R(\hat{\mathbf{1}}, a_n)\, R\left(\hat{\mathbf{3}}, n\pi - \sum_{m=1}^{n} A_m\right) \end{aligned} \tag{30}$$

All but the rightmost factor together must be equivalent to the transversal of the entire spherical polygon and hence equal to the identity matrix. It therefore follows

$$R = R(\hat{\mathbf{3}}, -E_n) \tag{31}$$

with

$$E_n = \sum_{m=1}^{n} A_m - n\pi \tag{32}$$

Q.E.D.

Conclusion

We have derived the Gauss-Bonet theorem in terms of Euler rotations without the need to evaluate specific elements of the direction-cosine matrix. The theorem has been generalized for a spherical polygon of n sides.

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