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A simple global synchronization criterion for coupled chaotic systems

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Abstract

Based on the Lyapunov stabilization theory and Gerschgorin theorem, a simple generic criterion is derived for global synchronization of two coupled chaotic systems with a unidirectional linear error feedback coupling. This simple criterion is applicable to a large class of chaotic systems, where only a few algebraic inequalities are involved. To demonstrate the efficiency of design, the suggested approach is applied to some typical chaotic systems with different types of nonlinearities, such as the original Chua's circuit, the modified Chua's circuit with a sine function, and the Rössler chaotic system. It is proved that these synchronizations are ensured by suitably designing the coupling parameters.

1. Introduction

Chaos synchronization [3,26] has been investigated for a decade, for which many effective methods have been presented [1–4,6–11,14,15,17–20,23–26,29,30,33–35]. Due to the simple configuration and ease of implementation in real systems, the unidirectional linear error feedback coupling scheme turns out to be one of the most efficient methods for chaos synchronization [10,11,14,15,19,20,29]. In order to design a response (or slave) chaotic system based on the unidirectional linear error feedback methodology, the choice of the feedback gain (or coupling parameters) is the key problem in consideration. For Lur'e systems, some LMI conditions have been suggested for determining the feedback gains (or coupling parameters) [8,29,30]. In [20], the in-phase solution decomposition method has been introduced to determine the feedback gains for Lorenz system, Chen system and newly found Lü system. For a general chaotic system, a generic condition of global chaos synchronization has also been established via a Riccati matrix inequality with some time-varying parameters related to the nonlinearity of the chaotic system [14].

The aim of this paper is to further develop a simple but generic criterion for the global synchronization of two coupled general chaotic systems, along with a simple configuration for the corresponding implementation. More precisely, in this paper, the synchronization of two coupled chaotic systems using the unidirectional linear error feedback scheme is studied based on the Lyapunov stability theory [16,22] and Gerschgorin's theorem [12], and a simple generic condition for global chaos synchronization of two coupled chaotic systems is derived. This condition for chaos synchronization is in the form of a few algebraic inequalities, which is very convenient to verify.

The layout of this paper is as follows. In Section 2, based on the Lyapunov stability theory and Gerschgorin's theorem, the generic condition for global synchronization is derived for two coupled chaotic systems using the

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unidirectional linear error feedback coupling scheme. A criterion for the global synchronization is then established in the form of a few algebraic inequalities. In Section 3, this criterion is applied to some typical chaotic systems with different types of nonlinearities, such as the original Chua's circuit, the modified Chua's circuit with a sine function, and the Rössler system. To that end, conditions for choosing the feedback gain (or coupling parameters) are devised to ensure the global synchronization for these chaotic systems. Finally, some concluding remarks are given in Section 4.

2. A criterion for global chaos synchronization

Consider a chaotic system in the form of

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + g(\mathbf{x}) + \mathbf{u}, \tag{1}$$

where $\mathbf{x} \in R^n$ is the state vector, $\mathbf{u} \in R^n$ is the external input vector, $\mathbf{A} \in R^{n\times n}$ is a constant matrix, and $g(\mathbf{x})$ is a continuous nonlinear function. Assuming that

$$g(\mathbf{x}) - g(\tilde{\mathbf{x}}) = \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}}(\mathbf{x} - \tilde{\mathbf{x}}) \tag{2}$$

for a bounded matrix $\mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}}$, in which the elements are dependent on $\mathbf{x}$ and $\tilde{\mathbf{x}}$.

Remark 1. Most of chaotic systems, including all Lur'e nonlinear systems and Lipschitz nonlinear systems, can be described by (1) and (2), which will be further illustrated by concrete examples in Section 3.

From the unidirectional linear coupling approach, a slave system for (1) is constructed as follows:

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + g(\tilde{\mathbf{x}}) + \mathbf{u} + \mathbf{K}(\mathbf{x} - \tilde{\mathbf{x}}), \tag{3}$$

where $\mathbf{K} = \mathrm{diag}(k_1, k_2, \ldots, k_n)$, with $k_i \in R$, $i = 1, 2, \ldots, n$, is a feedback matrix to be designed later.

From (1) and (3), the following error system equation can be obtained:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + g(\mathbf{x}) - g(\tilde{\mathbf{x}}) - \mathbf{K}(\mathbf{x} - \tilde{\mathbf{x}}) = \mathbf{A}\mathbf{e} - \mathbf{K}\mathbf{e} + g(\mathbf{x}) - g(\tilde{\mathbf{x}}) = (\mathbf{A} - \mathbf{K})\mathbf{e} + g(\mathbf{x}) - g(\tilde{\mathbf{x}}), \tag{4}$$

where $\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}$ is the error term.

Theorem 1. *If the feedback gain matrix* $\mathbf{K}$ *is chosen such that*

$$\lambda_i \leqslant \mu < 0, \quad i = 1, 2, \ldots, n, \tag{5}$$

where λ_i *are the eigenvalues of the matrix* $(\mathbf{A} - \mathbf{K} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{K} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}})$ *with a positive definite symmetric constant matrix* $\mathbf{P}$, *and* μ *is a negative constant*, *then the error dynamical system* (4) *is globally exponentially stable about the origin*, *implying that the two systems* (1) *and* (3) *are globally asymptotically synchronized.*

Proof. Choose the Lyapunov function

$$V = \mathbf{e}^{\mathrm{T}}\mathbf{P}\mathbf{e}, \tag{6}$$

where $\mathbf{P}$ is a positive definite symmetric constant matrix. Then, its derivative is

$$\begin{aligned}
\dot{V} &= \dot{\mathbf{e}}^{\mathrm{T}}\mathbf{P}\mathbf{e} + \mathbf{e}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{e}} = \Big[(\mathbf{A} - \mathbf{K})\mathbf{e} + g(\mathbf{x}) - g(\tilde{\mathbf{x}})\Big]^{\mathrm{T}}\mathbf{P}\mathbf{e} + \mathbf{e}^{\mathrm{T}}\mathbf{P}\Big[(\mathbf{A} - \mathbf{K})\mathbf{e} + g(\mathbf{x}) - g(\tilde{\mathbf{x}})\Big] \\
&= \mathbf{e}^{\mathrm{T}}\Big[(\mathbf{A} - \mathbf{K})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{K})\Big]\mathbf{e} + \Big[g(\mathbf{x}) - g(\tilde{\mathbf{x}})\Big]^{\mathrm{T}}\mathbf{P}\mathbf{e} + \mathbf{e}^{\mathrm{T}}\mathbf{P}\Big[g(\mathbf{x}) - g(\tilde{\mathbf{x}})\Big] \\
&= \mathbf{e}^{\mathrm{T}}\Big[(\mathbf{A} - \mathbf{K} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{K} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}})\Big]\mathbf{e} = \mathbf{e}^{\mathrm{T}}\mathbf{Q}\mathbf{e},
\end{aligned} \tag{7}$$

where $\mathbf{Q} = (\mathbf{A} - \mathbf{K} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{K} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}})$.

Since $\mathbf{Q} = \mathbf{Q}'$, let $\mathbf{Q} = \mathbf{U}^*\mathbf{\Lambda}\mathbf{U}$, where $\mathbf{U}$ is square unitary matrix and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then, (7) becomes

$$\dot{V} = \mathbf{e}^{\mathrm{T}}\mathbf{Q}\mathbf{e} = \mathbf{e}^{\mathrm{T}}\mathbf{U}^*\mathbf{\Lambda}\mathbf{U}\mathbf{e} = \mathbf{e}_1^{\mathrm{T}}\mathbf{\Lambda}\mathbf{e}_1 \leqslant \mu\mathbf{e}_1^{\mathrm{T}}\mathbf{e}_1 < 0, \tag{8}$$

where $\mathbf{e}_1 = \mathbf{U}\mathbf{e}$. Accounting to (8) and the Lyapunov stability theory, system (4) is globally exponentially stable about the origin, and hence, the two systems (1) and (3) are globally asymptotically synchronized. □

Based on the well-known Gerschgorin's theorem in matrix theory, the following result can be obtained.

Theorem 2. *Choose* $\mathbf{P} = \mathrm{diag}(p_1, p_2, \ldots, p_n)$, *and let*

$$\mathbf{P}(\mathbf{A} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}}) + (\mathbf{A} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}})^{\mathrm{T}}\mathbf{P} = [\bar{a}_{ij}] \quad \text{and} \quad R_i = \sum_{j=1, j\neq i}^{n} |\bar{a}_{ij}|. \tag{9}$$

If a suitable $\mathbf{K}$ *is chosen such that*

$$k_i \geqslant \frac{1}{2p_i}(\bar{a}_{ii} + R_i - \mu), \quad i = 1, 2, \ldots, n, \tag{10}$$

then (5) *is satisfied, implying that the two coupled chaotic systems* (1) *and* (3) *are globally synchronized.*

Remark 2. If $\mathbf{P} = \mathbf{I}$, then according to Theorems 1 and 2, one obtains the following algebraic inequalities for choosing the coupling parameters:

$$k_i \geqslant \tfrac{1}{2}(\bar{a}_{ii} + R_i - \mu), \quad i = 1, 2, \ldots, n. \tag{11}$$

Remark 3. If $R' = \max_{1 \leqslant i \leqslant n} \sum_{j=1, j\neq i}^{n} |\bar{a}_{ij}|$, then based on (9) one has $R' \geqslant R_i$ and according to Gerschgorin's theorem one has

$$k_i' \geqslant \frac{1}{2p_i}(\bar{a}_{ii} + R' - \mu), \quad i = 1, 2, \ldots, n. \tag{12}$$

However, the range for $\mathbf{K}$ in (12) is reduced as compared to (10).

Remark 4. For coupled chaotic systems of the Lur'e type, the corresponding algebraic inequality conditions can also be derived for determining the coupling parameters to ensure global chaos synchronization.

3. Synchronization of some typical chaotic systems

To illustrate the use of the chaos synchronization criterion derived above, three typical yet topologically different examples of chaotic systems are discussed.

3.1. The original Chua's circuit

Chua's circuit [28] is described by

$$\begin{cases} \dot{x} = \alpha(y - x - f(x)), \\ \dot{y} = x - y + z, \\ \dot{z} = -\beta y, \end{cases} \tag{13}$$

where $\alpha > 0$, $\beta > 0$, $a < b < 0$, $f(\cdot)$ is a piecewise linear function described by

$$f(x) = bx + \tfrac{1}{2}(a - b)(|x + 1| - |x - 1|), \tag{14}$$

In (14), we have

$$f(x) - f(\tilde{x}) = k_{x,\tilde{x}}(x - \tilde{x}), \tag{15}$$

where $k_{x,\tilde{x}}$ is dependent on x and $\tilde{x}$, and varies within the interval $[a, b]$ for $t \geqslant 0$, that is, $k_{x,\tilde{x}}$ is bounded by constants as $a \leqslant k_{x,\tilde{x}} \leqslant b < 0$ (see Fig. 1).

Refering to (3), the following slave system is constructed for the drive (13) with a linear unidirectional coupling:

$$\begin{cases} \dot{\tilde{x}} = \alpha(\tilde{y} - \tilde{x} - f(\tilde{x})) + k_1(x - \tilde{x}), \\ \dot{\tilde{y}} = \tilde{x} - \tilde{y} + \tilde{z} + k_2(y - \tilde{y}), \\ \dot{\tilde{z}} = -\beta\tilde{y} + k_3(z - \tilde{z}). \end{cases} \tag{16}$$

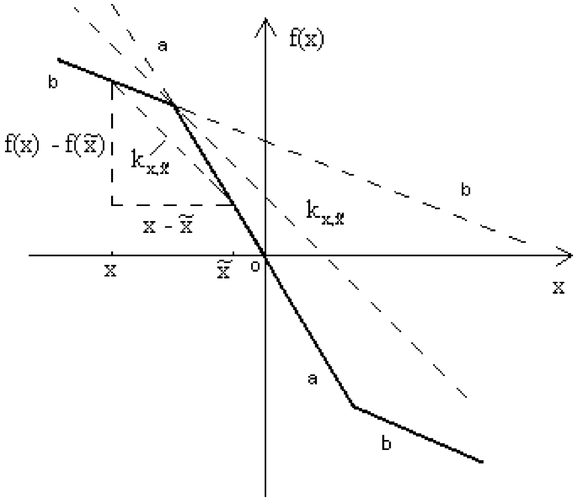

Fig. 1. Graphical representation of $k_{x,\tilde{x}}$ and equality (15).

Subtracting (16) from (13) gives

$$\begin{cases} \dot{e}_x = \alpha(e_y - e_x - k_{x,\tilde{x}}e_x) - k_1 e_x, \\ \dot{e}_y = e_x - e_y + e_z - k_2 e_y, \\ \dot{e}_z = -\beta e_y - k_3 e_z, \end{cases} \tag{17}$$

where $e_x = x - \tilde{x}$, $e_y = y - \tilde{y}$, $e_z = z - \tilde{z}$. System (17) can be rewritten as

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + g(\mathbf{x}) - g(\tilde{\mathbf{x}}) - \mathbf{K}\mathbf{e}, \tag{18}$$

where

$$\mathbf{A} = \begin{bmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}, \qquad \mathbf{e} = \begin{bmatrix} x - \tilde{x} \\ y - \tilde{y} \\ z - \tilde{z} \end{bmatrix} \quad \text{and} \quad g(\mathbf{x}) = \begin{bmatrix} -\alpha f(x) \\ 0 \\ 0 \end{bmatrix}.$$

Observe that

$$g(\mathbf{x}) - g(\tilde{\mathbf{x}}) = \begin{bmatrix} -\alpha(f(x) - f(\tilde{x})) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\alpha k_{x,\tilde{x}}(x - \tilde{x}) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\alpha k_{x,\tilde{x}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x - \tilde{x} \\ y - \tilde{y} \\ z - \tilde{z} \end{bmatrix} = \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}}\mathbf{e}, \tag{19}$$

where

$$\mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}} = \begin{bmatrix} -\alpha k_{x,\tilde{x}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows from (18) and (19) that

$$(\mathbf{A} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}}) + (\mathbf{A} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}})^{\mathrm{T}} = \begin{bmatrix} (-2\alpha - 2\alpha k_{x,\tilde{x}}) & \alpha + 1 & 0 \\ \alpha + 1 & -2 & 1 - \beta \\ 0 & 1 - \beta & 0 \end{bmatrix}. \tag{20}$$

One may then choose

$$\begin{aligned} k_1 &\geqslant \tfrac{1}{2}(-2\alpha - 2\alpha k_{x,\tilde{x}} + |1 + \alpha| - \mu), \\ k_2 &\geqslant \tfrac{1}{2}(-2 + |1 + \alpha| + |1 - \beta| - \mu), \\ k_3 &\geqslant \tfrac{1}{2}(|1 - \beta| - \mu). \end{aligned} \tag{21}$$

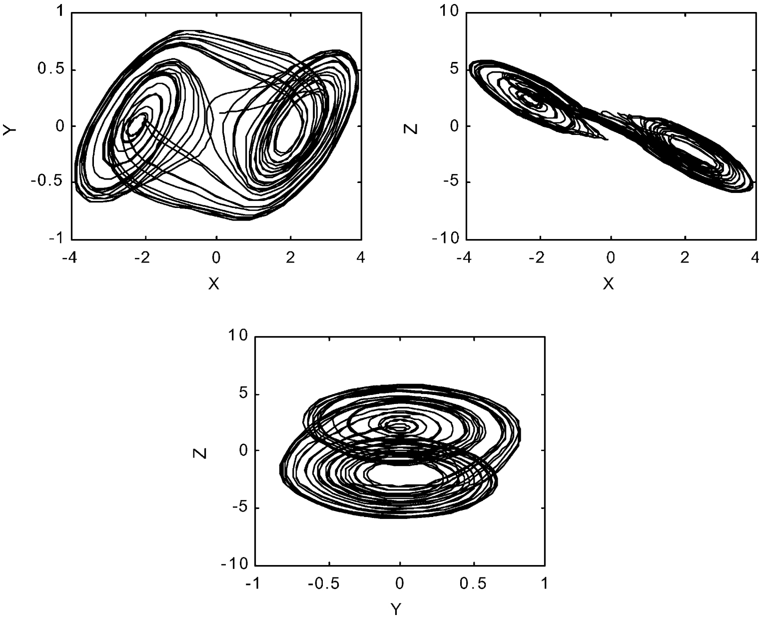

Fig. 2. The double scroll attractors of Chua's circuit.

According to Theorem 2 and Remark 2, the two coupled Chua's systems (13) and (16) are globally asymptotically synchronized. Since $\alpha > 0$ and $a \leqslant k_{x\tilde{x}} \leqslant b < 0$, from (21), one can choose

$$\begin{aligned} k_1 &\geqslant \tfrac{1}{2}(1 - \alpha - 2a\alpha - \mu), \\ k_2 &\geqslant \tfrac{1}{2}(\alpha - 1 + |1 - \beta| - \mu), \\ k_3 &\geqslant \tfrac{1}{2}(|1 - \beta| - \mu). \end{aligned} \tag{22}$$

Corollary 1. *For the two coupled Chua's systems* (13) *and* (16), *if the feedback matrix* **K** *is chosen such that inequality* (22) *holds, then they are globally asymptotically synchronized.*

When $\alpha = 9.78$, $\beta = 14.97$, $a = -1.31$, $b = -0.75$, system (13) exhibits chaotic behavior (see Fig. 2). By selecting $\mu = -0.5$ and the coupling parameters as $k_1 = 9$, $k_2 = 12$, $k_3 = 8$, the inequality (22) holds. Based on Corollary 1, the two coupled Chua's circuits (13) and (16), with the above-chosen parameters are globally asymptotically synchronized, as shown in Fig. 3.

3.2. Modified Chua's circuit with a sine function

Unlike the original Chua's circuit, the modified Chua's circuit uses a sine function [32]. For this circuit, *n*-scroll attractors can be obtained, as shown in Fig. 4.

The dimensionless state equation of the circuit is

$$\begin{cases} \dot{x} = \alpha(y - f(x)), \\ \dot{y} = x - y + z, \\ \dot{z} = -\beta y, \end{cases} \tag{23}$$

where

$$f(x) = \begin{cases} \frac{b\pi}{2a}(x - 2ac) & \text{if } x \geqslant 2ac, \\ -b\sin(\frac{\pi x}{2a} + d) & \text{if } -2ac < x < 2ac, \\ \frac{b\pi}{2a}(x + 2ac) & \text{if } x \leqslant -2ac. \end{cases} \tag{24}$$

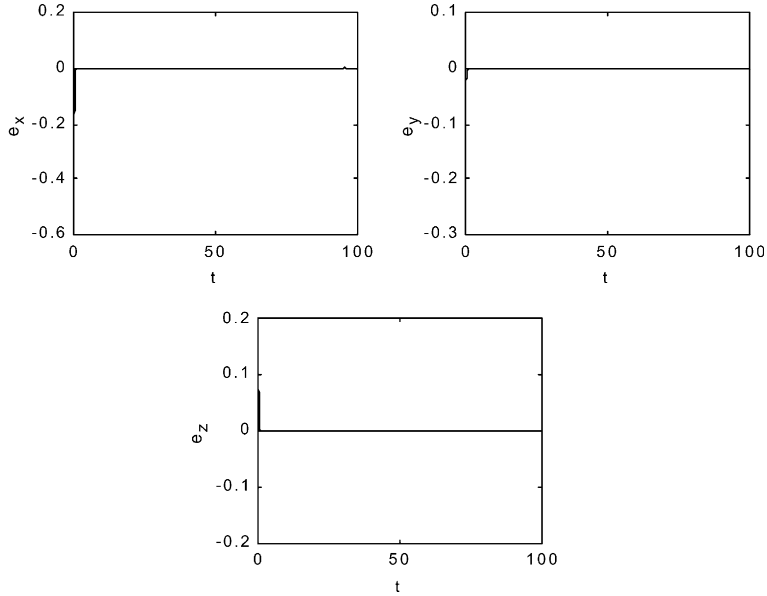

Fig. 3. The difference signal $e_x = x - \tilde{x}$, $e_y = y - \tilde{y}$, $e_z = z - \tilde{z}$ in two coupled Chua's circuits with the coupling parameters $k_1 = 9$, $k_2 = 12$ and $k_3 = 8$.

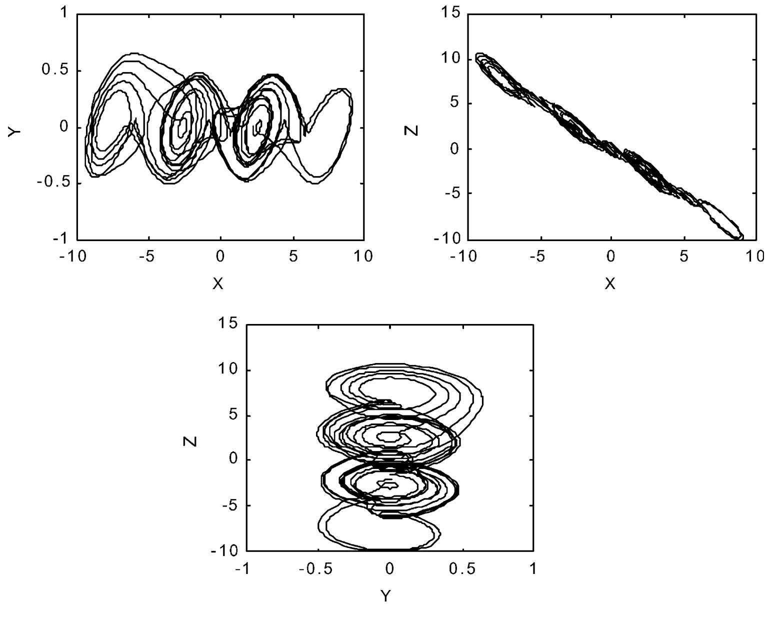

Fig. 4. Four-scroll attractors of the modified Chua's circuit with sine function.

Here, in (23) and (24), α, β, a, b, c, d are suitable constants, and $\alpha > 0$, $\beta > 0$, $a > 0$, $b > 0$. An n-scroll attractor is generated under the following constraints:

$$n = c + 1 \tag{25}$$

and

$$d = \begin{cases} \pi & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases} \tag{26}$$

In (24), one has

$$f(x) - f(\tilde{x}) = k_{x,\tilde{x}}(x - \tilde{x}), \tag{27}$$

where $k_{x,\tilde{x}}$ is dependent on x and $\tilde{x}$, and satisfies the condition of $-\pi b/2a \leqslant k_{x,\tilde{x}} \leqslant \pi b/2a$ (similar to Fig. 1).

The slave system for system (23), via a linear unidirectional coupling, is

$$\begin{cases} \dot{\tilde{x}} = \alpha(\tilde{y} - f(\tilde{x})) + k_1(x - \tilde{x}), \\ \dot{\tilde{y}} = \tilde{x} - \tilde{y} + \tilde{z} + k_2(y - \tilde{y}), \\ \dot{\tilde{z}} = -\beta\tilde{y} + k_3(z - \tilde{z}). \end{cases} \tag{28}$$

Subtracting (28) from (23), the following error dynamical system is obtained:

$$\begin{cases} \dot{e}_x = \alpha(e_y - k_{x,\tilde{x}}e_x) - k_1 e_x, \\ \dot{e}_y = e_x - e_y + e_z - k_2 e_y, \\ \dot{e}_z = -\beta e_y - k_3 e_z, \end{cases} \tag{29}$$

where $e_x = x - \tilde{x}$, $e_y = y - \tilde{y}$, $e_z = z - \tilde{z}$. System (29) can be rewritten as

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + g(\mathbf{x}) - g(\tilde{\mathbf{x}}) - \mathbf{K}\mathbf{e}, \tag{30}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} x - \tilde{x} \\ y - \tilde{y} \\ z - \tilde{z} \end{bmatrix} \quad \text{and} \quad g(\mathbf{x}) = \begin{bmatrix} -\alpha f(x) \\ 0 \\ 0 \end{bmatrix}.$$

Hence,

$$g(\mathbf{x}) - g(\tilde{\mathbf{x}}) = \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}}\mathbf{e} \quad \text{and} \quad \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}} = \begin{bmatrix} -\alpha k_{x,\tilde{x}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It then follows that

$$(\mathbf{A} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}}) + (\mathbf{A} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}})^{\mathrm{T}} = \begin{bmatrix} -2ak_{x,\tilde{x}} & \alpha + 1 & 0 \\ \alpha + 1 & -2 & 1 - \beta \\ 0 & 1 - \beta & 0 \end{bmatrix}. \tag{31}$$

It then follows from Theorems 1 and 2 that with

$$\begin{aligned} k_1 &\geqslant \tfrac{1}{2}(\pi b + 1 + \alpha - \mu), \\ k_2 &\geqslant \tfrac{1}{2}(-1 + \alpha + |1 - \beta| - \mu), \\ k_3 &\geqslant \tfrac{1}{2}(|1 - \beta| - \mu), \end{aligned} \tag{32}$$

the two coupled modified Chua's systems (23) and (28) are globally asymptotically synchronized.

Corollary 2. *For the two coupled modified Chua's systems* (23) *and* (28), *if* k_1, k_2, k_3 *are chosen such that the inequality* (32) *holds*, *then they are globally asymptotically synchronized.*

Let $\alpha = 10.814$, $\beta = 14.0$, $a = 1.3$, $b = 0.11$, $c = 3$, $d = 0$. Then, system (23) exhibits chaotic behavior (see Fig. 4). Select $\mu = -0.5$ and the coupling parameters as $k_1 = 8$, $k_2 = 12$, $k_3 = 8$. Then, the inequality (32) holds. Hence, it follows from Corollary 2 that the two coupled modified Chua's circuits (23) and (28), with the above-chosen parameters, are globally asymptotically synchronized, as shown in Fig. 5.

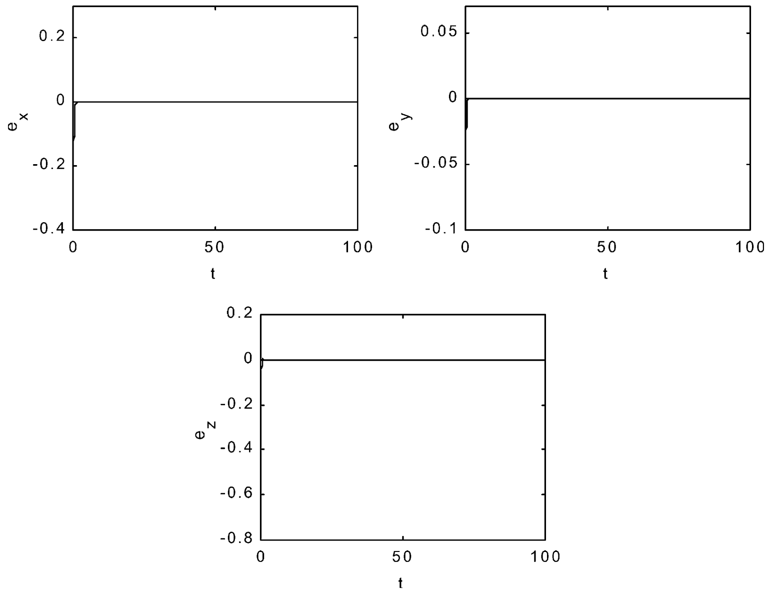

Fig. 5. The difference signal $e_x = x - \tilde{x}$, $e_y = y - \tilde{y}$, $e_z = z - \tilde{z}$ in two coupled modified Chua's circuits with the coupling parameters $k_1 = 8$, $k_2 = 12$ and $k_3 = 8$.

4. Rössler system

Rössler system [27] is described by the following equation:

$$\begin{cases} \dot{x} = -(y + z), \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - c), \end{cases} \tag{33}$$

where a, b and c denote positive parameters. According to the unidirectional linear error feedback coupling approach, the slave system of (33) is constructed as follows:

$$\begin{cases} \dot{\tilde{x}} = -(\tilde{y} + \tilde{z}) + k_1(x - \tilde{x}), \\ \dot{\tilde{y}} = \tilde{x} + a\tilde{y} + k_2(y - \tilde{y}), \\ \dot{\tilde{z}} = b + \tilde{z}(\tilde{x} - c) + k_3(z - \tilde{z}). \end{cases} \tag{34}$$

It follows from (33) and (34) that

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + g(\mathbf{x}) - g(\tilde{\mathbf{x}}) - \mathbf{K}\mathbf{e}, \tag{35}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} x - \tilde{x} \\ y - \tilde{y} \\ z - \tilde{z} \end{bmatrix} \quad \text{and} \quad g(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ xz \end{bmatrix}.$$

Hence, one has

$$g(\mathbf{x}) - g(\tilde{\mathbf{x}}) = \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}}\mathbf{e} \quad \text{and} \quad \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{z} & 0 & x \end{bmatrix},$$

so that

$$(\mathbf{A} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}}) + (\mathbf{A} + \mathbf{M}_{\mathbf{x},\tilde{\mathbf{x}}})^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & \tilde{z} - 1 \\ 0 & 2a & 0 \\ \tilde{z} - 1 & 0 & 2x - 2c \end{bmatrix}. \tag{36}$$

It follows from Theorems 1 and 2 that if

$$\begin{aligned} &k_1 \geqslant \tfrac{1}{2}(|\tilde{z} - 1| - \mu),\\ &k_2 \geqslant \tfrac{1}{2}(2a - \mu),\\ &k_3 \geqslant \tfrac{1}{2}(|\tilde{z} - 1| + 2x - 2c - \mu), \end{aligned} \tag{37}$$

then the two coupled Rössler systems (33) and (34) are globally asymptotically synchronized.

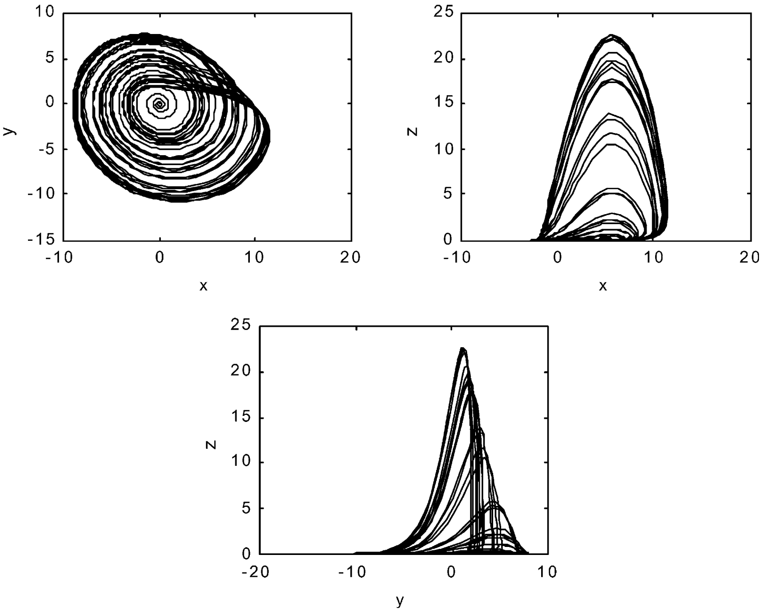

Fig. 6. The attractors of Rössler chaotic system.

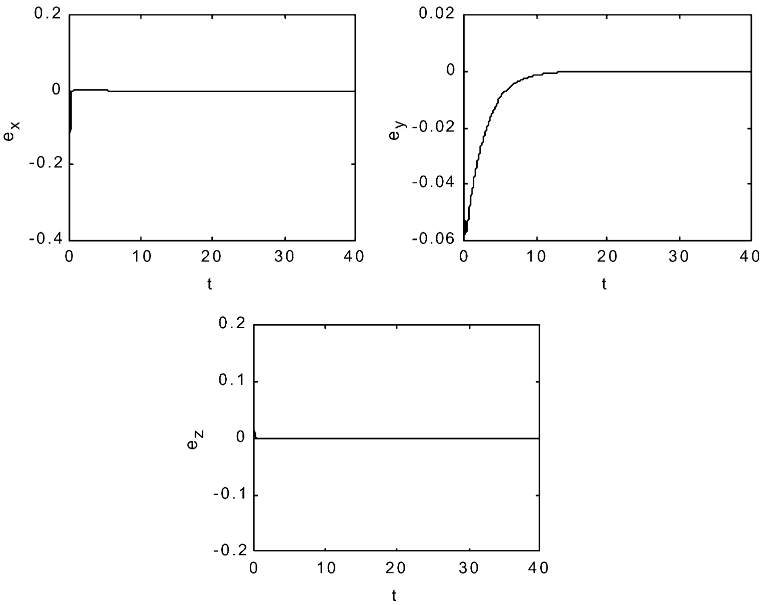

Fig. 7. The difference signal $e_x = x - \tilde{x}$, $e_y = y - \tilde{y}$, $e_z = z - \tilde{z}$ in two coupled Rössler chaotic systems with the coupling parameters $k_1 = 12$, $k_2 = 0.5$ and $k_3 = 19.5$.

Corollary 3. *For the two coupled Rössler systems* (33) *and* (34), *if* k_1, k_2, k_3 *are chosen such that the inequality* (37) *holds, then they are globally asymptotically synchronized.*

Remark 5. Since the trajectory of a chaotic system is bounded, inequality (37) holds for large enough values of k_1, k_2, k_3.

Selecting $a = 0.2$, $b = 0.2$, $c = 5.7$ gives a chaotic behavior of the system as depicted in Fig. 6. From the figure, one can see that $-10 < x < 13$, $-12 < y < 8$, $0 < z < 24$. Choosing $\mu = -0.5$ and the coupling parameters as $k_1 = 12$, $k_2 = 0.5$, $k_3 = 19.5$ will satisfy inequality (37). Hence, by Corollary 3, the two coupled Rössler systems (33) and (34) are globally asymptotically synchronized, as shown in Fig. 7.

5. Concluding remarks

In this paper, a simple algebraic condition is derived for the global synchronization of two coupled general chaotic systems with a unidirectional linear error feedback coupling. Suitable coupling parameters can be easily designed accounting to the given condition to ensure the global chaos synchronization. This simple criterion is applicable to a large class of chaotic systems. Simulations have shown its effectiveness in application to some typical chaotic systems with different types of nonlinearities. The new result can also be applied to many other chaotic systems, such as the Murali–Lakshmanan–Chua (MLC) circuit [23], Chua's circuit with cubic nonlinearity [13], modified Chua's circuit with nonlinear quadratic function $x|x|$ [31], Lorenz system [21], Chen's system [5], and so on. Further simplification of the conditions is still quite possible, which will be investigated elsewhere.

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