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Eric J. Olson,* Department of Mathematics and Statistics, University of Nevada, Reno, NV 89557, USA. email: ejolson@unr.edu

James C. Robinson,† Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK. email: j.c.robinson@warwick.ac.uk

A SIMPLE EXAMPLE CONCERNING THE UPPER BOX-COUNTING DIMENSION OF A CARTESIAN PRODUCT

Abstract

We give a simple example of two countable sets X and Y of real numbers such that their upper box-counting dimension satisfies the strict inequality $\dim_{\mathrm{B}}(X \times Y) < \dim_{\mathrm{B}}(X) + \dim_{\mathrm{B}}(Y)$.

1 Introduction

The behaviour of any notion of 'dimension' under the action of taking products is a fundamental property, and it is of particular interest to determine whether (and when) equality holds in the formula

$$\dim(X \times Y) = \dim X + \dim Y.$$

In general, additional conditions are required to ensure equality; this is illustrated by what is perhaps the primary inequality for the dimension of products: if A and B are Borel subsets of Euclidean space, then

$$\dim_{\mathrm{H}}(A) + \dim_{\mathrm{H}}(B) \le \dim_{\mathrm{H}}(A \times B) \le \dim_{\mathrm{H}}(A) + \dim_{\mathrm{P}}(B),$$

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where $\dim_{\mathrm{H}}$ is the Hausdorff dimension and $\dim_{\mathrm{P}}$ the packing dimension (see Falconer [2], for example).

Here we consider this property for the upper box-counting dimension, which we denote by $\dim_{\mathrm{B}}$. It was shown by Tricot [5] that, in general,

$$\dim_{\mathrm{B}}(X \times Y) \le \dim_{\mathrm{B}}(X) + \dim_{\mathrm{B}}(Y); \tag{1}$$

here we provide a very simple example of two countable subsets of the real line, X and Y, such that the inequality in (1) is strict.

Robinson and Sharples [4] gave a significantly more involved example of two generalised Cantor sets X and Y of real numbers for which the inequality in (1) is strict. The more complicated construction there allows significantly more flexibility: one can construct two sets X and Y such that their upper and lower box-counting dimensions take any values allowed by the chain of inequalities

$$\begin{aligned}
\dim_{\mathrm{LB}}(X) + \dim_{\mathrm{LB}}(Y) &\le \dim_{\mathrm{LB}}(X \times Y) \\
&\le \min(\dim_{\mathrm{LB}}(X) + \dim_{\mathrm{B}}(Y), \dim_{\mathrm{B}}(X) + \dim_{\mathrm{LB}}(Y)) \\
&\le \max(\dim_{\mathrm{LB}}(X) + \dim_{\mathrm{B}}(Y), \dim_{\mathrm{B}}(X) + \dim_{\mathrm{LB}}(Y)) \\
&\le \dim_{\mathrm{B}}(X \times Y) \\
&\le \dim_{\mathrm{B}}(X) + \dim_{\mathrm{B}}(Y).
\end{aligned}$$

While the existence of sets X and Y such that strict inequality holds in (1) is thus a particular case of the result in [4], the example presented here is very much more straightforward.

We now make some of the terminology used above and below more precise.

Given a metric space X with metric d_X, the upper box-counting dimension of X, $\dim_{\mathrm{B}}(X)$, is defined by

$$\dim_{\mathrm{B}}(X) = \limsup_{r \to 0} \frac{\log N(X, r)}{-\log r},$$

where $N(X, r)$ denotes the minimum number of balls of radius r required to cover X, see Falconer [2], Robinson [3], or Tricot [5], for example. (Note that some authors refer to this as the 'fractal dimension,' see [1], for example.)

If Y is another metric space with metric d_Y, then the metric space $X \times Y$ is the Cartesian product of X and Y, along with a metric $d_{X \times Y}$ which we assume to be equivalent to $d_X + d_Y$.

2 The example

For convenience, we use the notation

$$\operatorname{sll} t = \sin \log \log t \quad \text{and} \quad \operatorname{cll} t = \cos \log \log t.$$

We show that the two sets

$$X = \{\, f(n) : n \in \mathbb{N} \text{ and } n \geq 25 \,\} \cup \{\, 0 \,\}, \quad \text{where} \quad f(t) = t^{-8-\operatorname{sll} t},$$

and

$$Y = \{\, g(n) : n \in \mathbb{N} \text{ and } n \geq 25 \,\} \cup \{\, 0 \,\}, \quad \text{where} \quad g(t) = t^{-8+\operatorname{sll} t},$$

satisfy $\dim_{\mathrm{B}}(X \times Y) < \dim_{\mathrm{B}}(X) + \dim_{\mathrm{B}}(Y)$. Specifically, we will show that

$$\dim_{\mathrm{B}}(X) \geq 1/8, \quad \dim_{\mathrm{B}}(Y) \geq 1/8, \quad \text{and} \quad \dim_{\mathrm{B}}(X \times Y) < 1/4.$$

We begin with a preliminary lemma that gives upper and lower bounds for certain coverings of subsets of X and Y.

Lemma 1. *Choose $r < 5^{-20}$ and let t_1 be such that $r = t_1^{-9-\operatorname{sll} t_1}$. If*

$$B = \{f(n) : \ 25 \leq n < t_1\},$$

then $t_1 - 26 \leq N(B, r/2) \leq t_1 - 24$.

PROOF. First note that $t_1 = r^{-1/(9+\operatorname{sll} t_1)} > \left(5^{20}\right)^{1/(9+\operatorname{sll} t_1)} \geq 5^2 = 25$. Since

$$f'(t) = -t^{-9-\operatorname{sll} t}(8 + \operatorname{sll} t + \operatorname{cll} t) < 0, \tag{2}$$

the sequence $f(n)$ is decreasing. So we can bound the distance between points in B by considering $|f(n+1) - f(n)|$. To bound this, we write

$$|f(n+1) - f(n)| = |f'(n) + \tfrac{1}{2} f''(\xi)|$$

for some $\xi \in (n, n+1)$, using Taylor's Theorem. Since $\xi > n \geq 25$, certainly

$$\begin{aligned} f''(\xi) &= \xi^{-10-\operatorname{sll} \xi} \Big\{ (9 + \operatorname{sll} \xi + \operatorname{cll} \xi)(8 + \operatorname{sll} \xi + \operatorname{cll} \xi) - \frac{\operatorname{cll} \xi - \operatorname{sll} \xi}{\log \xi} \Big\} \\ &\leq 112\, \xi^{-10-\operatorname{sll} \xi} \\ &\leq 5\xi^{-9-\operatorname{sll} \xi} \leq 5n^{-9-\operatorname{sll} n}, \end{aligned}$$

since $\xi \mapsto \xi^{-9-\operatorname{sll}\xi}$ is a decreasing function (see (2)) and $f''(\xi) \geq 40\,\xi^{-10-\operatorname{sll}\xi} > 0$. We therefore obtain the upper bound

$$|f(n+1) - f(n)| = |f'(n) + \tfrac{1}{2} f''(\xi)| \leq 13\, n^{-9-\operatorname{sll} n}.$$

Since $f'(n) < -6n^{-9-\operatorname{sll} n}$ by (2), we also obtain the lower bound

$$|f'(n) + \tfrac{1}{2} f''(\xi)| \geq |f'(n)| - \tfrac{1}{2} f''(\xi) \geq 6\, n^{-9-\operatorname{sll} n} - 5\, n^{-9-\operatorname{sll} n} = n^{-9-\operatorname{sll} n}.$$

It follows that exactly one $r/2$-ball is required to cover each of the points in B. Therefore,

$$N(B, r/2) = \operatorname{card}\{\, n \in \mathbb{N} : 25 \leq n < t_1 \,\}$$

and the lemma follows. □

The slow fluctuation in these upper and lower bounds allows us to prove our main result.

Theorem 2. $\dim_{\mathrm{B}}(X) \geq 1/8$, $\dim_{\mathrm{B}}(Y) \geq 1/8$, *and* $\dim_{\mathrm{B}}(X \times Y) < 1/4 \leq \dim_{\mathrm{B}}(X) + \dim_{\mathrm{B}}(Y)$.

Proof. First we bound the dimension of X; the bound for Y follows similarly. Let $r < 5^{-20}$ and let t_1 be such that $r = t_1^{-9-\operatorname{sll} t_1}$. Let

$$B = \{\, f(n) : 25 \leq n < t_1 \} \quad \text{and} \quad C = \{\, f(n) : n \geq t_1 \,\},$$

so that $X = B \cup C$. Taking $r \to 0$ along a sequence such that $\operatorname{sll} t_1 = -1$, we can use the result of the lemma to obtain the lower bound

$$N(X, r/2) \geq N(B, r/2) \geq t_1 - 26 \geq r^{-1/(9+\operatorname{sll} t_1)} - 26 \geq r^{-1/8} - 26,$$

and therefore, $\dim_{\mathrm{B}}(X) \geq 1/8$. The lower bound on $\dim_{\mathrm{B}}(Y)$ follows similarly.

To deal with the product set $X \times Y$, notice that since $C \subseteq [0, f(t_1)]$, it follows that

$$N(C, r/2) \leq \frac{f(t_1)}{r/2} = 2t_1 = 2r^{-1/(9+\operatorname{sll} t_1)}.$$

Lemma 1 provides an estimate on $N(B, r/2)$ from above, so we obtain

$$N(X, r/2) \leq N(B, r/2) + N(C, r/2) \leq K_1 r^{-1/(9+\operatorname{sll} t_1)}.$$

Defining t_2 so that $r = t_2^{-9+\operatorname{sll} t_2}$, a similar argument guarantees that

$$N(Y, r/2) \leq K_2 r^{-1/(9-\operatorname{sll} t_2)}.$$

Therefore,

$$N(X \times Y, r/2) \le N(Y, r/2) N(X, r/2) \le K_1 K_2 \left(\frac{1}{r}\right)^{\frac{1}{9-\operatorname{sll} t_1} + \frac{1}{9+\operatorname{sll} t_2}}.$$

Now, since $t_1^{9+\operatorname{sll} t_1} = t_2^{9-\operatorname{sll} t_2}$, taking logarithms once yields

$$\frac{\log t_1}{\log t_2} = \frac{9 - \operatorname{sll} t_2}{9 + \operatorname{sll} t_1} \le 5/4,$$

and taking logarithms again shows that $|\log\log t_1 - \log\log t_2| \le \log(5/4)$. It follows that $N(X \times Y, r/2) \le K_1 K_2 \, (2/r)^c$, where

$$c = \max\left\{\frac{1}{9 - \sin\theta_1} + \frac{1}{9 + \sin\theta_2} : \ |\theta_1 - \theta_2| \le \log(5/4)\right\} < 1/4 :$$

clearly $c \le 2 \times 1/8 = 1/4$, and equality cannot hold since this would require $\sin\theta_1 = 1$ and $\sin\theta_2 = -1$, which is impossible since $|\theta_1 - \theta_2| < \pi$. It follows that

$$\dim_{\mathrm{B}}(X \times Y) \le c < 1/4 \le \dim_{\mathrm{B}}(X) + \dim_{\mathrm{B}}(Y),$$

which finishes the proof. □

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