A Simple Method for Estimating Demand Systems under Separable Utility Assumptions [1]

The various separable utility assumptions imply special conditions for the Slutsky conditions of classical demand theory. These conditions may be used as non-linear parametric restrictions in the estimation of a complete system of linear demand equations. In the process of estimation the null hypothesis on the compatibility of the prior and sample information is tested and rejected. The procedure is applied to Barten's sixteen sector Dutch data and four forms of the separability hypothesis are investigated on a particular grouping.

I. THE SEPARABILITY HYPOTHESIS

The separability hypothesis, first advanced by Sono [16] and Leontief [12] takes two forms; a function may be either additive or separable. A utility function is additive if it is of the form

$$u(q) = F[u^1(q^1)+u^2(q^2)\ldots+u^m(q^m)] \qquad \ldots(1)$$

where the individual utilities u^i are a function of the associated quantities consumed and F is an additive function of the m utilities. A more general form of additivity, referred to here as groupwise additivity, exists when the primary utilities are functions of more than one commodity.

$$u(q) = F[u^1(q^1)+u^2(q^2)\ldots+u^m(q^m)], \qquad \ldots(2)$$

where q^i are the quantities consumed of the m groups of goods. A utility function is separable if it is of the form

$$u(q) = F[u^1(q^1),\, u^2(q^2),\, \ldots,\, u^m(q^m)]. \qquad \ldots(3)$$

Pearce [13] has shown that the Slutsky terms corresponding to a utility function which is groupwise additive are of the form

$$K_{ij} = \frac{\partial q_i}{\partial p_j} + q_j \frac{\partial q_i}{\partial M} = \phi \frac{\partial q_i}{\partial M}\frac{\partial q_j}{\partial M} = K_{ji} \qquad \ldots(4)$$

for $i \in I, j \in J$ and $I \neq J$. K_{ij} are the income compensated substitution terms, q are quantities, p are prices and M is income. In elasticity form (4) becomes

$$(e_{ij}/w_j)+E_i = (\phi/M)E_iE_j = \theta E_iE_j = (e_{ji}/w_i)+E_j, \qquad \ldots(5)$$

and (5) expressed as a non-linear restriction on the cross elasticity is

$$e_{ij} = w_j\theta E_iE_j - w_jE_i, \qquad \ldots(6)$$

where e_{ij} is the elasticity of demand for good (i) with respect to the price of good (j), E_i is the income elasticity of demand for good (i) and $w_j = p_jq_j/M$ is the proportion of total expenditure devoted to good (j). The Slutsky condition under additivity holds as (5) for all

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i and j where $i \neq j$. The additivity coefficient $(-M/\phi)$ is variously referred to by Frisch[1] as the " flexibility of the marginal utility of money " and by Barten[2] as " the elasticity of the marginal utility of income ". The Slutsky terms for a separable utility function are of the form

$$(e_{ij}/w_j)+E_i = \theta_{IJ}E_iE_j = \theta_{JI}E_jE_i = (e_{ji}/w_i)+E_j \qquad \ldots(7)$$

for $i \in I, j \in J$ and $I \neq J$. The non-linear restriction on the cross elasticity of demand is now

$$e_{ij} = w_j\theta_{IJ}E_iE_j - w_jE_i. \qquad \ldots(8)$$

Following Pearce,[3] the between group separability coefficients may be interpreted as measures of the general level of substitution between different groups representing different wants. This point is returned to in a discussion of the results.

Various combinations of the two separability patterns are possible, these may be described by utility trees of differing forms. Goldman and Uzawa [9] define one such form as " Pearce " separability. The class of utility functions corresponding to this definition of separability is of the form

$$u(g) = F\left[u^1\left(\sum_{i=1}^{N_1} g_i(q_i)\right), \ \ldots u^m\left(\sum_{i=1}^{N_m} g_i(q_i)\right)\right]. \qquad \ldots(9)$$

In other words, the utility function is separable between groups and additively separable within groups. The Slutsky conditions under Pearce separability are the same form as (7) but the grouping is now $i \in I, j \in J$ including $I = J$.

In the subsequent empirical work four types of separable utility functions are distinguished: additive, groupwise additive, separable and Pearce separable. The object of the present paper is to outline a method for estimating a system of log-linear demand equations subject to the parameters satisfying the special non-linear restrictions imposed by the various separability hypotheses. In addition, the homogeneity and Engel aggregation conditions are enforced on the parameters. The complete restrictions are then:[4]

homogeneity

$$e_{ii} = \left(-\sum_{\substack{j=1 \\ i \neq j}}^{n} e_{ij}\right) - E_i; \qquad \ldots(10)$$

Engel aggregation

$$E_n = \frac{1}{w_n} - \sum_{j=1}^{n-1} \frac{w_j}{w_n} E_j; \qquad \ldots(11)$$

Slutsky symmetry

$$e_{ij} = \frac{w_j}{w_i} e_{ji} + w_jE_j - w_jE_i; \qquad \ldots(12)$$

and separability

$$e_{ji} = w_i\theta_{JI}E_jE_i - w_iE_j. \qquad \ldots(13)$$

The last restriction depends clearly on the separability or additivity hypothesis imposed and the grouping chosen.

It may be argued that the question has been begged here as the Slutsky and separability restrictions are local conditions applying for small changes in prices under the assumption of unchanged tastes. The hard realities of economic statistics make it necessary to use data generated over a forty year time span for which the assumptions with regard to tastes and prices are clearly violated. In addition the exercise is applied to a log linear demand function so the generality placed on results and interpretations is further reduced. This is a general weakness in attempting to match economic theory to the available data and is present in all previous studies in this field. The Slutsky, separability and Engel restrictions were imposed

1 See Frisch [8], p. 183.
2 See Barten [4], p. 4.
3 See Pearce [14], p. 210.
4 See Frisch [8].

at the mean proportion of expenditure devoted to each good; this appeared a better solution than attempting to find a specific utility function for which these conditions were satisfied globally or applying the restrictions as a succession of local approximations to the observations as the latter would result in more restrictions than could be handled by the estimation procedures outlined below. The present objective is to extend the estimation of demand systems to include the most general separability case under the existing " rules of the game ". However, it is apparent that if consumer maximisation theory is to have relevance it must be set up to deal with data covering a span of years.

II. THE ESTIMATION PROCEDURE

The problem is to estimate a system of linear equations subject to linear and non-linear parametric restrictions. A difficulty, more aparent than real, is that the separability coefficients only enter the objective function (or likelihood function) through the restrictions. The most obvious way to estimate such a system of equations is to use Zellner's [19] application of Aitken's generalised least squares and minimise

$$u'\Omega^{-1}u = (y-Z\alpha)'\Omega^{-1}(y-Z\alpha), \qquad \ldots(14)$$

where there are n equations, T observations and the same $n+1$ explanatory variables in each equation. The notation in (14) is the same as Zellner's.

The approach chosen here is to substitute the restrictions into (14) and minimise the non-linear reduced form in terms of the remaining unrestricted parameters. Under additivity only $n-1$ income elasticities and θ the additivity coefficient will remain as free parameters. Let the n reduced form parameters in this case be written as β and the relation between the structural and the reduced form is

$$\alpha = F(\beta). \qquad \ldots(15)$$

In the present context the quadratic form to be minimised is

$$Q = u'\Omega^{-1}u = (y-ZF\beta)'\Omega^{-1}(y-ZF\beta). \qquad \ldots(16)$$

After the restrictions are introduced by substitution the quadratic form is non-linear with respect to β. To minimise, this function may be approximated in the usual way by the first two terms of a Taylor series expansion and the iteration algorithm becomes.[1]

$$\beta^{n+1} = \beta^{n} - G^{-1}g \qquad \ldots(17)$$

where G and g are the matrix of second and vector of first derivatives respectively.

$$\frac{\partial Q}{\partial \beta} = \frac{\partial \alpha}{\partial \beta'} Z'\Omega^{-1}(y-Z\alpha), \qquad \ldots(18)$$

$$\frac{\partial^2 Q}{\partial \beta \partial \beta'} = \frac{\partial \alpha}{\partial \beta'} Z'\Omega^{-1}Z \frac{\partial \alpha}{\partial \beta} - \frac{\partial^2 \alpha}{\partial \beta \partial \beta'} Z'\Omega^{-1}u. \qquad \ldots(19)$$

Since plim $Z'\Omega^{-1}u$ is $0(T^{-1})$ the final term may be deleted from (19) and the Gauss-Newton solution is now

$$\beta^{n+1} = \beta^{n} - \left[\frac{\partial \alpha}{\partial \beta'} Z'\Omega^{-1}Z \frac{\partial \alpha}{\partial \beta}\right]^{-1} \frac{\partial \alpha}{\partial \beta} Z'\Omega^{-1}u. \qquad \ldots(20)$$

The reduced form variance-covariance matrix is given by (19); however a useful asymptotic approximation is $\left[\frac{\partial \alpha}{\partial \beta'} Z'\Omega^{-1}Z \frac{\partial \alpha}{\partial \beta}\right]^{-1}$. Similarly $\frac{\partial \alpha}{\partial \beta}\left[\frac{\partial \alpha}{\partial \beta'} Z'\Omega^{-1}Z \frac{\partial \alpha}{\partial \beta}\right]^{-1} \frac{\partial \alpha}{\partial \beta'}$ may be used

[1] See Powell [15].

as an asymptotic approximation to the variance-covariance matrix of the structural parameters. In practice, as is well known, convergence is facilitated by searching the direction $G^{-1}g$ for a minimum on each iteration.

The dual of the restricted estimation problem is the accompanying hypothesis testing procedure to establish the compatibility of the prior and the sample information. Three tests have been discussed previously by the writer [7] for linear restrictions, they are treated more rigorously by Aitchison and Silvey [1], [2]. The simplest test, due to Wald [18], relies on the statistic $(R\hat{b}-s)'[R(Z'\Omega^{-1}Z)^{-1}R']^{-1}(R\hat{b}-s)$ being distributed under the usual assumptions as a χ^2 with k degrees of freedom for k linear restrictions; R being the restriction matrix, s the restriction vector and $\hat{b}$ referring to the unrestricted parameter estimates. With non-linear restriction R and s will depend on the ultimate optimum values of $\tilde{b}$, the parameter estimates satisfying the restrictions. A simpler test in the present context is Hotelling's T^2 [1] statistic which may be written tr $\Omega^{-1}(\tilde{B}-\hat{B})X'X(\tilde{B}-\hat{B})'$ where $\tilde{B}$ and $\hat{B}$ are $n\times(n+1)$ matrices of $\tilde{b}$ and $\hat{b}$ arranged sequentially. It is easy to show that[2]

$$T(\tilde{\Omega}-\hat{\Omega}) = (\tilde{B}-\hat{B})X'X(\tilde{B}-\hat{B})'.$$

Consequently, a likelihood ratio test statistic emerges $T\,\mathrm{tr}\,\Omega^{-1}(\tilde{\Omega}-\hat{\Omega})$ which in actual usage may be written $T\,\mathrm{tr}\,\hat{\Omega}^{-1}(\tilde{\Omega}-\hat{\Omega})$ and is asymptotically χ^2 with k degrees of freedom. These results, developed for linear restrictions, may be extended to the non-linear case under assumptions of local linearity.[3] The likelihood ratio test is used subsequently to test the compatibility of the prior and sample information.

III. THE APPLICATION: RESTRICTED ESTIMATION AND HYPOTHESIS TESTING

The procedure was applied to Barten's sixteen commodity Dutch expenditure data, [5]. The data, more fully reported elsewhere, consists of 34 annual observations covering the periods 1921-39 and 1948-62. The commodities, groups and the mean proportion of expenditure devoted to each are given in Table 1.

The grouping chosen; food, pleasure goods, durables and other expenditure, has been used by Barten [6] for estimation under groupwise additivity. The data has also been used by the writer [7] to estimate the elasticity coefficients of the sixteen demand equations subject to the Slutsky, homogeneity and Engel restrictions. In this paper the complete unrestricted price and income elasticities, the restricted price and income elasticities under the Slutsky, homogeneity and Engel conditions and the restricted estimates under the Slutsky restrictions alone were presented. As before, constants were included in the log linear demand equations, though for ease of presentation they are not given here. Though it would have been desirable and technically possible to split the data into pre- and post-war periods and perform the restricted estimation separately on both for evidence of taste changes, this was not done; instead a dummy variable was used and when its parameters were found to be generally insignificant and to introduce little or no change into the price and income elasticities this attempt to partition the data was not pursued further.

The demand equations were estimated subject to the restrictions on the cross elasticities posed by additivity, groupwise additivity, Pearce separability and separability. Additional Slutsky restrictions were imposed on the within-group cross elasticities under the groupwise additivity and separability hypotheses; the homogeneity restrictions were imposed on the own price elasticities and the Engel aggregation condition was used to eliminate one income elasticity. As mentioned, the restrictions were applied at the mean proportion of expenditure devoted to each good so the following results may be checked using Table 1.

In all applications convergence was rapid and unique. For additivity the SELS estimates of the income elasticities were used as the starting values and the initial value of θ was 0·5. Representative convergence paths are given in Table 2.

[1] See Anderson [3], Chap. 5. [2] See Byron [7]. [3] See Aitchison and Silvey [2].

The complete price and income elasticities under additivity, homogeneity and Engel restrictions are given in Table 3. The value of the objective function under the restrictions is tr $\hat{\Omega}^{-1}\Omega$, where $\hat{\Omega} = \hat{U}\hat{U}/T$, $\hat{U}$ being a $T \times n$ matrix of the SELS residuals. The value of

TABLE 1

Grouping of the Dutch Sixteen Sector Model

Group Number	Group	Variable Number	Variable	Average Proportion of Expenditure
1	Food	1	groceries	0·0557
		2	dairy products	0·0732
		3	potatoes, fruit and vegetables	0·0451
		4	meat, meat products	0·0756
		5	fish, preserved fish	0·0071
		6	bread	0·0354
2	Pleasure Goods	7	pastry, chocolate and ice cream	0·0316
		8	tobacco products	0·0371
		9	beverages	0·0275
3	Durables	10	clothing and textiles	0·1344
		11	footwear	0·0146
		12	household durables	0·0734
		13	other durables	0·0203
4	Other	14	rent	0·0779
		15	fuel, electricity, gas and water	0·0534
		16	other goods and services	0·2377

the likelihood ratio test statistic is 41489 which, on a χ^2 at 257 degrees of freedom, leads to a clear rejection of the null hypothesis. The estimate of the separability coefficient θ of $\underset{(0{\cdot}121)}{0{\cdot}707}$ results in an estimate of the elasticity of the marginal utility of income of $\underset{(0{\cdot}48)}{-1{\cdot}41}$.

TABLE 2

Additivity: Convergence of Estimates

E_1	E_9	E_{15}	θ	$\tilde{u}'\hat{\Omega}^{-1}\tilde{u}$
1·145	0·047	1·250	0·500	3238·44
0·638	1·351	0·856	0·696	1220·93
0·634	1·329	0·831	0·708	1201·44
0·627	1·323	0·835	0·707	1201·40
0·627	1·323	0·834	0·707	1201·40

The second hypothesis tested was Pearce separability: between groups the cross elasticities obeyed the separability form of the Slutsky hypothesis, whilst within groups they obeyed the additivity form of the hypothesis. The starting values were the income elasticities obtained under additivity while $\theta_{IJ} = 0{\cdot}5$ were used for all the initial separability coefficients. Convergence took 10 iterations, probably because of poor initial starting values. The separability coefficients and their standard errors obtained under Pearce separability are given in Table 4 while the accompanying price and income elasticities follow in Table 5.

TABLE 3

Price and income elasticities and standard errors under additivity

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	M
q_1	−0·463 (0·009)	−0·008 (0·0003)	0·008 (0·0004)	0·001 (0·0008)	0·003 (0·0003)	−0·028 (0·0006)	−0·014 (0·0003)	−0·004 (0·0002)	−0·001 (0·0005)	−0·001 (0·0007)	0·003 (0·0001)	0·012 (0·0006)	0·009 (0·0003)	−0·044 (0·0009)	−0·014 (0·0005)	−0·086 (0·002)	0·628 (0·013)
q_2	−0·036 (0·0007)	−0·829 (0·007)	0·016 (0·0007)	0·002 (0·002)	0·006 (0·0005)	−0·052 (0·0005)	−0·026 (0·0004)	−0·008 (0·0004)	−0·002 (0·0005)	−0·002 (0·001)	0·005 (0·0002)	0·023 (0·001)	0·017 (0·0004)	−0·081 (0·0007)	−0·025 (0·0007)	−0·158 (0·002)	1·151 (0·010)
q_3	−0·057 (0·001)	−0·025 (0·001)	−1·274 (0·012)	0·004 (0·003)	0·010 (0·001)	−0·083 (0·001)	−0·042 (0·001)	−0·012 (0·001)	−0·003 (0·001)	−0·003 (0·002)	0·008 (0·0003)	0·037 (0·002)	0·027 (0·0007)	−0·130 (0·001)	−0·040 (0·001)	−0·253 (0·004)	1·837 (0·018)
q_4	−0·045 (0·001)	−0·019 (0·001)	0·019 (0·001)	−1·024 (0·016)	0·008 (0·001)	−0·066 (0·001)	−0·034 (0·001)	−0·010 (0·0005)	−0·002 (0·001)	−0·003 (0·002)	0·007 (0·0002)	0·029 (0·001)	0·021 (0·001)	−0·103 (0·002)	−0·032 (0·001)	−0·200 (0·004)	1·452 (0·023)
q_5	−0·077 (0·003)	−0·034 (0·002)	0·033 (0·002)	0·005 (0·003)	−1·748 (0·061)	−0·113 (0·004)	−0·058 (0·002)	−0·017 (0·001)	−0·004 (0·001)	−0·004 (0·003)	0·011 (0·001)	0·050 (0·003)	0·037 (0·002)	−0·176 (0·006)	−0·054 (0·002)	−0·343 (0·013)	2·492 (0·088)
q_6	0·012 (0·0004)	0·005 (0·0002)	−0·005 (0·0003)	−0·001 (0·0005)	−0·002 (0·0002)	0·301 (0·005)	0·009 (0·0002)	0·003 (0·0001)	0·001 (0·0002)	−0·001 (0·0004)	−0·002 (0·0001)	−0·008 (0·0004)	−0·006 (0·0002)	0·028 (0·0005)	0·009 (0·0002)	0·055 (0·0012)	−0·400 (0·007)
q_7	−0·012 (0·0004)	−0·005 (0·0003)	0·005 (0·0003)	0·001 (0·0005)	0·002 (0·0002)	−0·017 (0·0005)	−0·278 (0·009)	−0·002 (0·0001)	−0·001 (0·0002)	−0·001 (0·0004)	0·002 (0·0001)	0·008 (0·0003)	0·006 (0·0002)	−0·027 (0·001)	−0·008 (0·0004)	−0·053 (0·002)	0·381 (0·013)
q_8	−0·036 (0·0007)	−0·016 (0·0007)	0·016 (0·0007)	0·002 (0·002)	0·006 (0·001)	−0·052 (0·001)	−0·027 (0·0004)	−0·827 (0·008)	−0·002 (0·001)	−0·002 (0·001)	0·005 (0·0002)	0·023 (0·001)	0·017 (0·0004)	−0·082 (0·0008)	−0·025 (0·0007)	−0·160 (0·0003)	1·159 (0·012)
q_9	−0·041 (0·001)	−0·018 (0·001)	0·017 (0·001)	0·002 (0·002)	0·007 (0·001)	−0·060 (0·001)	−0·030 (0·001)	−0·008 (0·0005)	−0·938 (0·016)	−0·002 (0·001)	0·006 (0·0002)	0·027 (0·001)	0·019 (0·0005)	−0·093 (0·002)	−0·029 (0·001)	−0·182 (0·004)	1·323 (0·024)
q_{10}	−0·043 (0·001)	−0·019 (0·001)	0·019 (0·001)	0·003 (0·002)	0·007 (0·001)	−0·063 (0·0004)	−0·032 (0·0004)	−0·009 (0·001)	−0·002 (0·001)	−0·989 (0·0002)	0·006 (0·001)	0·028 (0·001)	0·020 (0·001)	−0·098 (0·001)	−0·030 (0·0003)	−0·192 (0·003)	1·396 (0·009)
q_{11}	−0·058 (0·001)	−0·025 (0·001)	0·025 (0·001)	0·004 (0·003)	0·010 (0·0001)	−0·085 (0·001)	−0·043 (0·001)	−0·012 (0·001)	−0·003 (0·001)	−0·003 (0·002)	−1·308 (0·011)	0·038 (0·002)	0·027 (0·001)	−0·132 (0·001)	−0·041 (0·001)	−0·257 (0·004)	1·864 (0·016)
q_{12}	−0·056 (0·001)	−0·024 (0·001)	0·024 (0·001)	0·003 (0·002)	0·010 (0·001)	−0·082 (0·001)	−0·042 (0·001)	−0·012 (0·001)	−0·003 (0·001)	−0·003 (0·002)	0·008 (0·0003)	−1·239 (0·010)	0·027 (0·001)	−0·127 (0·001)	−0·039 (0·001)	−0·249 (0·003)	1·805 (0·016)
q_{13}	−0·076 (0·001)	−0·033 (0·001)	0·033 (0·001)	0·005 (0·003)	0·013 (0·001)	−0·111 (0·001)	−0·056 (0·001)	−0·016 (0·001)	−0·004 (0·001)	−0·004 (0·003)	0·011 (0·0004)	0·049 (0·002)	−1·693 (0·016)	−0·173 (0·002)	−0·054 (0·002)	−0·337 (0·005)	2·448 (0·023)
q_{14}	−0·004 (0·0001)	−0·002 (0·0001)	0·002 (0·0001)	0·000 (0·0002)	0·001 (0·0001)	−0·006 (0·0001)	−0·003 (0·0001)	−0·001 (0·0001)	−0·000 (0·0001)	−0·000 (0·0002)	0·001 (0·0000)	0·003 (0·0001)	0·001 (0·0000)	−0·102 (0·001)	−0·003 (0·0001)	−0·018 (0·0003)	0·131 (0·0016)
q_{15}	−0·026 (0·0007)	−0·011 (0·0005)	0·011 (0·0005)	0·002 (0·001)	0·004 (0·0004)	−0·037 (0·0007)	−0·019 (0·0004)	−0·006 (0·0003)	−0·001 (0·0034)	−0·001 (0·0009)	0·004 (0·0001)	0·017 (0·001)	0·012 (0·0002)	−0·059 (0·001)	−0·607 (0·001)	−0·115 (0·002)	0·833 (0·014)
q_{16}	−0·018 (0·0005)	−0·008 (0·0003)	0·008 (0·0005)	0·001 (0·0008)	0·003 (0·0003)	−0·026 (0·0002)	−0·014 (0·0002)	−0·004 (0·0003)	−0·001 (0·0003)	−0·001 (0·0007)	0·003 (0·0001)	0·013 (0·0005)	0·009 (0·0002)	−0·042 (0·0005)	−0·013 (0·0003)	−0·025 (0·006)	0·594 (0·007)

The starting values chosen for the imposition of the separability hypothesis were the income, within-group cross elasticities and between-group separability coefficients derived under Pearce separability. The between-group cross elasticities were constrained to the Slutsky restrictions under the separability hypothesis while the within-group cross elasticities were simply constrained by the Slutsky conditions. Convergence was achieved in six iterations. The separability coefficients and their standard errors are given in Table 6 while the accompanying price and income elasticities follow in Table 7.

The value of the objective function under separability of 551·06 was a considerable reduction on the result under either additivity or Pearce separability. However, when converted to a likelihood ratio test on 230 degrees of freedom the null hypothesis was again rejected.

TABLE 4

"Pearce" separability coefficients and standard errors

	1	2	3	4
1	13·369 (0·732)	2·701 (0·167)	1·639 (0·062)	0·429 (0·028)
2		5·660 (0·409)	0·197 (0·079)	−0·004 (0·034)
3			2·843 (0·050)	0·252 (0·009)
4				−0·174 (0·009)

The final application was to impose the groupwise additivity hypothesis. The initial estimates of the income and within-group cross elasticities were the values under the separability hypothesis. The starting value for θ was Barten's estimate of 0·555. Convergence was achieved in five iterations. At convergence the additivity coefficient was $\underset{(0{\cdot}006)}{0{\cdot}352}$ and the accompanying estimate of the elasticity of the marginal utility of income was $\underset{(0{\cdot}38)}{-2{\cdot}84}$ which is considerably higher than Barten's estimate of $-1{\cdot}80$. The price and income elasticities under groupwise additivity are given in Table 8. The value of the objective function achieved was 625·98, which results again in rejection of the null hypothesis on the likelihood ratio test at 230 degrees of freedom.

A comparison of the objective values under the four separability hypotheses as well as under the Slutsky, homogeneity and Engel conditions is made in Table 9.

With such a mass of results it becomes difficult to discuss their meaning in great detail; certain general points emerge however. First, if the gross cross price elasticities of substitution are converted to Hicks-Allen[1] elasticities of substitution and then separated into net elasticities and income effects a general dominance of the income effects over the substitution effects emerges,

$$e_{ij} = w_j E_i + w_j \sigma_{ij}, \qquad \text{...(21)}$$

where σ_{ij} is the Hicks-Allen elasticity of substitution. An examination of Table 3, the price and income elasticities under additivity indicates a scatter of results with 60 per cent of the cross elasticities exhibiting gross complementarity. However, if the conversion is made to remove the income effects, it emerges that all the commodities are net substitutes with the exception of group 6 which, not surprisingly, is bread. Referring back to Table 3, the

[1] See Hicks and Allen [11].

TABLE 5

Price and income elasticities and standard errors: " Pearce " separability

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	M
q_1	−0·522 (0·013)	0·140 (0·005)	0·040 (0·001)	0·078 (0·003)	0·012 (0·001)	−0·021 (0·001)	0·014 (0·001)	0·012 (0·001)	0·011 (0·001)	−0·003 (0·001)	0·002 (0·002)	0·047 (0·003)	0·012 (0·001)	−0·011 (0·001)	−0·003 (0·001)	−0·022 (0·002)	0·212 (0·007)
q_2	0·077 (0·004)	−1·428 (0·014)	0·140 (0·004)	0·277 (0·010)	0·044 (0·002)	−0·074 (0·002)	0·050 (0·004)	0·042 (0·004)	0·039 (0·004)	−0·010 (0·003)	0·008 (0·001)	0·168 (0·008)	0·043 (0·002)	−0·041 (0·002)	−0·009 (0·002)	−0·077 (0·007)	0·752 (0·012)
q_3	0·039 (0·002)	0·255 (0·007)	−0·913 (0·011)	0·142 (0·005)	0·022 (0·001)	−0·038 (0·001)	0·025 (0·002)	0·022 (0·002)	0·020 (0·002)	−0·005 (0·002)	0·004 (0·0003)	0·086 (0·004)	0·022 (0·001)	−0·021 (0·001)	−0·005 (0·001)	0·039 (0·004)	0·384 (0·011)
q_4	0·045 (0·002)	0·291 (0·010)	0·082 (0·103)	−0·963 (0·024)	0·026 (0·001)	−0·043 (0·001)	0·029 (0·003)	0·025 (0·002)	0·023 (0·002)	−0·006 (0·002)	0·005 (0·0004)	0·098 (0·005)	0·025 (0·001)	−0·024 (0·001)	−0·005 (0·001)	−0·045 (0·005)	0·439 (0·016)
q_5	0·071 (0·004)	0·457 (0·021)	0·129 (0·006)	0·253 (0·013)	−1·725 (0·069)	−0·068 (0·003)	0·046 (0·004)	0·039 (0·004)	0·036 (0·004)	−0·009 (0·003)	0·007 (0·001)	0·153 (0·010)	0·039 (0·003)	−0·037 (0·003)	−0·008 (0·002)	−0·071 (0·008)	0·689 (0·034)
q_6	−0·014 (0·001)	−0·088 (0·003)	−0·025 (0·001)	−0·049 (0·002)	−0·008 (0·002)	0·356 (0·001)	−0·009 (0·008)	−0·007 (0·001)	−0·007 (0·001)	0·002 (0·001)	−0·001 (0·001)	−0·030 (0·002)	−0·008 (0·0004)	0·007 (0·0005)	0·002 (0·0004)	0·014 (0·001)	−0·134 (0·004)
q_7	−0·027 (0·003)	0·086 (0·010)	0·002 (0·003)	0·016 (0·007)	0·007 (0·001)	−0·055 (0·001)	−0·882 (0·016)	0·181 (0·015)	0·158 (0·012)	−0·137 (0·007)	−0·013 (0·001)	−0·043 (0·016)	−0·013 (0·004)	−0·089 (0·002)	−0·061 (0·004)	−0·273 (0·013)	1·145 (0·016)
q_8	−0·022 (0·002)	0·070 (0·008)	0·002 (0·003)	0·013 (0·006)	0·006 (0·001)	−0·045 (0·001)	0·161 (0·013)	−0·730 (0·014)	0·129 (0·010)	−0·111 (0·003)	−0·011 (0·001)	−0·035 (0·013)	−0·030 (0·003)	−0·072 (0·002)	−0·050 (0·003)	−0·222 (0·010)	0·929 (0·014)
q_9	−0·025 (0·002)	0·080 (0·009)	0·002 (0·008)	0·015 (0·006)	0·007 (0·001)	−0·051 (0·001)	0·184 (0·014)	0·168 (0·013)	−0·858 (0·019)	−0·128 (0·006)	−0·012 (0·001)	−0·040 (0·015)	−0·012 (0·004)	−0·083 (0·002)	−0·057 (0·003)	−0·254 (0·012)	1·065 (0·084)
q_{10}	−0·020 (0·001)	0·009 (0·002)	−0·009 (0·001)	−0·012 (0·002)	0·001 (0·0003)	−0·023 (0·0004)	−0·013 (0·002)	−0·017 (0·002)	−0·012 (0·001)	−0·638 (0·007)	0·016 (0·0004)	0·243 (0·004)	0·063 (0·001)	−0·035 (0·0006)	−0·016 (0·0006)	−0·087 (0·002)	0·552 (0·009)
q_{11}	−0·035 (0·001)	0·018 (0·004)	−0·017 (0·001)	−0·022 (0·003)	−0·001 (0·0005)	−0·044 (0·0007)	−0·025 (0·003)	−0·031 (0·003)	−0·022 (0·002)	0·079 (0·004)	−1·245 (0·013)	0·456 (0·009)	0·117 (0·002)	−0·066 (0·001)	−0·030 (0·001)	−0·164 (0·004)	1·034 (0·016)
q_{12}	−0·090 (0·003)	0·042 (0·009)	−0·041 (0·003)	−0·052 (0·007)	0·002 (0·001)	−0·106 (0·001)	−0·060 (0·007)	−0·075 (0·007)	−0·053 (0·006)	0·189 (0·008)	0·070 (0·002)	−1·949 (0·018)	0·279 (0·005)	−0·157 (0·002)	−0·072 (0·002)	−0·390 (0·008)	2·464 (0·018)
q_{13}	−0·084 (0·002)	0·039 (0·003)	−0·039 (0·003)	−0·049 (0·006)	0·002 (0·001)	−0·100 (0·007)	−0·057 (0·006)	−0·070 (0·005)	−0·050 (0·006)	0·177 (0·008)	0·065 (0·001)	1·021 (0·010)	−2·591 (0·028)	−0·148 (0·002)	−0·068 (0·002)	−0·367 (0·007)	2·316 (0·022)
q_{14}	−0·036 (0·0004)	−0·056 (0·001)	−0·027 (0·0004)	−0·043 (0·001)	−0·004 (0·0001)	−0·027 (0·0003)	−0·023 (0·001)	−0·028 (0·001)	−0·020 (0·001)	−0·082 (0·001)	−0·008 (0·0001)	−0·020 (0·001)	−0·006 (0·0003)	−0·097 (0·002)	−0·050 (0·001)	−0·208 (0·003)	0·713 (0·007)
q_{15}	−0·091 (0·001)	−0·089 (0·003)	−0·068 (0·001)	−0·110 (0·002)	−0·009 (0·0003)	−0·068 (0·001)	−0·057 (0·002)	−0·067 (0·002)	−0·050 (0·002)	−0·208 (0·003)	−0·019 (0·0004)	−0·050 (0·003)	−0·015 (0·0009)	−0·157 (0·002)	−0·214 (0·004)	−0·526 (0·008)	1·798 (0·023)
q_{16}	−0·067 (0·001)	−0·065 (0·002)	−0·050 (0·001)	−0·081 (0·001)	−0·007 (0·0002)	−0·050 (0·0003)	−0·042 (0·002)	−0·049 (0·002)	−0·036 (0·001)	−0·153 (0·002)	−0·014 (0·0002)	−0·037 (0·002)	−0·011 (0·001)	−0·115 (0·001)	−0·092 (0·001)	−0·451 (0·004)	1·317 (0·008)

income elasticity of demand for bread is $\dfrac{-0{\cdot}400}{(0{\cdot}007)}$. Under additivity then, the constraints produce one inferior good. However, in the unrestricted case[1], bread was not an inferior good, though indeed the result was not significantly different from zero. Again, with apologies, referring to results too detailed to present here, when the Slutsky, homogeneity and Engel restrictions only were introduced there resulted a scatter of net complementarity and substitutability with no overall domination of the income effect.

The simplest way to examine the dominance of the income effects is in a tabular fashion. The income effect is dominant throughout under additivity and groupwise additivity while under Pearce separability and separability it dominates the elasticities of groups 2, 3 and 4. The reason why this is so is readily apparent; the imposition of the various types of separability tends to make the income elasticities larger and the cross price elasticities smaller than in the unrestricted case.

TABLE 6

Separability coefficients and standard errors

	1	2	3	4
1		1·672 (0·120)	1·809 (0·071)	1·798 (0·032)
2			0·987 (0·080)	−0·340 (0·031)
3				0·213 (0·011)
4				

It can be readily ascertained that normal goods can be expected to be net substitutes under separability restrictions. The presence of gross complementarity or substitutability depends on the magnitude of the income elasticities relative to the separability coefficient.

$$(e_{ij}/w_j)+E_i = \theta_{IJ}E_IE_J = (e_{ji}/w_i)+E_j, \qquad \text{...(22)}$$

where e_{ij}, e_{ji} are gross elasticities and $e_{ij}+w_jE_j$, $e_{ji}+w_iE_j$ are the net elasticities. For normal goods, E_i, $E_j>0$ and e_{ij} will be a gross complement only if $\theta_{IJ}E_j<1$. Furthermore, to be net substitutes all that is required is that $\theta_{IJ}>0$.

It is easy to ascertain that θ_{IJ} becomes a measure of the degree of substitutability between groups of goods. Using Theil's [17] notation the price and income derivatives may be expressed as

$$\begin{bmatrix} U & p \\ p' & 0 \end{bmatrix}\begin{bmatrix} Q_p & M \\ -\lambda_p & -\lambda_M \end{bmatrix} = \begin{bmatrix} I & 0 \\ -q' & 1 \end{bmatrix}, \qquad \text{...(23)}$$

where $U = \dfrac{\partial^2 u}{\partial q_i \partial q_j}$, $q_M = \dfrac{\partial q_i}{\partial M}$, $Q_p = \dfrac{\partial q_i}{\partial p_j}$, $\lambda_M = \dfrac{\partial \lambda}{\partial M}$ and $\lambda_p = \dfrac{\partial \lambda}{\partial p_i}$.

The solution for the Slutsky equation is

$$\frac{\partial q_i}{\partial p_j}+q_j\frac{\partial q_i}{\partial M} = \phi_{IJ}\frac{\partial q_i}{\partial M}\frac{\partial q_j}{\partial M} = \lambda U^{ij}-\left(\lambda\Big/\frac{\partial \lambda}{\partial M}\right)\frac{\partial q_i}{\partial M}\frac{\partial q_j}{\partial M}. \qquad \text{...(24)}$$

[1] See Byron [7].

TABLE 7

Price and income elasticities and standard errors: weak separability

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	M
q_1	−0·796	0·116	−0·004	−0·071	0·017	0·323	0·007	−0·017	−0·010	0·019	0·007	0·069	0·016	−0·009	0·010	−0·008	0·331
	(0·019)	(0·015)	(0·012)	(0·017)	(0·008)	(0·012)	(0·002)	(0·002)	(0·001)	(0·003)	(0·001)	(0·006)	(0·001)	(0·001)	(0·001)	(0·003)	(0·020)
q_2	0·085	−1·270	0·147	0·392	0·098	0·020	0·008	0·020	−0·012	0·023	0·008	0·082	0·019	−0·010	0·011	−0·010	0·389
	(0·012)	(0·023)	(0·012)	(0·018)	(0·010)	(0·013)	(0·002)	(0·002)	(0·001)	(0·003)	(0·001)	(0·006)	(0·001)	(0·001)	(0·001)	(0·004)	(0·017)
q_3	−0·031	0·209	−1·018	−0·151	0·165	−0·264	0·017	0·042	−0·024	0·046	0·017	0·069	−0·040	−0·021	−0·024	−0·020	0·803
	(0·014)	(0·019)	(0·019)	(0·020)	(0·011)	(0·013)	(0·003)	(0·005)	(0·001)	(0·006)	(0·001)	(0·009)	(0·002)	(0·002)	(0·002)	(0·008)	(0·029)
q_4	−0·091	0·352	−0·100	−1·150	−0·022	−0·160	0·021	−0·053	−0·031	0·059	0·021	0·215	0·050	−0·027	0·030	−0·026	1·023
	(0·013)	(0·017)	(0·012)	(0·030)	(0·009)	(0·009)	(0·004)	(0·006)	(0·002)	(0·008)	(0·002)	(0·011)	(0·003)	(0·003)	(0·003)	(0·010)	(0·035)
q_5	−0·132	1·011	1·065	−2·295	−0·927	0·878	0·006	0·015	−0·009	0·017	0·006	0·062	0·014	−0·008	0·009	−0·007	0·294
	(0·069)	(0·099)	(0·070)	(0·100)	(0·085)	(0·005)	(0·002)	(0·006)	(0·003)	(0·007)	(0·002)	(0·022)	(0·003)	(0·003)	(0·003)	(0·004)	(0·105)
q_6	0·528	0·071	−0·298	−0·261	0·179	−0·178	−0·001	−0·002	0·001	−0·001	−0·001	−0·006	−0·002	0·001	−0·001	0·001	−0·030
	(0·018)	(0·025)	(0·019)	(0·019)	(0·013)	(0·024)	(0·0004)	(0·001)	(0·0006)	(0·001)	(0·0004)	(0·004)	(0·001)	(0·0005)	(0·0005)	(0·0006)	(0·019)
q_7	−0·024	−0·025	0·015	0·053	−0·004	−0·037	−0·671	−0·138	0·278	−0·029	0·005	0·081	0·017	−0·099	−0·088	−0·326	0·994
	(0·003)	(0·004)	(0·004)	(0·008)	(0·001)	(0·001)	(0·016)	(0·020)	(0·011)	(0·008)	(0·002)	(0·013)	(0·003)	(0·003)	(0·004)	(0·010)	(0·021)
q_8	−0·038	−0·037	0·022	0·077	−0·005	−0·053	−0·132	−0·833	−0·197	−0·042	0·007	0·116	0·025	−0·143	−0·127	−0·470	1·434
	(0·004)	(0·006)	(0·006)	(0·012)	(0·002)	(0·002)	(0·011)	(0·026)	(0·010)	(0·012)	(0·002)	(0·018)	(0·005)	(0·003)	(0·003)	(0·012)	(0·021)
q_9	0·001	0·002	−0·001	−0·003	0·000	0·002	0·352	0·321	−0·641	0·002	−0·000	−0·005	−0·001	0·006	0·005	0·020	−0·060
	(0·001)	(0·0009)	(0·0006)	(0·002)	(0·0001)	(0·000)	(0·013)	(0·014)	(0·017)	(0·001)	(0·0002)	(0·003)	(0·0006)	(0·003)	(0·003)	(0·011)	(0·033)
q_{10}	−0·017	−0·017	0·016	0·051	−0·003	−0·029	−0·000	0·012	−0·023	−0·706	0·099	0·044	0·002	−0·051	−0·025	−0·143	0·791
	(0·002)	(0·002)	(0·002)	(0·005)	(0·001)	(0·001)	(0·002)	(0·003)	(0·0009)	(0·010)	(0·003)	(0·010)	(0·007)	(0·001)	(0·001)	(0·003)	(0·015)
q_{11}	−0·030	−0·029	0·027	0·086	−0·005	−0·050	−0·001	0·021	−0·039	0·833	−0·527	0·327	−1·582	−0·086	−0·042	−0·242	1·337
	(0·003)	(0·004)	(0·004)	(0·008)	(0·002)	(0·002)	(0·003)	(0·006)	(0·001)	(0·080)	(0·027)	(0·034)	(0·034)	(0·002)	(0·002)	(0·006)	(0·027)
q_{12}	−0·048	−0·046	0·044	0·137	−0·007	−0·079	−0·001	0·033	−0·006	−0·100	0·053	−1·730	0·263	−0·137	−0·067	−0·386	2·134
	(0·006)	(0·007)	(0·006)	(0·013)	(0·003)	(0·003)	(0·006)	(0·009)	(0·002)	(0·011)	(0·007)	(0·026)	(0·013)	(0·003)	(0·003)	(0·009)	(0·025)
q_{13}	−0·042	−0·041	0·039	0·122	−0·006	−0·071	−0·001	0·029	−0·055	−0·133	−1·151	0·969	−1·029	−0·122	−0·059	−0·343	1·895
	(0·005)	(0·006)	(0·005)	(0·012)	(0·003)	(0·003)	(0·005)	(0·008)	(0·002)	(0·046)	(0·025)	(0·049)	(0·050)	(0·008)	(0·003)	(0·008)	(0·034)
q_{14}	−0·034	−0·041	−0·013	−0·011	−0·005	−0·030	−0·035	−0·046	−0·022	−0·093	−0·009	−0·033	−0·010	−0·129	−0·100	0·216	0·827
	(0·001)	(0·001)	(0·001)	(0·002)	(0·0005)	(0·001)	(0·001)	(0·001)	(0·0003)	(0·001)	(0·000)	(0·001)	(0·0004)	(0·003)	(0·004)	(0·001)	(0·009)
q_{15}	−0·080	−0·098	−0·032	−0·027	−0·011	−0·070	−0·082	−0·107	−0·052	−0·217	−0·020	−0·078	−0·023	−0·232	−0·272	−0·541	1·942
	(0·002)	(0·003)	(0·002)	(0·005)	(0·002)	(0·001)	(0·002)	(0·003)	(0·001)	(0·004)	(0·0005)	(0·003)	(0·0009)	(0·006)	(0·020)	(0·027)	(0·026)
q_{16}	−0·046	−0·057	−0·018	−0·016	−0·006	−0·001	−0·047	−0·072	−0·030	−0·125	−0·012	−0·045	−0·014	−0·094	−0·078	−0·432	1·121
	(0·001)	(0·001)	(0·001)	(0·003)	(0·001)	(0·001)	(0·001)	(0·002)	(0·0004)	(0·002)	(0·0003)	(0·002)	(0·001)	(0·002)	(0·005)	(0·009)	(0·012)

TABLE 8

Price and income elasticities and standard errors: groupwise additivity

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	M
q_1	−0·813 (0·015)	0·230 (0·015)	−0·060 (0·010)	−0·039 (0·014)	−0·005 (0·008)	0·308 (0·013)	−0·014 (0·0005)	−0·011 (0·0004)	−0·014 (0·0006)	−0·056 (0·002)	−0·004 (0·0002)	−0·012 (0·001)	−0·002 (0·0002)	−0·038 (0·001)	−0·013 (0·001)	−0·108 (0·005)	0·654 (0·026)
q_2	0·189 (0·011)	−1·207 (0·023)	0·192 (0·012)	0·521 (0·017)	0·114 (0·010)	−0·044 (0·014)	−0·009 (0·0004)	−0·007 (0·0004)	−0·009 (0·0004)	−0·034 (0·002)	−0·003 (0·0002)	−0·007 (0·0005)	−0·002 (0·0002)	−0·023 (0·001)	−0·008 (0·0005)	−0·067 (0·003)	0·405 (0·021)
q_3	−0·096 (0·012)	0·265 (0·019)	−0·956 (0·018)	0·135 (0·018)	0·206 (0·010)	−0·056 (0·013)	−0·022 (0·0008)	−0·017 (0·0006)	−0·022 (0·0008)	−0·089 (0·003)	−0·007 (0·0003)	−0·019 (0·001)	−0·004 (0·0003)	−0·061 (0·002)	−0·021 (0·001)	−0·171 (0·004)	1·036 (0·033)
q_4	−0·048 (0·010)	0·460 (0·016)	0·082 (0·031)	−0·644 (0·021)	−0·252 (0·009)	−0·183 (0·009)	−0·022 (0·001)	−0·017 (0·0007)	−0·022 (0·001)	−0·036 (0·004)	−0·008 (0·0003)	−0·018 (0·001)	−0·004 (0·0004)	−0·059 (0·002)	−0·021 (0·001)	−0·166 (0·008)	1·009 (0·040)
q_5	−0·019 (0·064)	1·184 (0·099)	1·342 (0·068)	−2·616 (0·092)	−0·829 (0·082)	0·793 (0·062)	−0·005 (0·003)	−0·004 (0·002)	−0·005 (0·003)	−0·022 (0·010)	−0·002 (0·001)	−0·005 (0·002)	−0·001 (0·0004)	−0·015 (0·008)	−0·005 (0·002)	−0·041 (0·020)	0·251 (0·122)
q_6	0·535 (0·019)	−0·043 (0·028)	−0·140 (0·017)	−0·296 (0·018)	0·164 (0·012)	−0·070 (0·025)	0·006 (0·0005	0·004 (0·0004)	0·005 (0·0005)	0·022 (0·002)	0·002 (0·0002)	0·005 (0·0005)	0·001 (0·0001)	0·015 (0·001)	0·005 (0·0005)	0·042 (0·004)	−0·257 (0·023)
q_7	−0·038 (0·009)	−0·056 (0·001)	−0·026 (0·001)	−0·044 (0·002)	−0·005 (0·0003)	−0·034 (0·0008)	−0·609 (0·011)	0·022 (0·010)	0·210 (0·010)	−0·077 (0·002)	−0·006 (0·0002)	−0·016 (0·001)	−0·003 (0·0004)	−0·053 (0·001)	−0·018 (0·0007)	−0·147 (0·0038)	0·896 (0·0219)
q_8	−0·066 (0·001)	−0·097 (0·002)	−0·044 (0·001)	−0·075 (0·002)	−0·010 (0·0004)	−0·059 (0·0009)	−0·001 (0·008)	−0·701 (0·011)	−0·070 (0·009)	−0·132 (0·002)	−0·011 (0·0003)	−0·028 (0·002)	−0·006 (0·0006)	−0·091 (0·001)	−0·032 (0·001)	−0·254 (0·004)	1·544 (0·020)
q_9	−0·026 (0·002)	−0·039 (0·002)	−0·018 (0·001)	−0·030 (0·002)	−0·004 (0·0004)	−0·023 (0·002)	0·249 (0·012)	0·128 (0·012)	−0·634 (0·017)	−0·052 (0·004)	−0·004 (0·003)	−0·011 (0·0008)	−0·002 (0·0002)	−0·036 (0·002)	−0·013 (0·001)	−0·101 (0·007)	0·618 (0·042)
q_{10}	−0·044 (0·0008)	−0·064 (0·001)	−0·029 (0·0008)	−0·050 (0·002)	−0·007 (0·0003)	−0·040 (0·0008)	−0·022 (0·0005)	−0·017 (0·0005)	−0·022 (0·0006)	−0·739 (0·009)	0·085 (0·003)	0·106 (0·008)	0·066 (0·006)	−0·060 (0·001)	−0·021 (0·0007)	−0·169 (0·003)	1·028 (0·017)
q_{11}	−0·063 (0·002)	−0·092 (0·002)	−0·042 (0·001)	−0·072 (0·002)	−0·009 (0·0005)	−0·057 (0·001)	−0·032 (0·0007)	−0·025 (0·0007)	−0·032 (0·0009)	0·724 (0·029)	−0·604 (0·027)	0·533 (0·030)	−1·342 (0·034)	−0·086 (0·002)	−0·031 (0·001)	−0·242 (0·005)	1·473 (0·029)
q_{12}	−0·091 (0·002)	−0·133 (0·002)	−0·061 0·002)	−0·103 (0·004)	−0·014 0·0006)	−0·082 (0·001)	−0·046 (0·0009)	−0·036 (0·001)	−0·046 (0·001)	0·047 (0·014)	0·096 (0·005)	−1·353 (0·019)	0·216 (0·012)	−0·124 (0·002)	−0·044 (0·001)	−0·349 (0·005)	2·121 (0·026)
q_{13}	−0·099 (0·002)	−0·144 (0·003)	−0·066 (0·002)	−0·112 0·003)	−0·015 (0·0007)	−0·089 (0·002)	−0·050 (0·001)	−0·039 (0·001)	−0·050 (0·001)	0·268 (0·042)	−0·981 (0·025)	0·769 (0·046)	−1·132 (0·054)	−0·135 (0·002)	−0·048 (0·002)	−0·379 (0·007)	2·304 (0·038)
q_{14}	−0·030 (0·0006)	−0·044 (0·0007)	−0·020 (0·0005)	−0·034 (0·001)	−0·004 (0·0002)	−0·027 (0·0004)	−0·015 (0·0002)	−0·012 (0·0003)	−0·015 (0·0004)	−0·059 (0·001)	−0·005 (0·002)	−0·012 (0·0007)	−0·002 (0·0002)	−0·057 (0·002)	−0·090 (0·004)	−0·268 (0·007)	0·697 (0·010)
q_{15}	−0·074 (0·002)	−0·108 (0·002)	−0·049 (0·001)	−0·084 (0·003)	−0·011 (0·003)	−0·067 (0·0006)	−0·037 (0·001)	−0·029 (0·0008)	−0·037 (0·0009)	−0·148 (0·001)	−0·012 (0·003)	−0·031 (0·001)	−0·007 (0·0007)	−0·211 (0·006)	−0·132 (0·019)	−0·687 (0·028)	1·728 (0·027)
q_{16}	−0·037 (0·001)	−0·054 (0·001)	−0·025 (0·001)	−0·042 (0·001)	−0·005 (0·0003)	−0·033 (0·0006)	−0·019 (0·0004)	−0·015 (0·0005)	−0·019 (0·0005)	−0·074 (0·0019)	−0·006 (0·0002)	−0·016 (0·0008)	−0·003 (0·0003)	−0·101 (0·0002)	−0·108 (0·010)	−0·309 (0·010)	0·870 (0·014)

Multiplying through by $M/(q_i q_j)$,

$$\frac{1}{w_j} e_{ij} + E_i = \frac{\phi_{IJ}}{M} E_i E_j = \frac{M}{q_i q_j} \lambda U^{ij} - \frac{\lambda E_i E_j}{M \dfrac{\partial \lambda}{\partial M}}, \qquad \text{...(25)}$$

$$\frac{U^{ij}}{\left(\theta_{IJ} - \dfrac{1}{\mu_M}\right)} = \frac{q_i q_j}{\lambda M} E_i E_j. \qquad \text{...(26)}$$

TABLE 9

Objective values achieved under various assumptions

Slutsky conditions[1]	113·48
Separability	551·06
groupwise additivity	625·98
" Pearce " separability	839·32
additivity	1201·40

[1] See Byron [7].

TABLE 10

Between and within group elasticities

	Pearce Separability	Separability	Groupwise Additivity
11	*GS/NS*	*—/—*	*GC/NS*
12	*GS/NS*	*GS/NS*	*GC/NS*
13	*GS/NS*	*GS/NS*	*GC/NS*
14	*GC/NS*	*GC/NS*	*GC/NS*
21	*GS/NS*	*GC/NS*	*GC/NS*
22	*GS/NS*	*GC/NS*	*GC/NS*
23	*GC/NS*	*—/NS*	*GC/NS*
24	*GC/NC*	*GC/NC*	*GC/NS*
31	*GC/NS*	*GC/NS*	*GC/NS*
32	*GC/NS*	*—/NS*	*GC/NS*
33	*GS/NS*	*—/NS*	*GC/NS*
34	*GC/NS*	*GC/NS*	*GC/NS*
41	*GC/NS*	*GC/NS*	*GC/NS*
42	*GC/NC*	*GC/NC*	*GC/NS*
43	*GC/NS*	*GC/NS*	*GC/NS*
44	*GC/NC*	*GC/NC*	*GC/NC*

where $\mu_M = \dfrac{\partial \lambda}{\partial M} \dfrac{M}{\lambda}$ is the income elasticity of the marginal utility of money. Since

$$\frac{U^{ij}}{\left(\theta_{IJ} - \dfrac{1}{\mu_M}\right)} > 0$$

for normal goods, it follows that if $\theta_{IJ} < \dfrac{1}{\mu_M}$ then $U^{ij} < 0$ and if $\theta_{IJ} > \dfrac{1}{\mu_M}$ then $U^{ij} > 0$. Clearly, the separability conditions place restrictions on the curvature of the utility hyper-surface as θ_{IJ} is held constant for all $i \in I$ and $j \in J$. The separability coefficient is a measure of the utility interaction between goods in the various groups at a given level of utility indicated by the income elasticity of the marginal utility of money. Information on this elasticity and on the utility interactions cannot be derived from the present results. As

$$\mu_M = \lambda_M \frac{M}{\lambda} = \frac{1}{p'U^{ij}p} \frac{M}{\lambda}$$

it would appear that little more of a quantitative or qualitative nature can be said about the restrictiveness of the separability hypothesis for the utility function or the apparent inevitability of the dominance of the income effects under these conditions.

It is worth noting that the results under additivity do not support Green's[1] result. The hypothesis of direct additivity implies that either (*a*) all goods are normal and substitutes for each other; or (*b*) one good is normal and a substitute for every other good, the remaining goods are either all inferior and complementary with each other or all neutral and unrelated to each other. One good (bread) was inferior and was a net complement for all other goods. This divergence might be taken as evidence that consumer's do not engage in maximising behaviour. An alternative conclusion would be that the consumer's utility function is not, in fact, additive and in view of the likelihood ratio tests it is this which the author would favour.

However, it is still possible that the basic hypothesis is incorrect—that consumers do not maximize utility. This conclusion would be supported by the application of a likelihood ratio test to the results under the Slutsky restrictions alone (Table 9). The Slutsky restrictions were rejected and it therefore became impossible for the various separability hypothesis to be accepted as they impose more severe restrictions on the income compensated substitution terms than mere symmetry. If this were not so it would be possible that the consumers do maximise utility but not according to the budgeting pattern implied by the separability hypotheses considered here.

It is quite likely that the method adopted to test the hypotheses was unduly severe. The restrictions are imposed at the mean proportion of expenditure devoted to each good and do not hold exactly at any point in time. Anyhow, with changing tastes and a general shift up the utility function it is likely that such local conditions are not hypotheses worth testing. It is also possible that the choice of a log linear demand function affected the results as no global utility function can correspond to such a representation of the demand equations. Finally, it is possible that the grouping chosen was simply incorrect—a change of grouping may lead to more favourable results. The evidence presented here weighs pretty heavily against the Slutsky and separability hypotheses but the conclusion may well be that it is meaningless to test such hypotheses in a static framework given the data the econometrician is forced to work with.

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