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A SIMPLE FORMULA FOR AN ANALOGUE OF CONDITIONAL WIENER INTEGRALS AND ITS APPLICATIONS II

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Abstract. Let $C[0,T]$ denote the space of real-valued continuous functions on the interval $[0,T]$ with an analogue w_φ of Wiener measure and for a partition $0=t_0<t_1<\ldots<t_n<t_{n+1}=T$ of $[0,T]$, let $X_n\colon C[0,T]\to\mathbb{R}^{n+1}$ and $X_{n+1}\colon C[0,T]\to\mathbb{R}^{n+2}$ be given by $X_n(x)=(x(t_0),x(t_1),\ldots,x(t_n))$ and $X_{n+1}(x)=(x(t_0),x(t_1),\ldots,x(t_{n+1}))$, respectively.

In this paper, using a simple formula for the conditional w_φ-integral of functions on $C[0,T]$ with the conditioning function X_{n+1}, we derive a simple formula for the conditional w_φ-integral of the functions with the conditioning function X_n. As applications of the formula with the function X_n, we evaluate the conditional w_φ-integral of the functions of the form $F_m(x)=\int_0^T(x(t))^m\,\mathrm{d}t$ for $x\in C[0,T]$ and for any positive integer m. Moreover, with the conditioning X_n, we evaluate the conditional w_φ-integral of the functions in a Banach algebra $\mathscr{S}_{w_\varphi}$ which is an analogue of the Cameron and Storvick's Banach algebra $\mathscr{S}$. Finally, we derive the conditional analytic Feynman w_φ-integrals of the functions in $\mathscr{S}_{w_\varphi}$.

Keywords: analogue of Wiener measure, Cameron-Martin translation theorem, conditional analytic Feynman w_φ-integral, conditional Wiener integral, Kac-Feynman formula, simple formula for conditional w_φ-integral

MSC 2010: 28C20

1. Introduction and preliminaries

Let $C_0[0,T]$ be the space of real-valued continuous functions x on $[0,T]$ with $x(0)=0$. It is well-known that the space $C_0[0,T]$ is equipped with the Wiener measure which is a probability measure. On the space, Yeh introduced an inversion formula that a conditional expectation can be found by a Fourier-transform ([11]). As applications of the formula he obtained very useful results including the Kac-Feynman integral equation and the conditional Cameron-Martin translation theorem using the inversion formula ([12], [13]). But Yeh's inversion formula is very complicated in its applications when the conditioning function is vector-valued.

Let $\tau\colon 0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$ be a partition of the interval $[0, T]$. In [9], Park and Skoug derived a simple formula for conditional Wiener integrals on $C_0[0, T]$ with the conditioning function $X_\tau\colon C_0[0, T] \to \mathbb{R}^{n+1}$ given by

$$X_\tau(x) = (x(t_1), \ldots, x(t_n), x(t_{n+1})).$$

This formula expresses the conditional Wiener integrals directly in terms of ordinary Wiener integrals. Using the formula, they generalized the Kac-Feynman formula and obtained a Cameron-Martin type translation theorem for conditional Wiener integrals.

On the other hand, let $C[0, T]$ denote the space of real-valued continuous functions on the interval $[0, T]$. Im and Ryu introduced a probability measure w_φ on $(C[0, T], \mathscr{B}(C[0, T]))$ where $\mathscr{B}(C[0, T])$ denotes Borel σ-algebra on $C[0, T]$ and φ is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ ([7], [10]). This measure space is a generalization of the Wiener space. In [7], they derived a translation theorem of w_φ-integral, which corresponds to the Cameron-Martin's translation theorem on the Wiener space ([2]). And also, Im and Ryu evaluated the conditional w_φ-integral of functions of the form

$$F_m(x) = \int_0^T (x(t))^m \, \mathrm{d}t \quad (m = 1, 2) \tag{1.1}$$

on $C[0, T]$ with the conditioning function $X_n\colon C[0, T] \to \mathbb{R}^{n+1}$ given by

$$X_n(x) = (x(t_0), x(t_1), \ldots, x(t_n)) \tag{1.2}$$

and $X_{n+1}\colon C[0, T] \to \mathbb{R}^{n+2}$ given by

$$X_{n+1}(x) = (x(t_0), x(t_1), \ldots, x(t_n), x(t_{n+1})), \tag{1.3}$$

and derived a translation theorem of conditional w_φ-integral when the conditioning function is $X(x) = x(T)$. But their methods were complicated in the proofs.

In [5], the author derived a simple formula for the conditional w_φ-integral of the functions on $C[0, T]$ with the vector-valued conditioning function X_{n+1} given by (1.3). This formula expresses the conditional w_φ-integral directly in terms of non-conditional w_φ-integral. As applications of the formula, he evaluated the conditional w_φ-integrals of the functions given by (1.1) for any positive integer m and using the translation theorem of w_φ-integral in [7], he also derived a translation theorem for the conditional w_φ-integral of functions on $C[0, T]$. But, there are no known simple formulas for the conditional w_φ-integral with the conditioning function X_n given by (1.2).

In this paper, using the simple formula ([5]) for the conditional w_φ-integral of the functions on $C[0,T]$ with the conditioning function X_{n+1}, we derive a simple formula for the conditional w_φ-integral of the functions with the conditioning function X_n. As applications of the formula with the function X_n, we evaluate the conditional w_φ-integrals of the functions F_m given by (1.1) for any positive integer m. Moreover, on $C[0,T]$, we evaluate the conditional w_φ-integrals of the functions in a Banach algebra $\mathscr{S}_{w_\varphi}$ which is an analogue of the Cameron and Storvick's Banach algebra $\mathscr{S}$ in [3]. And then, we evaluate the conditional analytic Feynman w_φ-integrals of the functions in the Banach algebra $\mathscr{S}_{w_\varphi}$.

Throughout this paper, let $\mathbb{C}$ and $\mathbb{C}_+$ denote the set of complex numbers and that of complex numbers with positive real parts, respectively.

Now, we begin with introducing the probability space $(C[0,T], \mathscr{B}(C[0,T]), w_\varphi)$. For a positive real T, let $C = C[0,T]$ be the space of all real-valued continuous functions on the closed interval $[0,T]$ with the supremum norm. For $\vec{t} = (t_0, t_1, \ldots, t_n)$ with $0 = t_0 < t_1 < \ldots < t_n \leqslant T$, let $J_{\vec{t}}\colon C[0,T] \to \mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \ldots, x(t_n)).$$

For B_j $(j = 0, 1, \ldots, n)$ in $\mathscr{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}\Big(\prod_{j=0}^{n} B_j\Big)$ of $C[0,T]$ is called an interval and let $\mathscr{I}$ be the set of all such intervals. For a probability measure φ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, we let

$$m_\varphi\left[J_{\vec{t}}^{-1}\left(\prod_{j=0}^{n} B_j\right)\right] = \int_{B_0} \int_{\prod_{j=1}^{n} B_j} W_n(\vec{t}; u_0, u_1, \ldots, u_n) d(u_1, \ldots, u_n) \,\mathrm{d}\varphi(u_0),$$

where

$$(1.4) \quad W_n(\vec{t}; u_0, u_1, \ldots, u_n) = \left[\prod_{j=1}^{n} \frac{1}{2\pi(t_j - t_{j-1})}\right]^{1/2} \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\}.$$

It can be shown that $\mathscr{B}(C[0,T])$, the Borel σ-algebra of $C[0,T]$, coincides with the smallest σ-algebra generated by $\mathscr{I}$ and there exists a unique probability measure w_φ on $(C[0,T], \mathscr{B}(C[0,T]))$ such that $w_\varphi(I) = m_\varphi(I)$ for all I in $\mathscr{I}$ ([7], [10], [14]). This measure w_φ is called an analogue of Wiener measure associated with the probability measure φ.

By the change of variable theorem, we can easily prove the following theorem.

Theorem 1.1 ([7, Lemma 2.1]). *If $f\colon \mathbb{R}^{n+1}\to\mathbb{C}$ is a Borel measurable function then we have*

$$\int_C f(x(t_0),x(t_1),\ldots,x(t_n))\,\mathrm{d}w_\varphi(x) \stackrel{*}{=} \int_{\mathbb{R}}\int_{\mathbb{R}^n} f(u_0,u_1,\ldots,u_n)W_n(\vec{t};u_0,u_1,\ldots,u_n)\,\mathrm{d}(u_1,\ldots,u_n)\,\mathrm{d}\varphi(u_0)$$

where $\stackrel{*}{=}$ means that if either side exists then both sides exist and they are equal.

Let $\{e_k\colon k=1,2,\ldots\}$ be a complete orthonormal subset of $L_2[0,T]$ such that each e_k is of bounded variation. For f in $L_2[0,T]$ and x in $C[0,T]$, we let

$$(f,x)=\lim_{n\to\infty}\int_0^T\biggl[\sum_{k=1}^n e_k(t)\int_0^T f(s)e_k(s)\,\mathrm{d}s\biggr]\,\mathrm{d}x(t)$$

if the limit exists. (f,x) is called the Paley-Wiener-Zygmund integral of f according to x.

Applying Theorem 3.5 in [7], we can easily prove the following theorem.

Theorem 1.2. *Let $\{h_1,h_2,\ldots,h_n\}$ be an orthonormal system of $L_2[0,T]$. For $i=1,2,\ldots,n$, let $Z_i(x)=(h_i,x)$. Then $Z_1,Z_2,\ldots,Z_n$ are independent and each Z_i has the standard normal distribution. Moreover, if $f\colon \mathbb{R}^n\to\mathbb{R}$ is Borel measurable, then we have*

$$\int_C f(Z_1(x),Z_2(x),\ldots,Z_n(x))\,\mathrm{d}w_\varphi(x) \stackrel{*}{=} \Bigl(\frac{1}{2\pi}\Bigr)^{n/2}\int_{\mathbb{R}^n} f(u_1,u_2,\ldots,u_n)\exp\biggl\{-\frac{1}{2}\sum_{j=1}^n u_j^2\biggr\}\,\mathrm{d}(u_1,u_2,\ldots,u_n),$$

where $\stackrel{*}{=}$ means that if either side exists then both sides exist and they are equal.

Let $F\colon C[0,T]\to\mathbb{C}$ be integrable and let X be a random vector on $C[0,T]$. Then, we have the conditional expectation $E[F|X]$ of F given X from a well-known probability theory ([8]). Further, there exists a P_X-integrable complex-valued function ψ on the value space of X such that $E[F|X](x)=(\psi\circ X)(x)$ for w_φ-a.e. $x\in C[0,T]$, where P_X is the probability distribution of X on the value space of X. The function ψ is called the conditional w_φ-integral of F given X and it is also denoted by $E[F|X]$.

2. Simple formulas for conditional w_φ-integrals

In this section, we derive a simple formula for the conditional w_φ-integrals of the functions on $C[0,T]$ with the conditioning function X_n given by (1.2).

For a given partition $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$ of $[0,T]$ and for x in $C[0,T]$, define the polygonal function $[x]$ on $[0,T]$ by

$$[x](t) = x(t_{j-1}) + \frac{t-t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})), \quad t_{j-1} \leqslant t \leqslant t_j,\ j = 1, \ldots, n+1.$$

Similarly, for $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+2}$, define the polygonal function $[\vec{\xi}_{n+1}]$ on $[0,T]$ by

$$[\vec{\xi}_{n+1}](t) = \xi_{j-1} + \frac{t-t_j}{t_j - t_{j-1}}(\xi_j - \xi_{j-1}), \quad t_{j-1} \leqslant t \leqslant t_{j-1},\ j = 1, \ldots, n+1.$$

Then both $[x]$ and $[\vec{\xi}_{n+1}]$ are continuous on $[0,T]$, their graphs are line segments on each subinterval $[t_{j-1}, t_j]$ and $[x](t_j) = x(t_j)$ and $[\vec{\xi}_{n+1}](t_j) = \xi_j$ at each t_j.

To derive the desired simple formula, we begin with letting for $t_{j-1} \leqslant t \leqslant t_j$

$$\Gamma_j(t) = \frac{(t_j - t)(t - t_{j-1})}{t_j - t_{j-1}} \tag{2.1}$$

and

$$X_j(t,x) = x(t) - [x](t), \quad x \in C[0,T] \tag{2.2}$$

for each $j = 1, \ldots, n+1$.

The following theorem gives an interesting observation for the process $x(t) - [x](t)$ on $[0,T] \times C[0,T]$. In fact, $x(t) - [x](t)$ is a Brownian bridge motion on each subinterval and the detailed proof is given as in Theorem 2.4 of [5].

Theorem 2.1. *For each $j = 1, \ldots, n+1$, let X_j be given by* (2.2). *Then, X_j is a Brownian bridge motion process on $[t_{j-1}, t_j]$. Moreover, for $t \in (t_{j-1}, t_j)$, $X_j(t, \cdot)$ is normally distributed with mean* 0 *and variance $\Gamma_j(t)$ which is given by* (2.1).

Using Theorem 2.1, we can prove the following theorem which plays the key role in deriving the desired simple formula. We emphasize that the proof of the theorem is different from Theorem 1 of [9].

Theorem 2.2 ([5, Theorem 2.6]). *Let $Y_{n+1}\colon C[0,T] \to \mathbb{R}^{n+2}$ be given by*

$$Y_{n+1}(x) = (x(t_0), x(t_1) - x(t_0), \ldots, x(t_{n+1}) - x(t_0)).$$

Then the processes $\{x(t) - [x](t)\colon 0 \leqslant t \leqslant T\}$ and Y_{n+1} are stochastically independent.

Using Theorem 2.2, we can prove the following theorem. The detailed proof is given as in Theorem 2.8 of [5].

Theorem 2.3. *Let $X_{n+1}\colon C[0,T] \to \mathbb{R}^{n+2}$ be given by*

$$X_{n+1}(x) = (x(t_0), x(t_1), \ldots, x(t_{n+1})). \tag{2.3}$$

Then the processes $\{x(t) - [x](t)\colon 0 \leqslant t \leqslant T\}$ and X_{n+1} are stochastically independent.

Applying the same method used in the proof of Theorem 2 of [9] with an aid of Problem 4 in [1, p. 216], we have the following theorem from Theorem 2.3.

Theorem 2.4. *Let $F\colon C[0,T] \to \mathbb{C}$ be integrable and X_{n+1} be given by (2.3) of Theorem 2.3. Then for a Borel subset B of $\mathbb{R}^{n+2}$ we have*

$$\int_{X_{n+1}^{-1}(B)} F(x)\,\mathrm{d}w_\varphi(x) = \int_B E[F(x - [x] + [\vec{\xi}_{n+1}])]\,\mathrm{d}P_{X_{n+1}}(\vec{\xi}_{n+1})$$

where $P_{X_{n+1}}$ is the probability distribution of X_{n+1} on $(\mathbb{R}^{n+2}, \mathscr{B}(\mathbb{R}^{n+2}))$. Moreover, by the definition of the conditional w_φ-integral, we have for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$E[F|X_{n+1}](\vec{\xi}_{n+1}) = E[F(x - [x] + [\vec{\xi}_{n+1}])]. \tag{2.4}$$

Note that both $[x](t_0) = x(t_0)$ and $[\vec{\xi}_{n+1}](t_0) = \xi_0$ need not be 0 in Theorem 2.4. In the following theorem, we derive the desired simple formula by removing the component $x(t_{n+1})$ in the conditioning function X_{n+1} given by (2.3).

Theorem 2.5. *Let $X_n\colon\ C[0,T]\to\mathbb{R}^{n+1}$ be given by*

$$X_n(x)=(x(t_0),x(t_1),\dots,x(t_n)) \tag{2.5}$$

and X_{n+1} by (2.3). Moreover let F be defined and integrable on $C[0,T]$ and P_{X_n} be a probability distribution of X_n on $(\mathbb{R}^{n+1},\mathscr{B}(\mathbb{R}^{n+1}))$. Then for any Borel subset B of $\mathbb{R}^{n+1}$, we have

$$\int_{X_n^{-1}(B)}F(x)\,\mathrm{d}w_\varphi(x)=\Big[\frac{1}{2\pi(T-t_n)}\Big]^{1/2}\int_B\int_{\mathbb{R}}E[F(x-[x]+[\vec{\xi}_{n+1}])]\\ \times\exp\Big\{-\frac{(\xi_{n+1}-\xi_n)^2}{2(T-t_n)}\Big\}\,\mathrm{d}\xi_{n+1}\,\mathrm{d}P_{X_n}(\vec{\xi}_n)$$

where $\vec{\xi}_n=(\xi_0,\xi_1,\dots,\xi_n)$ and $\vec{\xi}_{n+1}=(\xi_0,\xi_1,\dots,\xi_n,\xi_{n+1})$. Hence we have by Theorem 2.4 and the definition of the conditional w_φ-integral

$$\begin{aligned}(2.6)\quad &E[F|X_n](\vec{\xi}_n)\\ &=\Big[\frac{1}{2\pi(T-t_n)}\Big]^{1/2}\int_{\mathbb{R}}E[F(x-[x]+[\vec{\xi}_{n+1}])]\exp\Big\{-\frac{(\xi_{n+1}-\xi_n)^2}{2(T-t_n)}\Big\}\,\mathrm{d}\xi_{n+1}\\ &=\Big[\frac{1}{2\pi(T-t_n)}\Big]^{1/2}\int_{\mathbb{R}}E[F|X_{n+1}](\vec{\xi}_{n+1})\exp\Big\{-\frac{(\xi_{n+1}-\xi_n)^2}{2(T-t_n)}\Big\}\,\mathrm{d}\xi_{n+1}\end{aligned}$$

for P_{X_n}-a.e. $\vec{\xi}_n\in\mathbb{R}^{n+1}$.

Proof. Let $P_{X_{n+1}}$ be the probability distribution of X_{n+1} on $(\mathbb{R}^{n+2},\mathscr{B}(\mathbb{R}^{n+2}))$. Then for any Borel subset B of $\mathbb{R}^{n+1}$, we have $X_n^{-1}(B)=X_{n+1}^{-1}(B\times\mathbb{R})$ so that we also have by Theorem 2.4

$$\begin{aligned}\int_{X_n^{-1}(B)}F(x)\,\mathrm{d}w_\varphi(x)&=\int_{X_{n+1}^{-1}(B\times\mathbb{R})}F(x)\,\mathrm{d}w_\varphi(x)\\ &=\int_{B\times\mathbb{R}}E[F|X_{n+1}](\vec{\xi}_{n+1})\,\mathrm{d}P_{X_{n+1}}(\vec{\xi}_{n+1})\\ &=\int_{B\times\mathbb{R}}E[F(x-[x]+[\vec{\xi}_{n+1}])]\,\mathrm{d}P_{X_{n+1}}(\vec{\xi}_{n+1}).\end{aligned}$$

By Theorem 1.1 and Fubini's theorem, we have

$$\begin{aligned}\int_{X_n^{-1}(B)}F(x)\,\mathrm{d}w_\varphi(x)&=\int_{\mathbb{R}}\int_{\mathbb{R}^n}\chi_B(\vec{\xi}_n)\Big[\int_{\mathbb{R}}E[F(x-[x]+[\vec{\xi}_{n+1}])]W_{n+1}((t_0,\dots,t_{n+1});\\ &\qquad\xi_0,\dots,\xi_n,\xi_{n+1})\,\mathrm{d}\xi_{n+1}\Big]\,\mathrm{d}(\xi_1,\dots,\xi_n)\,\mathrm{d}\varphi(\xi_0)\\ &=\Big[\frac{1}{2\pi(T-t_n)}\Big]^{1/2}\int_B\int_{\mathbb{R}}E[F(x-[x]+[\vec{\xi}_{n+1}])]\\ &\qquad\times\exp\Big\{-\frac{(\xi_{n+1}-\xi_n)^2}{2(T-t_n)}\Big\}\,\mathrm{d}\xi_{n+1}\,\mathrm{d}P_{X_n}(\vec{\xi}_n)\end{aligned}$$

where χ_B denotes the indicator function of B and W_{n+1} is given by (1.4) replacing n by $n+1$. Now, the proof is completed. $\square$

For a function $F\colon C[0,T] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-1/2}x)$ and $X_{n+1}^\lambda(x) = X_{n+1}(\lambda^{-1/2}x)$, $X_n^\lambda(x) = X_n(\lambda^{-1/2}x)$, where X_{n+1} and X_n are given by (2.3) and (2.5), respectively. Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. By the definition of conditional w_φ-integral and (2.4), we have

$$E[F^\lambda | X_{n+1}^\lambda](\vec{\xi}_{n+1}) = E[F(\lambda^{-1/2}(x - [x]) + [\vec{\xi}_{n+1}])]$$

for $P_{X_{n+1}^\lambda}$-a.e. $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$, where $P_{X_{n+1}^\lambda}$ is the probability distribution of X_{n+1}^λ on $(\mathbb{R}^{n+2}, \mathscr{B}(\mathbb{R}^{n+2}))$. For $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$ and $\xi_{n+1} \in \mathbb{R}$, let $\vec{\xi}_{n+1}^\lambda = (\lambda^{1/2}\xi_0, \lambda^{1/2}\xi_1, \ldots, \lambda^{1/2}\xi_n, \xi_{n+1})$. Then we have by (2.6) and the change of variable theorem

$$\begin{aligned} E[F^\lambda | X_n^\lambda](\vec{\xi}_n) &= \Big[\frac{1}{2\pi(T - t_n)}\Big]^{1/2} \int_{\mathbb{R}} E[F^\lambda(x - [x] + [\vec{\xi}_{n+1}^\lambda])] \\ &\qquad \times \exp\Big\{ -\frac{(\xi_{n+1} - \lambda^{1/2}\xi_n)^2}{2(T - t_n)} \Big\} \, \mathrm{d}\xi_{n+1} \\ &= \Big[\frac{\lambda}{2\pi(T - t_n)}\Big]^{1/2} \int_{\mathbb{R}} E[F(\lambda^{-1/2}(x - [x]) + [\vec{\xi}_{n+1}])] \\ &\qquad \times \exp\Big\{ -\frac{\lambda(\xi_{n+1} - \xi_n)^2}{2(T - t_n)} \Big\} \, \mathrm{d}\xi_{n+1} \end{aligned} \tag{2.7}$$

for $P_{X_n^\lambda}$-a.e. $\vec{\xi}_n$, where $P_{X_n^\lambda}$ is the probability distribution of X_n^λ on $(\mathbb{R}^{n+1}, \mathscr{B}(\mathbb{R}^{n+1}))$. If $E[F(\lambda^{-1/2}(x - [x]) + [\vec{\xi}_{n+1}])]$ has the analytic extension $J_\lambda^*(F)(\vec{\xi}_{n+1})$ on $\mathbb{C}_+$ as a function of λ, then it is called the conditional analytic Wiener w_φ-integral of F given X_{n+1} with parameter λ and denoted by

$$E^{anw_\lambda}[F | X_{n+1}](\vec{\xi}_{n+1}) = J_\lambda^*(F)(\vec{\xi}_{n+1})$$

for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. Moreover, if for a non-zero real q, $E^{anw_\lambda}[F | X_{n+1}](\vec{\xi}_{n+1})$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ-integral of F given X_{n+1} with parameter q and denoted by

$$E^{anf_q}[F | X_{n+1}](\vec{\xi}_{n+1}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F | X_{n+1}](\vec{\xi}_{n+1}).$$

Similar definitions are understood with (2.7) if we replace X_{n+1} by X_n.

Throughout the remainder of this paper, let X_{n+1} and X_n be given by (2.3) and (2.5), respectively. Moreover, let $P_{X_{n+1}}$ and P_{X_n} denote the probability distributions of X_{n+1} and X_n on the Borel σ-algebras of $\mathbb{R}^{n+2}$ and $\mathbb{R}^{n+1}$, respectively.

We now evaluate the conditional w_φ-integrals of the functions on $C[0,T]$ as applications of (2.4). For this purpose, we modify the result of [5, Theorem 3.1] in the following theorem.

Theorem 3.1. *Let $F_m(x) = \int_0^T (x(t))^m \,\mathrm{d}t$ ($m \in \mathbb{N}$) for $x \in C[0,T]$ and suppose that $\int_{\mathbb{R}} |u|^m \,\mathrm{d}\varphi(u) < \infty$. Then F_m is w_φ-integrable. Moreover, $E[F_m|X_{n+1}](\vec{\xi}_{n+1})$ exists for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$ and it is given by*

$$E[F_m|X_{n+1}](\vec{\xi}_{n+1}) = \sum_{j=1}^{n+1}\sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{m-2k} \frac{m!(l+k)!(t_j-t_{j-1})^{k+1}\xi_{j-1}^{m-2k-l}(\xi_j-\xi_{j-1})^l}{2^k l!(m-2k-l)!(l+2k+1)!}$$

where $[\,\cdot\,]$ denotes the greatest integer function.

P r o o f. Using Theorem 2.4 directly or by Theorem 3.1 in [5], we can prove for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$\tag{3.1} E[F_m|X_{n+1}](\vec{\xi}_{n+1}) = \sum_{j=1}^{n+1}\sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{2^k k!(m-2k)!} \times \int_{t_{j-1}}^{t_j} ([\vec{\xi}_{n+1}](t))^{m-2k}(\Gamma_j(t))^k \,\mathrm{d}t$$

where $\Gamma_j(t)$ is given by (2.1). For $j = 1, \ldots, n+1$ and $k = 0, \ldots, [\frac{m}{2}]$, we have

$$\begin{aligned}
&\int_{t_{j-1}}^{t_j} ([\vec{\xi}_{n+1}](t))^{m-2k}(\Gamma_j(t))^k \,\mathrm{d}t \\
&= \int_{t_{j-1}}^{t_j} \Big(\frac{\xi_j-\xi_{j-1}}{t_j-t_{j-1}}(t-t_{j-1})+\xi_{j-1}\Big)^{m-2k}\Big(\frac{(t_j-t)(t-t_{j-1})}{t_j-t_{j-1}}\Big)^k \,\mathrm{d}t \\
&= \sum_{l=0}^{m-2k}\binom{m-2k}{l}(t_j-t_{j-1})^{-l-k}(\xi_j-\xi_{j-1})^l\xi_{j-1}^{m-2k-l} \\
&\quad\times \int_{t_{j-1}}^{t_j} (t_j-t)^k(t-t_{j-1})^{l+k}\,\mathrm{d}t
\end{aligned}$$

by the binomial expansion. For $l = 0, \ldots, m-2k$, we now have by repeated applications of the integration by parts formula

$$
\begin{aligned}
&\int_{t_{j-1}}^{t_j} (t_j - t)^k (t - t_{j-1})^{l+k} \, \mathrm{d}t \\
&\qquad = \frac{k}{l+k+1} \int_{t_{j-1}}^{t_j} (t_j - t)^{k-1} (t - t_{j-1})^{l+k+1} \, \mathrm{d}t \\
&\qquad\qquad \vdots \\
&\qquad = \frac{k!}{(l+k+1)(l+k+2)\ldots(l+2k)} \int_{t_{j-1}}^{t_j} (t - t_{j-1})^{l+2k} \, \mathrm{d}t \\
&\qquad = \frac{k!}{(l+k+1)(l+k+2)\ldots(l+2k)(l+2k+1)} (t_j - t_{j-1})^{l+2k+1}
\end{aligned}
$$

so that we have

$$
\begin{aligned}
&E[F_m|X_{n+1}](\vec{\xi}_{n+1}) \\
&\quad = \sum_{j=1}^{n+1} \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{2^k k!(m-2k)!} \sum_{l=0}^{m-2k} \binom{m-2k}{l} (t_j - t_{j-1})^{-l-k} (\xi_j - \xi_{j-1})^l \xi_{j-1}^{m-2k-l} \\
&\qquad \times \frac{k!(t_j - t_{j-1})^{l+2k+1}}{(l+k+1)(l+k+2)\ldots(l+2k)(l+2k+1)} \\
&\quad = \sum_{j=1}^{n+1} \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{m-2k} \frac{m!\,(l+k)!\,(t_j - t_{j-1})^{k+1} \xi_{j-1}^{m-2k-l} (\xi_j - \xi_{j-1})^l}{2^k l!(m-2k-l)!(l+2k+1)!}
\end{aligned}
$$

which is the desired result. □

In the following example, we evaluate $E[F_m|X_{n+1}]$ $(m = 1, 2, 3)$ as special cases of Theorem 3.1.

Example 3.1. For $m = 1, 2, 3$, let $F_m(x) = \int_0^T (x(t))^m \, \mathrm{d}t$ for $x \in C[0,T]$ and suppose that $\int_{\mathbb{R}} |u|^m \, \mathrm{d}\varphi(u) < \infty$. Then for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$, we have by Theorem 3.1

$$
E[F_1|X_{n+1}](\vec{\xi}_{n+1}) = \frac{1}{2} \sum_{j=1}^{n+1} (t_j - t_{j-1})(\xi_j + \xi_{j-1})
$$

which can be also obtained by an application of Corollary 4.5 in [7]. We also have

$$
E[F_2|X_{n+1}](\vec{\xi}_{n+1}) = \frac{1}{6} \sum_{j=1}^{n+1} (t_j - t_{j-1})(t_j - t_{j-1} + 2\xi_j^2 + 2\xi_j \xi_{j-1} + 2\xi_{j-1}^2)
$$

which is the result given by Corollary 4.10 of [7]. Moreover we have

$$E[F_3|X_{n+1}](\vec{\xi}_{n+1}) = \frac{1}{4}\sum_{j=1}^{n+1}(t_j - t_{j-1})[(t_j - t_{j-1})(\xi_j + \xi_{j-1}) + \xi_j^3 + \xi_j^2\xi_{j-1} + \xi_j\xi_{j-1}^2 + \xi_{j-1}^3].$$

Remark 3.1. The results of Example 3.1 are also given by Example 3.3 in [5]. We emphasize that the evaluations of Example 3.1 depend on Theorem 3.1, but the evaluations of Example 3.3 in [5] depend on (3.1).

Now we evaluate the conditional w_φ-integral $E[F_m|X_n]$ of F_m which is given as in Theorem 3.1.

Theorem 3.2. *Under the conditions and notations given as in Theorem* 3.1, *we have for* P_{X_n}*-a.e.* $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$

$$E[F_m|X_n](\vec{\xi}_n) = \sum_{j=1}^{n}\sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{m-2k}\frac{m!(l+k)!(t_j - t_{j-1})^{k+1}\xi_{j-1}^{m-2k-l}(\xi_j - \xi_{j-1})^l}{2^k l!(m-2k-l)!(l+2k+1)!} + \sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{[\frac{m-2k}{2}]}\frac{m!(2l+k)!\xi_n^{m-2k-2l}(T-t_n)^{l+k+1}}{2^{l+k}l!(m-2k-2l)!(2l+2k+1)!}.$$

Proof. For convenience let

$$K = \sum_{j=1}^{n}\sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{m-2k}\frac{m!(l+k)!(t_j - t_{j-1})^{k+1}\xi_{j-1}^{m-2k-l}(\xi_j - \xi_{j-1})^l}{2^k l!(m-2k-l)!(l+2k+1)!}.$$

By Theorems 2.5 and 3.1, we have

$$\begin{aligned}E[F_m|X_n](\vec{\xi}_n) &= \Big[\frac{1}{2\pi(T-t_n)}\Big]^{1/2}\int_{\mathbb{R}} E[F_m(x - [x] + [\vec{\xi}_{n+1}])] \\ &\qquad\times \exp\Big\{-\frac{(\xi_{n+1}-\xi_n)^2}{2(T-t_n)}\Big\}\,\mathrm{d}\xi_{n+1} \\ &= K + \sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{m-2k}\frac{m!(l+k)!(T-t_n)^{k+1}\xi_n^{m-2k-l}}{2^k l!(m-2k-l)!(l+2k+1)!} \\ &\quad\times\Big[\frac{1}{2\pi(T-t_n)}\Big]^{1/2}\int_{\mathbb{R}}(\xi_{n+1}-\xi_n)^l\exp\Big\{-\frac{(\xi_{n+1}-\xi_n)^2}{2(T-t_n)}\Big\}\,\mathrm{d}\xi_{n+1}\end{aligned}$$

where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{n+1})$. Let $v = \xi_{n+1} - \xi_n$. By the change of variable theorem, we have

$$E[F_m|X_n](\vec{\xi}_n) = K + \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{m-2k} \frac{m!(l+k)!(T-t_n)^{k+1}}{2^k l!(m-2k-l)!(l+2k+1)!} \times \xi_n^{m-2k-l} \Big[\frac{1}{2\pi(T-t_n)}\Big]^{1/2} \int_{\mathbb{R}} v^l \exp\Big\{-\frac{v^2}{2(T-t_n)}\Big\} \, dv$$

$$= K + 2\sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m-2k}{2}]} \frac{m!(2l+k)!(T-t_n)^{k+1}}{2^k (2l)!(m-2k-2l)!(2l+2k+1)!} \times \xi_n^{m-2k-2l} \Big[\frac{1}{2\pi(T-t_n)}\Big]^{1/2} \int_0^\infty v^{2l} \exp\Big\{-\frac{v^2}{2(T-t_n)}\Big\} \, dv$$

replacing l by $2l$. Let $u = \frac{1}{2}v^2/(T-t_n)$. Again, we have by the change of variable theorem

$$E[F_m|X_n](\vec{\xi}_n) = K + \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m-2k}{2}]} \frac{m!(2l+k)!(T-t_n)^{k+1}}{2^k (2l)!(m-2k-2l)!(2l+2k+1)!} \times \xi_n^{m-2k-2l} 2^l (T-t_n)^l \Big(\frac{1}{\pi}\Big)^{1/2} \int_0^\infty u^{(2l+1)/2-1} \exp\{-u\} \, du$$

$$= K + \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m-2k}{2}]} \frac{2^l m!(2l+k)!}{2^k (2l)!(m-2k-2l)!(2l+2k+1)!} \times (T-t_n)^{l+k+1} \xi_n^{m-2k-2l} \Big(\frac{1}{\pi}\Big)^{1/2} \Gamma\Big(\frac{2l+1}{2}\Big)$$

$$= K + \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m-2k}{2}]} \frac{m!(2l+k)!(T-t_n)^{l+k+1} \xi_n^{m-2k-2l}}{2^{k+l} l!(m-2k-2l)!(2l+2k+1)!}$$

where Γ denotes the gamma function. Now the proof is completed. □

In the following example, we evaluate $E[F_m|X_n]$ $(m = 1, 2, 3)$ as applications of Theorem 3.2.

Example 3.2. For $m = 1, 2, 3$, let $F_m(x) = \int_0^T (x(t))^m \, dt$ for $x \in C[0, T]$ and suppose that $\int_{\mathbb{R}} |u|^m \, d\varphi(u) < \infty$. Moreover, let $Z_1(x) = T^{-1}F_1(x)$ and $Z_2(x) = \sum_{j=1}^{n+1} (t_j - t_{j-1})^{-1} \int_{t_{j-1}}^{t_j} x(t) \, dt$ for $x \in C[0, T]$. Then for P_{X_n}-a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, we have

$$E[F_1|X_n](\vec{\xi}_n) = \frac{1}{2} \sum_{j=1}^{n} (t_j - t_{j-1})(\xi_j + \xi_{j-1}) + (T - t_n)\xi_n.$$

Hence we have

$$E[Z_1|X_n](\vec{\xi_n}) = \frac{1}{2T}\sum_{j=1}^{n}(t_j - t_{j-1})(\xi_j + \xi_{j-1}) + \frac{1}{T}(T - t_n)\xi_n$$

and

$$E[Z_2|X_n](\vec{\xi_n}) = \frac{1}{2}\sum_{j=1}^{n}(\xi_j + \xi_{j-1}) + \xi_n$$

which are also given by Theorems 4.3 and 4.6 in [7], respectively. Further, we have

$$E[F_2|X_n](\vec{\xi_n}) = \frac{1}{6}\sum_{j=1}^{n}(t_j - t_{j-1})(t_j - t_{j-1} + 2\xi_j^2 + 2\xi_j\xi_{j-1} + 2\xi_{j-1}^2) + (T - t_n)\xi_n^2 + \frac{1}{2}(T - t_n)^2$$

which is also given by Theorem 4.8 in [7]. Finally, we have

$$\begin{aligned} E[F_3|X_n](\vec{\xi_n}) &= \frac{1}{4}\sum_{j=1}^{n}(t_j - t_{j-1})[(t_j - t_{j-1})(\xi_j + \xi_{j-1}) + \xi_j^3 + \xi_j^2\xi_{j-1} + \xi_j\xi_{j-1}^2 + \xi_{j-1}^3] \\ &\quad + \sum_{k=0}^{1}\sum_{l=0}^{[\frac{3-2k}{2}]}\frac{3!(2l+k)!(T-t_n)^{l+k+1}\xi_n^{3-2k-2l}}{2^{k+l}l!(3-2k-2l)!(2l+2k+1)!} \\ &= \frac{1}{4}\sum_{j=1}^{n}(t_j - t_{j-1})[(t_j - t_{j-1})(\xi_j + \xi_{j-1}) + \xi_j^3 + \xi_j^2\xi_{j-1} + \xi_j\xi_{j-1}^2 + \xi_{j-1}^3] \\ &\quad + (T - t_n)\xi_n^3 + \frac{3}{2}(T - t_n)^2\xi_n. \end{aligned}$$

Now, we generalize the result of Example 4 in [9], which is also considered by Chang and Chang ([4]).

Theorem 3.3. *Let $F(x) = \exp\{\int_0^T x(t)\,\mathrm{d}t\}$ for $x \in C[0,T]$. Then, for a.e. $y \in C[0,T]$, we have*

$$\lim_{\|\tau\|\to 0} E[F(x)|x(t_0) = y(t_0), x(t_1) = y(t_1), \ldots, x(t_n) = y(t_n)] = F(y)$$

where $\tau\colon\ 0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$ is any partition of the interval $[0,T]$.

Proof. For a.e. $y \in C[0,T]$, we have by Theorem 2.5

$$\begin{aligned} &E[F(x)|x(t_0) = y(t_0), x(t_1) = y(t_1), \ldots, x(t_n) = y(t_n)] \\ &\quad = \exp\biggl\{\frac{1}{2}\sum_{j=1}^{n}(t_j - t_{j-1})(y(t_{j-1}) + y(t_j))\biggr\}\int_C \exp\biggl\{\int_0^T (x(t) - [x](t))\,\mathrm{d}t\biggr\}\,\mathrm{d}w_\varphi(x) \\ &\qquad \times \Bigl[\frac{1}{2\pi(T - t_n)}\Bigr]^{1/2}\int_{\mathbb{R}} \exp\Bigl\{\frac{1}{2}(T - t_n)(y(t_n) + \xi_{n+1}) - \frac{(\xi_{n+1} - y(t_n))^2}{2(T - t_n)}\Bigr\}\,\mathrm{d}\xi_{n+1} \end{aligned}$$

where $[x]$ is the polygonal function of x given by τ. Then by the change of variable theorem, we have

$$
\begin{aligned}
&E[F(x)|x(t_0)=y(t_0),x(t_1)=y(t_1),\ldots,x(t_n)=y(t_n)]\\
&\quad=\exp\Big\{\frac{1}{2}\sum_{j=1}^{n}(t_j-t_{j-1})(y(t_{j-1})+y(t_j))+(T-t_n)y(t_n)\Big\}\\
&\qquad\times\Big[\int_C\exp\Big\{\int_0^T(x(t)-[x](t))\,\mathrm{d}t\Big\}\,\mathrm{d}w_\varphi(x)\Big]\\
&\qquad\times\Big[\Big[\frac{1}{2\pi(T-t_n)}\Big]^{1/2}\int_{\mathbb{R}}\exp\Big\{\frac{1}{2}(T-t_n)v-\frac{v^2}{2(T-t_n)}\Big\}\,\mathrm{d}v\Big]\\
&\quad=\exp\Big\{\frac{1}{2}\sum_{j=1}^{n}(t_j-t_{j-1})(y(t_{j-1})+y(t_j))+(T-t_n)y(t_n)\Big\}\exp\Big\{\frac{(T-t_n)^3}{8}\Big\}\\
&\qquad\times\int_C\exp\Big\{\int_0^T(x(t)-[x](t))\,\mathrm{d}t\Big\}\,\mathrm{d}w_\varphi(x).
\end{aligned}
$$

Letting $\|\tau\|\to 0$, we have the theorem because $\lim\limits_{\|\tau\|\to 0}(x(t)-[x](t))=0$ for $x\in C[0,T]$. □

4. Translation theorems for conditional w_φ-integrals

In this section, we derive a translation theorem for conditional w_φ-integrals, which is a generalization of Theorem 4 of [9].

Let $h\in L_2[0,T]$, $\alpha\in\mathbb{R}$ and let $x_0(t)=\int_0^t h(s)\,\mathrm{d}s+\alpha$ for $0\leqslant t\leqslant T$. Let φ_α be a measure on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ such that $\varphi_\alpha(B)=\varphi(B+\alpha)$ for $B\in\mathscr{B}(\mathbb{R})$. Moreover, let E_{w_φ} and $E_{w_{\varphi_\alpha}}$ denote the conditional w_φ-integral and the conditional w_{φ_α}-integral, respectively. The following theorems are translation theorems for the conditional w_φ-integrals.

Theorem 4.1 ([5, Theorem 4.2])**.** *Let X_{n+1} be given by* (2.3)*. Moreover, let F be defined and w_φ-integrable on $C[0,T]$. Then we have for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}=(\xi_0,\xi_1,\ldots,\xi_{n+1})\in\mathbb{R}^{n+2}$*

$$
\begin{aligned}
&E_{w_\varphi}[F|X_{n+1}](\vec{\xi}_{n+1})\\
&\quad=\exp\Big\{\sum_{j=1}^{n+1}\frac{x_0(t_j)-x_0(t_{j-1})}{t_j-t_{j-1}}\Big[(\xi_j-\xi_{j-1})-\frac{1}{2}(x_0(t_j)-x_0(t_{j-1}))\Big]\Big\}\\
&\qquad\times E_{w_{\varphi_\alpha}}[F(x_0+\cdot)J|X_{n+1}](\xi_0-x_0(t_0),\xi_1-x_0(t_1),\ldots,\xi_{n+1}-x_0(t_{n+1}))
\end{aligned}
$$

where $t_0=0$, $t_{n+1}=T$ and $J(x)=\exp\{-\frac{1}{2}[\|h\|_2^2+2(h,x)]\}$ for $x\in C[0,T]$.

Theorem 4.2 ([7, Theorem 5.4]). *Let $X\colon C[0,T]\to\mathbb{R}$ be defined by $X(x)=x(T)$. Then, under the assumptions and notations given as in Theorem 4.1, we have for a.e. $\xi\in\mathbb{R}$*

$$E_{w_\varphi}[F|X](\xi)=E_{w_\varphi}[F(x_0+\cdot)J|X](\xi-x_0(T))\int_{\mathbb{R}}\exp\Big\{-\frac{(\xi-x_0(T)-\xi_0)^2}{2T}\Big\}\,\mathrm{d}\varphi(\xi_0)$$
$$\times\Big[\int_{\mathbb{R}}\exp\Big\{-\frac{(\xi-\xi_0)^2}{2T}\Big\}\,\mathrm{d}\varphi(\xi_0)\Big]^{-1}.$$

Theorem 4.3. *Let $X_\tau\colon C[0,T]\to\mathbb{R}$ be defined by $X_\tau(x)=(x(t_0),x(T))$, where $t_0=0$. Then, under the assumptions and notations given as in Theorem 4.2, we have for a.e. $\xi\in\mathbb{R}$*

$$E_{w_\varphi}[F|X](\xi)=\Big(\frac{1}{2\pi T}\Big)^{1/2}\int_{\mathbb{R}}\exp\Big\{\frac{1}{T}\int_0^T h(s)\,\mathrm{d}s\Big[(\xi-\xi_0)-\frac{1}{2}\int_0^T h(s)\,\mathrm{d}s\Big]\Big\}$$
$$\times E_{w_{\varphi_\alpha}}[F(x_0+\cdot)J|X_\tau](\xi_0-\alpha,\xi-x_0(T))\exp\Big\{-\frac{(\xi_0-\xi)^2}{2T}\Big\}\,\mathrm{d}\varphi(\xi_0)$$
$$=E_{w_\varphi}[F(x_0+\cdot)J|X](\xi-x_0(T))\int_{\mathbb{R}}\exp\Big\{-\frac{(\xi-x_0(T)-\xi_0)^2}{2T}\Big\}\,\mathrm{d}\varphi(\xi_0)$$
$$\times\Big[\int_{\mathbb{R}}\exp\Big\{-\frac{(\xi-\xi_0)^2}{2T}\Big\}\,\mathrm{d}\varphi(\xi_0)\Big]^{-1}.$$

Proof. For a Borel subset B of $\mathbb{R}$, we have by Theorem 1.1 and Fubini's theorem

$$\int_{X^{-1}(B)}F(x)\,\mathrm{d}w_\varphi(x)=\int_{X_\tau^{-1}(\mathbb{R}\times B)}F(x)\,\mathrm{d}w_\varphi(x)=\int_{\mathbb{R}\times B}E[F|X_\tau](\vec{\xi})\,\mathrm{d}P_{X_\tau}(\vec{\xi})$$
$$=\Big(\frac{1}{2\pi T}\Big)^{1/2}\int_B\int_{\mathbb{R}}E[F|X_\tau](\vec{\xi})\exp\Big\{-\frac{(\xi_0-\xi)^2}{2T}\Big\}\,\mathrm{d}\varphi(\xi_0)\,\mathrm{d}\xi$$

where $\vec{\xi}=(\xi_0,\xi)$ and P_{X_τ} is the probability distribution of X_τ on $(\mathbb{R}^2,\mathscr{B}(\mathbb{R}^2))$. By the definition of conditional expectation and Theorem 4.1, we also have for a.e. $\xi\in\mathbb{R}$

$$E_{w_\varphi}[F|X](\xi)=\Big(\frac{1}{2\pi T}\Big)^{1/2}\int_{\mathbb{R}}\exp\Big\{\frac{1}{T}\int_0^T h(s)\,\mathrm{d}s\Big[(\xi-\xi_0)-\frac{1}{2}\int_0^T h(s)\,\mathrm{d}s\Big]\Big\}$$
$$\times E_{w_{\varphi_\alpha}}[F(x_0+\cdot)J|X_\tau](\xi_0-\alpha,\xi-x_0(T))\exp\Big\{-\frac{(\xi_0-\xi)^2}{2T}\Big\}\,\mathrm{d}\varphi(\xi_0)$$

which is the first equality in the theorem. The second equality in the theorem immediately follows from Theorem 4.2. □

Combining Theorems 2.5 and 4.1, we have the following theorem.

Theorem 4.4. *Let X_n be given by (2.5). Under the assumptions and notations given as in Theorem 4.1, we have for P_{X_n}-a.e. $\vec{\xi_n} = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$*

$$
\begin{aligned}
&E_{w_\varphi}[F|X_n](\vec{\xi_n}) \\
&= \Big[\frac{1}{2\pi(T-t_n)}\Big]^{\frac{1}{2}} \exp\Big\{\sum_{j=1}^{n} \frac{x_0(t_j)-x_0(t_{j-1})}{t_j-t_{j-1}}\Big[(\xi_j-\xi_{j-1}) - \frac{1}{2}(x_0(t_j)-x_0(t_{j-1}))\Big]\Big\} \\
&\quad \times \int_{\mathbb{R}} E_{w_{\varphi_\alpha}}[F(x_0+\cdot)J|X_{n+1}](\xi_0-x_0(t_0), \xi_1-x_0(t_1), \ldots, \\
&\qquad\qquad \xi_n-x_0(t_n), \xi_n-x_0(t_n)+v)\exp\Big\{-\frac{v^2}{2(T-t_n)}\Big\}\,\mathrm{d}v.
\end{aligned}
$$

Proof. By Theorems 2.5 and 4.1, we have for P_{X_n}-a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$

$$
\begin{aligned}
&E_{w_\varphi}[F|X_n](\vec{\xi_n}) \\
&= \Big[\frac{1}{2\pi(T-t_n)}\Big]^{\frac{1}{2}} \exp\Big\{\sum_{j=1}^{n} \frac{(x_0(t_j)-x_0(t_{j-1}))(\xi_j-\xi_{j-1})}{t_j-t_{j-1}} - \sum_{j=1}^{n+1}\frac{(x_0(t_j)-x_0(t_{j-1}))^2}{2(t_j-t_{j-1})}\Big\} \\
&\times \int_{\mathbb{R}} E_{w_{\varphi_\alpha}}[F(x_0+\cdot)J|X_{n+1}](\xi_0-x_0(t_0), \xi_1-x_0(t_1), \ldots, \xi_n-x_0(t_n), \xi_{n+1}-x_0(T)) \\
&\quad \times \exp\Big\{-\frac{(\xi_{n+1}-\xi_n)^2}{2(T-t_n)} + \frac{x_0(T)-x_0(t_n)}{T-t_n}(\xi_{n+1}-\xi_n)\Big\}\,\mathrm{d}\xi_{n+1} \\
&= \Big[\frac{1}{2\pi(T-t_n)}\Big]^{\frac{1}{2}} \exp\Big\{\sum_{j=1}^{n} \frac{(x_0(t_j)-x_0(t_{j-1}))}{t_j-t_{j-1}}\Big[(\xi_j-\xi_{j-1}) - \frac{1}{2}(x_0(t_j)-x_0(t_{j-1}))\Big]\Big\} \\
&\times \int_{\mathbb{R}} E_{w_{\varphi_\alpha}}[F(x_0+\cdot)J|X_{n+1}](\xi_0-x_0(t_0), \xi_1-x_0(t_1), \ldots, \xi_n-x_0(t_n), \xi_{n+1}-x_0(T)) \\
&\quad \times \exp\Big\{-\frac{[(\xi_{n+1}-\xi_n)-(x_0(T)-x_0(t_n))]^2}{2(T-t_n)}\Big\}\,\mathrm{d}\xi_{n+1}
\end{aligned}
$$

where $t_{n+1} = T$. Let $v = (\xi_{n+1}-\xi_n) - (x_0(T)-x_0(t_n))$. Then by the change of variable theorem, we have the theorem as desired. □

Letting $\alpha = 0$ in the result of Theorem 4.4, we have the following corollary since $\varphi_\alpha = \varphi$.

Corollary 4.1. *Under the assumptions and notations given as in Theorem 4.4 with one exception $\alpha = 0$, we have $x_0(t) = \int_0^t h(s)\,\mathrm{d}s$ for $t \in [0,T]$ and*

$$
\begin{aligned}
&E_{w_\varphi}[F|X_n](\vec{\xi_n}) \\
&= \Big[\frac{1}{2\pi(T-t_n)}\Big]^{\frac{1}{2}} \exp\Big\{\sum_{j=1}^{n} \frac{(x_0(t_j)-x_0(t_{j-1}))}{t_j - t_{j-1}} \Big[(\xi_j - \xi_{j-1}) - \frac{1}{2}(x_0(t_j)-x_0(t_{j-1}))\Big]\Big\} \\
&\quad \times \int_{\mathbb{R}} E_{w_\varphi}[F(x_0+\cdot)J|X_{n+1}](\xi_0 - x_0(t_0), \xi_1 - x_0(t_1), \ldots, \\
&\qquad\qquad \xi_n - x_0(t_n), \xi_n - x_0(t_n) + v) \exp\Big\{-\frac{v^2}{2(T-t_n)}\Big\}\,\mathrm{d}v
\end{aligned}
$$

for P_{X_n}-a.e. $\vec{\xi_n} = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$.

Suppose that V is a nonnegative continuous function on $\mathbb{R}$ satisfying the condition

$$\int_{\mathbb{R}} V(\xi) \exp\Big\{-\frac{\xi^2}{2t}\Big\}\,\mathrm{d}\xi < \infty$$

for every $t > 0$. For $\xi_{j-1}, \xi_j \in \mathbb{R}$ and $0 = t_0 < t_{j-1} < t_j$, let

$$
\begin{aligned}
U(\xi_{j-1}, \xi_j, t_{j-1}, t_j) &= \Big[\frac{1}{2\pi(t_j - t_{j-1})}\Big]^{1/2} \exp\Big\{-\frac{(\xi_j - \xi_{j-1})^2}{2(t_j - t_{j-1})}\Big\} \\
&\quad \times E_{w_\varphi}\Big[\exp\Big\{-\int_{t_{j-1}}^{t_j} V(x(s))\,\mathrm{d}s\Big\} \;\Big|\; x(t_0) = 0, x(t_{j-1}) = \xi_{j-1}, x(t_j) = \xi_j\Big].
\end{aligned}
$$

By (4.6) in [9] and Theorem 2.4, it is not difficult show that for a.e. $(\xi_1, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+1}$

$$
\begin{aligned}
&E_{w_\varphi}\Big[\exp\Big\{-\int_0^T V(x(s))\,\mathrm{d}s\Big\} \;\Big|\; x(t_0) = \xi_0, x(t_1) = \xi_1, \ldots, x(t_{n+1}) = \xi_{n+1}\Big] \\
&\quad = \prod_{j=1}^{n+1} [2\pi(t_j - t_{j-1})]^{1/2} U(\xi_{j-1}, \xi_j, t_{j-1}, t_j) \exp\Big\{\frac{(\xi_j - \xi_{j-1})^2}{2(t_j - t_{j-1})}\Big\}
\end{aligned}
$$

where $\xi_0 = 0$ and $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$.

We are now ready to write out an expression for the multi-conditional expectation. Indeed, by Theorem 2.5, we can write

$$
\begin{aligned}
&E_{w_\varphi}\Big[\exp\Big\{-\int_0^T V(x(s))\,\mathrm{d}s\Big\}\Big| x(t_0) = \xi_0, x(t_1) = \xi_1, \ldots, x(t_n) = \xi_n\Big] \\
&\quad = \Big[\prod_{j=1}^{n} [2\pi(t_j - t_{j-1})]^{1/2} \exp\Big\{\frac{(\xi_j - \xi_{j-1})^2}{2(t_j - t_{j-1})}\Big\} U(\xi_{j-1}, \xi_j, t_{j-1}, t_j)\Big] \\
&\qquad \times \int_{\mathbb{R}} U(\xi_n, \xi_{n+1}, t_n, T)\,\mathrm{d}\xi_{n+1}
\end{aligned}
$$

for a.e. $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.

5. Evaluation of conditional analytic Feynman w_φ-integrals

In this section, we evaluate the conditional analytic Feynman w_φ-integrals of several functions on $C[0,T]$.

In the following two theorems, we evaluate the conditional analytic Feynman w_φ-integrals of the function F_m given as in Theorems 3.1 and 3.2. Using similar methods in the proofs of Theorems 3.1 and 3.2, we can easily prove the theorems.

Theorem 5.1. *Let X_{n+1} be given by (2.3). Then, under the assumptions and notations given as in Theorem 3.1, $E^{anw_\lambda}[F_m|X_{n+1}](\vec{\xi}_{n+1})$ exists for $\lambda \in \mathbb{C}_+$ and for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. Moreover, for a non-zero real q, $E^{anf_q}[F_m|X_{n+1}](\vec{\xi}_{n+1})$ exists and it is given by*

$$E^{anf_q}[F_m|X_{n+1}](\vec{\xi}_{n+1}) = \sum_{j=1}^{n+1}\sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{m-2k}\Big(\frac{i}{q}\Big)^k \frac{m!(l+k)!(t_j-t_{j-1})^{k+1}\xi_{j-1}^{m-2k-l}(\xi_j-\xi_{j-1})^l}{2^k l!(m-2k-l)!(l+2k+1)!}.$$

Theorem 5.2. *Let X_n be given by (2.5). Then, under the assumptions and notations given as in Theorem 3.2, $E^{anw_\lambda}[F_m|X_n](\vec{\xi}_n)$ exists for $\lambda \in \mathbb{C}_+$ and for P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$. Moreover, for a non-zero real q, $E^{anf_q}[F_m|X_n](\vec{\xi}_n)$ exists and it is given by*

$$\begin{aligned}E^{anf_q}[F_m|X_n](\vec{\xi}_n) &= \sum_{j=1}^{n}\sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{m-2k}\Big(\frac{i}{q}\Big)^k \frac{m!(l+k)!(t_j-t_{j-1})^{k+1}\xi_{j-1}^{m-2k-l}(\xi_j-\xi_{j-1})^l}{2^k l!(m-2k-l)!(l+2k+1)!}\\ &\quad+\sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{[\frac{m-2k}{2}]}\Big(\frac{i}{q}\Big)^{k+l}\frac{m!(2l+k)!\xi_n^{m-2k-2l}(T-t_n)^{l+k+1}}{2^{l+k}l!(m-2k-2l)!(2l+2k+1)!}.\end{aligned}$$

Let $\mathscr{M}(L_2[0,T])$ be the class of $\mathbb{C}$-valued Borel measures of finite variation on $L_2[0,T]$ and let $\mathscr{S}_{w_\varphi}$ be the space of functions F of the form for $\sigma \in \mathscr{M}(L_2[0,T])$

$$F(x) = \int_{L_2[0,T]} \exp\{\mathrm{i}(v,x)\}\,\mathrm{d}\sigma(v) \tag{5.1}$$

for $x \in C[0,T]$. Let $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$ be a partition of the interval $[0,T]$ and for $v \in L_2[0,T]$ define the sectional average $\overline{v}$ of v by letting

$$\overline{v}(t) = \frac{1}{t_j - t_{j-1}}\int_{t_{j-1}}^{t_j} v(s)\,\mathrm{d}s \tag{5.2}$$

on each subinterval $(t_{j-1}, t_j]$ and by letting $\overline{v}(0) = 0$ ([6]). Then, we have for $v \in L_2[0,T]$ and $x \in C[0,T]$

$$(5.3)\qquad (v,[x]) = \int_0^T v(t)d[x](t) = \sum_{j=1}^{n+1} \frac{x(t_j)-x(t_{j-1})}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} v(t)\,\mathrm{d}t$$
$$= \sum_{j=1}^{n+1} \overline{v}(t_j)(x(t_j)-x(t_{j-1})) = \sum_{j=1}^{n+1} \int_{t_{j-1}}^{t_j} \overline{v}(t)dx(t) = (\overline{v}, x).$$

We are now ready to evaluate the conditional analytic Feynman w_φ-integrals of the functions in $\mathscr{S}_{w_\varphi}$.

Theorem 5.3. *Let X_{n+1} and $F \in \mathscr{S}_{w_\varphi}$ be given by* (2.3) *and* (5.1), *respectively. Then, for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1})$ exists for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+2}$ and it is given by*

$$E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1}) = \int_{L_2[0,T]} \exp\biggl\{\mathrm{i}\sum_{j=1}^{n+1} \overline{v}(t_j)(\xi_j - \xi_{j-1}) - \frac{1}{2\lambda}\|v-\overline{v}\|_2^2\biggr\}\,\mathrm{d}\sigma(v)$$

where $\overline{v}$ is given by (5.2). *Moreover, for a non-zero real q, $E^{anf_q}[F|X_{n+1}](\vec{\xi}_{n+1})$ exists and it is given by*

$$E^{anf_q}[F|X_{n+1}](\vec{\xi}_{n+1}) = \int_{L_2[0,T]} \exp\biggl\{\mathrm{i}\sum_{j=1}^{n+1} \overline{v}(t_j)(\xi_j - \xi_{j-1}) + \frac{1}{2q\mathrm{i}}\|v-\overline{v}\|_2^2\biggr\}\,\mathrm{d}\sigma(v).$$

Proof. For $\lambda > 0$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, we have by Fubini's theorem

$$E[F(\lambda^{-1/2}(x-[x]) + [\vec{\xi}_{n+1}])]$$
$$= \int_{L_2[0,T]} \exp\{\mathrm{i}(v, [\vec{\xi}_{n+1}])\} \int_C \exp\{\mathrm{i}\lambda^{-1/2}(v-\overline{v}, x)\}\,\mathrm{d}w_\varphi(x)\,\mathrm{d}\sigma(v)$$

by (5.3). Using the following well-known integration formula

$$(5.4)\qquad \int_{\mathbb{R}} \exp\{-au^2 + \mathrm{i}bu\}\,\mathrm{d}u = \Bigl(\frac{\pi}{a}\Bigr)^{1/2} \exp\Bigl\{-\frac{b^2}{4a}\Bigr\}$$

for $a \in \mathbb{C}_+$ and any real b, we have

$$E[F(\lambda^{-1/2}(x-[x])+[\vec{\xi}_{n+1}])] = \int_{L_2[0,T]} \exp\biggl\{\mathrm{i}\sum_{j=1}^{n+1} \overline{v}(t_j)(\xi_j-\xi_{j-1}) - \frac{1}{2\lambda}\|v-\overline{v}\|_2^2\biggr\}\,\mathrm{d}\sigma(v)$$

since $(v-\overline{v}, \cdot)$ is mean zero Gaussian with variance $\|v-\overline{v}\|_2^2$ by Theorem 1.2. By Morera's theorem and the dominated convergence theorem, we have the results. □

Letting $\lambda = 1$ in the result of Theorem 5.3, we have the conditional w_φ-integral of F in $\mathscr{S}_{w_\varphi}$.

Corollary 5.1. *Under the assumptions and notations given as in Theorem* 5.3, *we have for* $P_{X_{n+1}}$*-a.e.* $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$E[F|X_{n+1}](\vec{\xi}_{n+1}) = \int_{L_2[0,T]} \exp\biggl\{\mathrm{i}\sum_{j=1}^{n+1} \overline{v}(t_j)(\xi_j - \xi_{j-1}) - \frac{1}{2}\|v - \overline{v}\|_2^2\biggr\}\,\mathrm{d}\sigma(v).$$

Theorem 5.4. *Let* X_n *and* $F \in \mathscr{S}_{w_\varphi}$ *be given by* (2.5) *and* (5.1), *respectively. Then, for* $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[F|X_n](\vec{\xi}_n)$ *exists for* P_{X_n}*-a.e.* $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$ *and it is given by*

$$\begin{aligned}&E^{anw_\lambda}[F|X_n](\vec{\xi}_n)\\ &\quad= \int_{L_2[0,T]} \exp\biggl\{\mathrm{i}\sum_{j=1}^{n} \overline{v}(t_j)(\xi_j - \xi_{j-1}) - \frac{1}{2\lambda}[\|v - \overline{v}\|_2^2 + (T - t_n)[\overline{v}(T)]^2]\biggr\}\,\mathrm{d}\sigma(v)\end{aligned}$$

where $\overline{v}$ *is given by* (5.2). *Moreover, for a non-zero real* q, $E^{anf_q}[F|X_n](\vec{\xi}_n)$ *exists and it is given by*

$$\begin{aligned}&E^{anf_q}[F|X_n](\vec{\xi}_n)\\ &\quad= \int_{L_2[0,T]} \exp\biggl\{\mathrm{i}\sum_{j=1}^{n} \overline{v}(t_j)(\xi_j - \xi_{j-1}) + \frac{1}{2q\mathrm{i}}[\|v - \overline{v}\|_2^2 + (T - t_n)[\overline{v}(T)]^2]\biggr\}\,\mathrm{d}\sigma(v).\end{aligned}$$

Proof. For notational convenience, let $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$ and $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{n+1})$ for $\xi_{n+1} \in \mathbb{R}$. Moreover, let $[\vec{\xi}_{n+1}]$ be the polygonal function of $\vec{\xi}_{n+1}$. For $\lambda > 0$ and P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, we have by Theorems 2.5 and 5.3

$$\begin{aligned}K_\lambda &\equiv \Bigl[\frac{\lambda}{2\pi(T - t_n)}\Bigr]^{1/2} \int_{\mathbb{R}} E[F(\lambda^{-1/2}(x - [x]) + [\vec{\xi}_{n+1}])]\\ &\qquad\qquad \times \exp\Bigl\{-\frac{\lambda(\xi_{n+1} - \xi_n)^2}{2(T - t_n)}\Bigr\}\,\mathrm{d}\xi_{n+1}\\ &= \Bigl[\frac{\lambda}{2\pi(T - t_n)}\Bigr]^{1/2} \int_{L_2[0,T]} \exp\biggl\{\mathrm{i}\sum_{j=1}^{n} \overline{v}(t_j)(\xi_j - \xi_{j-1}) - \frac{1}{2\lambda}\|v - \overline{v}\|_2^2\biggr\}\\ &\qquad \times \int_{\mathbb{R}} \exp\Bigl\{\mathrm{i}\overline{v}(T)(\xi_{n+1} - \xi_n) - \frac{\lambda(\xi_{n+1} - \xi_n)^2}{2(T - t_n)}\Bigr\}\,\mathrm{d}\xi_{n+1}\,\mathrm{d}\sigma(v)\end{aligned}$$

by Fubini's theorem where $\overline{v}$ is the sectional average of v given by (5.2). Let $u = \xi_{n+1} - \xi_n$. Then we have by the change of variable theorem

$$\begin{aligned} K_\lambda &= \Big[\frac{\lambda}{2\pi(T-t_n)}\Big]^{1/2} \int_{L_2[0,T]} \exp\Big\{\mathrm{i}\sum_{j=1}^n \overline{v}(t_j)(\xi_j - \xi_{j-1}) - \frac{1}{2\lambda}\|v-\overline{v}\|_2^2\Big\} \\ &\qquad\qquad \times \int_{\mathbb{R}} \exp\Big\{i[\overline{v}(T)]u - \frac{\lambda u^2}{2(T-t_n)}\Big\}\,\mathrm{d}u\,\mathrm{d}\sigma(v) \\ &= \int_{L_2[0,T]} \exp\Big\{\mathrm{i}\sum_{j=1}^n \overline{v}(t_j)(\xi_j-\xi_{j-1}) - \frac{1}{2\lambda}\|v-\overline{v}\|_2^2 - \frac{(T-t_n)[\overline{v}(T)]^2}{2\lambda}\Big\}\,\mathrm{d}\sigma(v) \end{aligned}$$

by (5.4). By Morera's theorem and the dominated convergence theorem, we have the results. □

Now, letting $\lambda = 1$ in the result of Theorem 5.4, we have the following corollary.

Corollary 5.2. *Under the assumptions and notations given as in Theorem* 5.4, *we have for* P_{X_n}*-a.e.* $\vec{\xi}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned} &E[F|X_n](\vec{\xi}_n) \\ &\quad = \int_{L_2[0,T]} \exp\Big\{\mathrm{i}\sum_{j=1}^n \overline{v}(t_j)(\xi_j-\xi_{j-1}) - \frac{1}{2}[\|v-\overline{v}\|_2^2 + (T-t_n)[\overline{v}(T)]^2]\Big\}\,\mathrm{d}\sigma(v). \end{aligned}$$

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