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A Simple Generator for Discrete Log-Concave Distributions

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Abstract — Zusammenfassung

A Simple Generator for Discrete Log-Concave Distributions. We give a short algorithm that can be used to generate random integers with a log-concave distribution (such as the binomial, Poisson, hypergeometric, negative binomial, geometric, logarithmic series or Polya-Eggenberger distributions). The expected time is uniformly bounded over all these distributions. The algorithm can be implemented if a few lines of high level language code.

AMS Subject Classifications: 65C10, 65C05, 68C55.

Key words: Random number generation. Log-concave distributions. Rejection algorithm.

Ein einfacher Generator für diskrete log-konkave Verteilungen. Wir geben einen raschen Algorithmus an, der Zufallszahlen mit log-konkaver Verteilung (z. B. binomisch, Poisson, hypergeometrisch, negativ binomisch, geometrisch, logarithmisch, Polya-Eggenberger) erzeugt. Die mittlere Rechenzeit ist bezüglich all dieser Verteilungen gleichmäßig beschränkt. Der Algorithmus kann in wenigen Zeilen einer höheren Programmiersprache implementiert werden.

1. Introduction

A distribution on the integers is said to be log-concave when the probabilities p_k satisfy the inequality

$$p_k^2 \geq p_{k-1}\, p_{k+1}$$

for all k. This class of distribution is so vast that it includes nearly all distributions mentioned in Johnson and Kotz (1969), including for example those given in the following table:

Distribution	p_k	Parameter(s)
Binomial (n, p)	$\binom{n}{k} p^k (1-p)^{n-k}\ (0 \leq k \leq n)$	$n \geq 1,\ p \in [0, 1]$
Poisson (λ)	$\frac{\lambda^k}{k!} e^{-\lambda}\ (k \geq 0)$	$\lambda > 0$
Negative binomial (n, p)	$\binom{n+k-1}{n-1} (1-p)^k p^n\ (k \geq 0)$	$n \geq 1,\ p \in (0, 1]$

Distribution	p_k	Parameter(s)
Geometric (p)	$(1-p)^k p \ (k \geq 0)$	$p \in (0, 1]$
Logarithmic series (p)	$\frac{p^k}{-\log(1-p)\,k!} \ (k \geq 1)$	$p \in (0, 1)$
Rectangular (a, b)	$\frac{1}{b-a+1} \ (a \leq k \leq b)$	$a \leq b$, both integer
Hyper-Poisson (θ, λ)	$C_{\lambda,\theta} \frac{\lambda^k}{\prod_{j=0}^{k-1} (\lambda+j)} \ (k \geq 0)$	$\lambda > 0, \ \theta > 0$
Polya-Eggenberger	see Johnson and Kotz (1968, p. 230)	

The normalization constant $C_{\lambda,\theta}$ for the hyper-Poisson distribution is given in Johnson and Knotz (1969, p. 248). For the four-parameter Polya-Eggenberger distribution defined e.g. on p. 230 of Johnson and Kotz (1969), log-concavity only follows if the parameter s of Johnson and Kotz is zero or negative (special cases include the hypergeometric, binomial and rectangular distributions). It should be mentioned that for all the distributions in the table shown above, fast algorithms are available. For the binomial distribution, see Devroye and Naderisamani (1980), Ahrens and Dieter (1980) and Devroye (1986) for various uniformly fast methods. For the Poisson distribution, we refer to Schmeiser and Kachitvichyanukul (1981), Ahrens and Dieter (1980, 1982), Ahrens, Kohrt and Dieter (1983), and Devroye (1981, 1986). The logarithmic series distribution can be dealt with by the methods of Kemp (1981). The negative binomial is best obtained as a Poisson random variable in which the Poisson parameter is gamma-distributed (see Devroye, 1986). For the hypergeometric distribution, one could use the algorithm of Kachitvichyanukul and Schmeiser (1985) (see also Devroye (1986, p. 545)). For the Polya-Eggenberger distribution, it is possible (for some parameter choices) to generate a binomial (n, Y) random variate where Y is beta distributed with certain parameters (see e.g. Kemp and Kemp (1956), Bosch (1963) or Johnson and Kotz (1969)).

The algorithm presented in this paper is a discrete counterpart of an algorithm developed by the author (Devroye, 1984) for log-concave densities. This implies that it is a "black box" algorithm requiring no a priori knowledge (except that the distribution is log-concave with a mode at integer m), that it is conceptually straightforward, and that the expected time per random variate is uniformly bounded over all log-concave distributions.

2. Properties of Discrete Log-Concave Distributions

From the fact that

$$\ldots \geq \frac{p_k}{p_{k-1}} \geq \frac{p_{k+1}}{p_k} \geq \ldots$$

we can easily see that the distribution is unimodal. We will assume without loss of generality that one of the modes is at m. As we will see in the proof of Theorem 1, the chain of inequalities also implies that the p_k's tend to zero as $k \to \pm\infty$ at an exponential or subexponential rate. Hence, all moments exist; in fact, the moments also satisfy interesting inequalities, see e.g. Keilson (1972). We will need

Theorem 1:

For any discrete log-concave distribution with a mode at m and probabilities p_k, we have

$$p_{m+k} \le \min(p_m, p_m\, e^{1-p_m|k|}), \text{ all } k.$$

Proof of Theorem 1:

We need only consider the case $k \ge 0$. By the definition of log-concavity, putting $c = e^{-a} = p_{m+1}/p_m$, we have

$$\frac{p_{m+k}}{p_m} = \prod_{i=1}^{k} \frac{p_{m+i}}{p_{m+i-1}} \le \left(\frac{p_{m+1}}{p_m}\right)^k = e^{-ak}.$$

If we keep p_m fixed, the value of p_{m+k} is maximized when we force $p_{m+k+1} = 0$, $p_{m-1} = 0$ (i.e., the distribution is concentrated on $\{m, m+1, \ldots, m+k\}$), and have equality in the defining chain of inequalities, i.e.

$$p_{m+i} = p_m\, e^{-ai}$$

for all $0 \le i \le k$, and some $a \ge 0$. The constant a can be determined by a simple normalization. One quickly realizes that for $k+1 \le 1/p_m$, we can take $a=0$, while for $k+1 > 1/p_m$, we need $a > 0$. In the latter case, the normalization equation is

$$1 = \sum_{i=0}^{k} p_{m+i} = p_m \sum_{i=0}^{k} e^{-ia} = p_m \frac{1-e^{-a(k+1)}}{1-e^{-a}}.$$

Unfortunately, the solution of this equation is not available in a convenient explicit form. Of course, we are allowed to take a smaller than the solution a^* of the equation. In fact, any a for which

$$1 \le p_m \frac{1-e^{-a(k+1)}}{1-e^{-a}}.$$

will do. In view of $1-e^{-a} \le a$, it also suffices to take any a for which

$$a \le p_m\, 1 - e^{-a(k+1)}.$$

Let us now define $u = a/p_m$ and $1+z = p_m(1+k)$ where $z > 0$ by assumption. The inequality we should try to achieve is rewritten as

$$u \le 1 - e^{-(1+z)u} \quad (z > 0, u > 0).$$

One possible u-value can be obtained by taking the point u_0 with the property that the tangent of $1-e^{-(1+z)u}$ at $u = u_0$ takes the value 1 at $u=1$. Indeed, if u_0 would make the inequality fail, then the tangent in question would have to be at an angle of more than 45 degrees, which is impossible. The tangential line is

$$1 - e^{-(1+z)u_0} + (u-u_0)(1+z)e^{-(1+z)u_0}.$$

Equating this with 1 at $u=1$ yields

$$u_0 = 1 - \frac{1}{1+z}.$$

Thus, we can take

$$a = p_m u_0 = p_m - \frac{1}{1+k}.$$

Hence for

$$1+k > 1/p_m, \quad p_{m+k} \le p_m e^{-p_m k} e^{k/(k+1)},$$

which concludes the proof of Theorem 1. ■

3. A Rejection Algorithm

Suppose that $p_{m+k} \le g(x)$ for all $k - \frac{1}{2} \le x \le k + \frac{1}{2}$ and all $x \in R$, where g is a nonnegative integrable function (hence, proportional to a density). Then, a random variate with probability vector $\{p_k\}$ can be generated as follows:

```
REPEAT
    Generate a uniform [0,1] random variate U.
    Generate Y with density proportional to g.
    X ← round(Y).
UNTIL Ug(Y) ≤ p_{m+X}.
RETURN m+X.
```

This algorithm will be used here with

$$g(x) = \min\left(p_m, p_m e^{1 - p_m(|x| - \frac{1}{2})}\right).$$

The validity of g as a dominating function follows from Theorem 1. Observe that g is a mixture of a rectangular function on $[-w/p_m, w/p_m]$ (of integral $2w$ where $w = 1 + p_m/2$) and two antisymmetric exponential tails outside $[-w/p_m, w/p_m]$ (of integral 2). When g is used in the rejection algorithm, the rejection constant (or, expected number of iterations before halting) is $2w + 2 = 4 + p_m$. We summarize as follows:

```
w ← 1 + p_m/2 (computed once)
REPEAT
    Generate iid uniform [0,1] random variates U, W and a random sign S.
    IF U ≤ w/(1+w)
        THEN Y ← Vw/p_m (where V is uniform [0,1])
        ELSE Y ← (w+E)/p_m (where E is exponential)
    X ← S round(Y)
UNTIL W min(1, e^{w - p_m Y}) ≤ p_{m+X}/p_m.
RETURN m+X.
```

The expected number of iterations is $4 + p_m$. The algorithm should be implemented with care; several shortcuts can be used to reduce the actual time taken by the algorithm; a part of the uniform random variate U can be recovered to generate W; additional speed-ups are possible if bounds are used that depend upon p_m and other ılues of the probability vector at carefully picked integers. Typically, such bounds onsist of piecewise exponential pieces, and the area under the upper bound pproaches one as the number of pieces grows large. This is not unlike some general ble methods for monotone distributions discussed in Devroye (1986).

/hen the distribution is monotone (on $[m, \infty)$), the expected number of iterations ın be reduced to $2+p_m$, if we use rejection based upon the inequality $p_{m+k} \le g(x)$ r $k \ge 0$, $k \le x \le k+1$, where $g(x) = \min(p_m, p_m e^{1+p_m-p_m x})$. The corresponding gorithm is shown below:

$w \leftarrow 1+p_m$ (computed once)
REPEAT
  Generate iid uniform [0, 1] random variates U, W.
  IF $U \le \dfrac{w}{1+w}$
    THEN $Y \leftarrow Vw/p_m$ (where V is uniform [0, 1])
    ELSE $Y \leftarrow (w+E)/p_m$ (where E is exponential)
  $X \leftarrow \lceil Y \rceil$.
UNTIL $W \min(1, e^{w-p_m Y}) \le p_{m+X}/p_m$.
RETURN $m+X$.

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