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# A SIMPLE NONLINEAR MODEL FOR THE RETURN TO ISOTROPY IN TURBULENCE

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## ABSTRACT

A quadratic nonlinear generalization of the linear Rotta model for the slow pressure-strain correlation of turbulence is developed. The model is shown to satisfy realizability and to give rise to no stable non-trivial equilibrium solutions for the anisotropy tensor in the case of vanishing mean velocity gradients. The absence of stable non-trivial equilibrium solutions is a necessary condition to ensure that the model predicts a return to isotropy for all relaxational turbulent flows. Both the phase space dynamics and the temporal behavior of the model are examined and compared against experimental data for the return to isotropy problem. It is demonstrated that the quadratic model successfully captures the experimental trends which clearly exhibit nonlinear behavior. Direct comparisons are also made with the predictions of the Rotta model and the Lumley model.

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# 1. Introduction

The *return to isotropy* problem in turbulence has served as a cornerstone for the calibration of models for the slow pressure-strain correlation. The slow pressure-strain correlation is the part of the pressure-strain correlation that is independent of the mean velocity gradients. It is experimentally observed that an initially anisotropic, homogeneous turbulence produced by the application of constant mean velocity gradients undergoes a relaxation to a state of isotropy when the mean velocity gradients are removed. Rotta<sup>1</sup> was the first to develop a turbulence model for the slow pressure-strain correlation which captured the return to isotropy behavior within the framework of second-order closure models. While the model (which is linear in the anisotropy tensor) performs reasonably well for small initial anisotropies, it can give rise to considerable errors for more general turbulent flows undergoing a relaxation from an initial state that is strongly anisotropic. This was pointed out by Lumley<sup>2</sup> who emphasized the need to incorporate nonlinear effects and developed a general nonlinear representation for the slow pressure-strain correlation based on invariant tensor representation theory. Subsequent experiments (see Gence and Mathieu<sup>3</sup> and Choi and Lumley<sup>4</sup>) have unequivocally confirmed the nonlinear nature of the return to isotropy problem.

In this paper, a quadratic model for the slow pressure-strain correlation, which contains only one independent constant, will be derived and calibrated. It will be demonstrated that this simple nonlinear model provides for a more complete description of the return to isotropy problem than either the linear Rotta model<sup>1</sup> or the quasi-linear model of Lumley<sup>2</sup> and Shih and Lumley<sup>5</sup> (where nonlinearity is introduced through the invariants of the anisotropy tensor). In addition to doing a fairly good job in matching the results of four independent sets of experimental data, the quadratic model will be shown to satisfy realizability and to possess no stable non-trivial fixed points for the anisotropy tensor in relaxational turbulent flows. The latter constraint, which appears to have been overlooked in the previous literature on the subject, is a crucial constraint that nonlinear models must satisfy in order to ensure the return to isotropy in relaxational turbulent flows.

The mathematical properties and dynamical performance of the proposed model will be discussed in detail.

## 2. Formulation of the problem

In conventional, second-order turbulence modeling, transport equations for the Reynolds stress tensor  $\overline{u_i u_j}$  and the dissipation rate  $\epsilon$  are solved in addition to solving the Reynolds averaged Navier-Stokes equations for the mean velocity  $\overline{U}_i$ . The exact transport equation for the Reynolds stress is

$$\partial_t \overline{u_i u_j} + \overline{U}_k (\overline{u_i u_j})_{,k} = P_{ij} - T_{ijk,k} - D_{ij} + \Pi_{ij} \quad (1)$$

where

$$\begin{aligned} P_{ij} &= -\overline{u_i u_k} \overline{U}_{j,k} - \overline{u_j u_k} \overline{U}_{i,k} \\ T_{ijk} &= \overline{u_i u_j u_k} + \frac{1}{\rho} [\overline{p u_i} \delta_{jk} + \overline{p u_j} \delta_{ik}] - \nu (\overline{u_i u_j})_{,k} \\ D_{ij} &= 2\nu \overline{u_{i,k} u_{j,k}} \\ \Pi_{ij} &= \frac{1}{\rho} (\overline{p u_{i,j}} + \overline{p u_{j,i}}) \end{aligned} \quad (2)$$

given that  $u_i$  is the fluctuating velocity,  $p$  is the fluctuating pressure,  $\rho$  is the mass density, and  $\nu$  is the kinematic viscosity. In (1),  $P_{ij}$  is the production,  $T_{ijk}$  is the diffusive transport,  $D_{ij}$  is the dissipation rate tensor,  $\Pi_{ij}$  is the pressure-strain correlation, an overbar represents an ensemble mean, and a comma denotes a partial derivative with respect to the spatial coordinates. A transport equation is also usually carried for the dissipation rate  $\epsilon$ , which is defined as follows

$$\epsilon = \nu \overline{u_{i,j} u_{i,j}} \quad (3)$$

The typical approach to modeling the unknown correlations  $T_{ijk}$ ,  $D_{ij}$  and  $\Pi_{ij}$  in (1) is to express them as functions of the Reynolds stress tensor  $\overline{u_i u_j}$ , the gradients of the Reynolds stress tensor  $(\overline{u_i u_j})_{,k}$ , the mean velocity gradients  $\overline{U}_{i,j}$ , the dissipation rate  $\epsilon$ , and the kinematic viscosity  $\nu$  at lower turbulence Reynolds numbers.

At high Reynolds numbers, the dissipation rate tensor is nearly isotropic so that we can set

$$D_{ij} = \frac{2}{3}\epsilon\delta_{ij} \quad (4)$$

The quantity  $\Pi_{ij}$  can be decomposed into a slow pressure-strain term  $\Pi_{ij}^S$  which is independent of  $\bar{U}_{i,j}$  and a rapid term  $\Pi_{ij}^R$  which is linear in  $\bar{U}_{i,j}$ . The dependence of the slow pressure-strain correlation  $\Pi_{ij}^S$  on  $\overline{u_i u_j}$  and  $\epsilon$  can be studied in isolation by considering the *return to isotropy* problem. The return to isotropy problem refers to the flow wherein *homogeneous, anisotropic* turbulence produced by mean velocity gradients is observed to relax towards isotropy when the mean velocity gradients are removed. This return to isotropy problem is an idealized case, but it is important nevertheless because it serves to extract and calibrate a part of the general functional dependence of  $\Pi_{ij}$  for more complex turbulent flows.

Eq. (1) for the Reynolds stress tensor simplifies considerably for the return to isotropy of homogeneous turbulence since  $\bar{U}_{i,j}$  and  $T_{ijk,k}$  are identically zero, and  $D_{ij}$  may be assumed to be isotropic at high turbulence Reynolds numbers. The simplified form of (1) is written as follows:

$$\overline{u_i \dot{u}_j} = \Pi_{ij}^S - \frac{2}{3}\epsilon\delta_{ij} \quad (5)$$

where we have made use of the fact that the rapid part of the pressure-strain correlation vanishes for vanishing  $\bar{U}_{i,j}$ .

The anisotropy tensor  $b_{ij}$  is defined by

$$b_{ij} = \frac{\overline{u_i u_j}}{q^2} - \frac{1}{3}\delta_{ij} \quad (6)$$

where

$$q^2 = \overline{u_i u_i} \quad (7)$$

The matrix associated with the anisotropy tensor is denoted by  $\mathbf{b}$ . Using (5) and its contraction, the following evolution equation for  $b_{ij}$  is obtained

$$\dot{b}_{ij} = \frac{\Pi_{ij}^S}{q^2} + 2\frac{\epsilon}{q^2}b_{ij} \quad (8)$$

By defining a dimensionless tensor  $\Phi_{ij}$  by

$$\Pi_{ij}^S = -\epsilon \Phi_{ij}, \quad (9)$$

(8) is rewritten as follows,

$$\dot{b}_{ij} = -\frac{\epsilon}{q^2}(\Phi_{ij} - 2b_{ij}) \quad (10)$$

Eq. (10) must be solved along with evolution equations for  $q^2$  and  $\epsilon$ . For the return to isotropy problem, the exact transport equation for  $q^2$  and the standard model equation for  $\epsilon$  are as follows

$$\begin{aligned} \dot{q}^2 &= -2\epsilon \\ \dot{\epsilon} &= -2C_{\epsilon 2} \frac{\epsilon^2}{q^2} \end{aligned} \quad (11)$$

where  $C_{\epsilon 2}$  is a constant that assumes a value of approximately 1.90.

The important variables in this homogeneous, high Reynolds number flow are  $b_{ij}$ ,  $q^2$  and  $\epsilon$ . Lumley<sup>2</sup> gave the general tensorial form of the function  $\Phi_{ij}(b_{ij}, q^2, \epsilon)$  which can be written as follows

$$\Phi_{ij} = \alpha_1(II, III)b_{ij} + \alpha_2(II, III)[b_{ik}b_{kj} - \frac{II}{3}\delta_{ij}] \quad (12)$$

In (12),  $\alpha_1$  and  $\alpha_2$  are general functions of the invariants  $II = b_{ik}b_{ki}$  and  $III = b_{ik}b_{kj}b_{ji}$ . It should be noted that  $II$  and  $III$  denote the traces of  $\mathbf{b}^2$  and  $\mathbf{b}^3$  respectively and differ from the principal invariants of  $\mathbf{b}$  by constant multiplicative factors since  $\mathbf{b}$  is traceless. On substituting the model of  $\Phi_{ij}$  from (12) into (10), we obtain

$$\dot{b}_{ij} = -\frac{\epsilon}{q^2} \left\{ [\alpha_1(II, III) - 2]b_{ij} + \alpha_2(II, III)[b_{ik}b_{kj} - \frac{II}{3}\delta_{ij}] \right\} \quad (13)$$

A prescription of the functions  $\alpha_1(II, III)$  and  $\alpha_2(II, III)$  is now required to close (13). The approach that is most widely used in practice is the linear model proposed by Rotta<sup>1</sup> in which  $\alpha_1(II, III)$  is assumed to be a constant and  $\alpha_2(II, III)$  is assumed to be zero. The commonly used second-order closure model of Launder, Reece and Rodi<sup>6</sup> is based on the Rotta model where

$$\begin{aligned} \alpha_1(II, III) &= C_1 \approx 3.0 \\ \alpha_2(II, III) &= 0 \end{aligned} \quad (14)$$

Lumley and Newman<sup>7</sup> and Lumley<sup>2</sup> have pointed out that experimental data does not support the linear model of Rotta. The problem with the linear model is twofold: (a) its prediction that each component of the anisotropy tensor decays at the same rate, and (b) its prediction that this decay rate is independent of the initial state of anisotropy. These predictions can be seen mathematically in the closed form solution of the return to isotropy problem for the Rotta model, given by

$$b_{ij} = b_{ij}(0) \left[ 1 + 2(C_{\epsilon 2} - 1) \frac{\epsilon_0}{q_0^2} t \right]^{-1/2(C_{\epsilon 2} - 1)} \quad (15)$$

In contradiction to (15), experiments indicate that the rate of the return to isotropy *decreases* with increasing values of the third invariant *III*, and that different components of the anisotropy tensor relax to zero at *different* rates. These are effects that are decidedly nonlinear.

In order to incorporate nonlinear effects, Lumley<sup>2</sup> developed a model within the framework of (12), and also allowed the coefficients  $\alpha_1$  and  $\alpha_2$  to additionally depend on the turbulence Reynolds number  $Re_t = q^4/9\epsilon\nu$ . This quasi-linear model of Lumley<sup>2</sup> is as follows

$$\begin{aligned} \alpha_1(II, III, Re_t) &= 2.0 + \frac{F}{9} \exp\left(\frac{-7.77}{\sqrt{Re_t}}\right) \left\{ \frac{72}{\sqrt{Re_t}} \right. \\ &\quad \left. + 80.1 \ln[1 + 62.4(-II^* + 2.3III^*)] \right\} \\ \alpha_2(II, III, Re_t) &= 0 \end{aligned} \quad (16)$$

where

$$\begin{aligned} F &= 1 + 9II^* + 27III^* \\ II^* &= -\frac{II}{2}, \quad III^* = \frac{III}{3} \end{aligned} \quad (17)$$

Here, we refer to this model as being quasi-linear since there is no tensorial nonlinearity, that is,

$$\Phi_{ij} = \alpha_1(II, III, Re_t) b_{ij} \quad (18)$$

It will be shown later that while this model does correctly mimic some of the nonlinear effects associated with the return to isotropy (see Shih and Lumley<sup>5</sup>) it yields an incorrect

representation of the problem in phase space. In the next section, a quadratic model will be developed that appears to capture the essence of the nonlinearity in phase space.

### 3. The quadratic nonlinear model

We will now develop a quadratic model, which is a special case of (12). Since each component energy  $\overline{u_\alpha u_\alpha}$  (where Greek indices indicate no summation) is bounded by

$$0 \leq \overline{u_\alpha u_\alpha} \leq q^2 \quad (19)$$

it is a simple matter to show from (6) that each principal value of  $b_{ij}$  (denoted by  $b^{(\alpha)}$ ) lies in the range

$$-\frac{1}{3} \leq b^{(\alpha)} \leq \frac{2}{3} \quad (20)$$

Consequently, the principal values of  $b_{ij}$  are less than unity, and it is possible that a lower order Taylor expansion of  $\Phi_{ij}(b_{ij})$  around the isotropic state of  $b_{ij} = 0$  can yield an adequate approximation. Carrying out such a Taylor expansion around  $b_{ij} = 0$ , while constraining  $\Phi_{ij}$  to be of zero trace, gives

$$\Phi = C_1 \mathbf{b} - C_2 (\mathbf{b}^2 - \frac{II}{3} \mathbf{I}) + C_3 (\mathbf{b}^3 - \frac{III}{3} \mathbf{I}) + \dots \quad (21)$$

where  $C_1, C_2, C_3, \dots$  are constants. In (21) we use coordinate-free notation, that is, we denote  $\Phi_{ij}, b_{ij}$  and  $\delta_{ij}$  by the associated matrices  $\Phi, \mathbf{b}$  and  $\mathbf{I}$ . Using the Cayley-Hamilton theorem to express  $\mathbf{b}^3$  and higher powers of  $\mathbf{b}$  in terms of  $II, III, \mathbf{b}$  and  $\mathbf{b}^2$  reduces (21) to (12) where  $\alpha_1(II, III)$  and  $\alpha_2(II, III)$  are polynomial functions of  $II$  and  $III$  of infinite degree.

The model of  $\Phi_{ij}$  that we propose in this paper is a quadratic truncation of (21), and is given by

$$\Phi_{ij} = C_1 b_{ij} - C_2 [b_{ik} b_{kj} - \frac{II}{3} \delta_{ij}] \quad (22)$$

where  $C_1$  and  $C_2$  are constants. Such a quadratic model is attractive because it is the simplest nonlinear model for  $\Phi_{ij}$  (of course a linear truncation of (21), which leads to



Rotta's model, would be even more simple but, unfortunately, is in conflict with experimental data). Furthermore, since for most problems of engineering interest the magnitude of  $b_{ij}$  tends to be less than  $1/3$ , there is a good chance for a quadratic model to yield an acceptable approximation for a variety of flows.

At this point, we will derive the evolution equations determining the phase space dynamics for the return to isotropy problem. It is useful to introduce the transformed time variable  $\tau$  defined by

$$d\tau = \frac{\epsilon}{q^2} dt \quad (23)$$

in order to free the formulation of the problem from any explicit dependence on  $\epsilon$  and  $q^2$ . Using the system of equations (11) to solve for  $q^2(t)$  and  $\epsilon(t)$ , and integrating (23) we obtain the following explicit relationship between  $\tau$  and  $t$ :

$$\tau = \frac{1}{2(C_{\epsilon 2} - 1)} \ln[1 + 2(C_{\epsilon 2} - 1) \frac{\epsilon_0}{q_0^2} t] \quad (24)$$

Using (23), the transport equation (13) for  $b_{ij}$  can then be rewritten as

$$\frac{db_{ij}}{d\tau} = -\{[\alpha_1(II, III) - 2]b_{ij} + \alpha_2(II, III)[b_{ik}b_{kj} - \frac{II}{3}\delta_{ij}]\} \quad (25)$$

which is clearly independent of  $q^2$  and  $\epsilon$ .

We now show that the phase space for (25) is two-dimensional. Let the principal axes of  $b_{ij}$  at  $\tau = 0$  be  $\{x_1, x_2, x_3\}$  and the corresponding eigenvalues of  $b_{ij}$  be denoted by  $b_{11}$ ,  $b_{22}$ , and  $b_{33}$ . The off-diagonal terms,  $b_{ij}$  ( $i \neq j$ ) are zero at  $\tau = 0$ . On examining (25) in principal axes form it is clear that

$$\frac{db_{ij}}{d\tau} = 0 \quad \text{for} \quad i \neq j \quad (26)$$

Since  $b_{ij}(0)$  is zero for  $i \neq j$ , we conclude from (26) that the off-diagonal terms of  $b_{ij}$  are identically zero for all times; and therefore, the principal axes of  $b_{ij}$  do not change in time. The time invariance of the principal axes has been experimentally observed, for example by Gence and Mathieu<sup>3</sup>. Therefore the only time varying components of  $b_{ij}$ , referred to the coordinate system  $\{x_1, x_2, x_3\}$ , are  $b_{11}$ ,  $b_{22}$ , and  $b_{33}$ . Since  $b_{ij}$  is traceless, only two of the three quantities  $b_{11}$ ,  $b_{22}$ , and  $b_{33}$ , are independent. Thus the phase space in this problem is

two-dimensional. The two phase space variables may be chosen to be any two independent functions of the eigenvalues of  $b_{ij}$ .

A possible choice for phase space variables is  $II$  and  $III$ . The variables  $II$  and  $III$  are desirable to use since they are uniquely determined by the principal values of  $b_{ij}$ , are invariant to a rotation of coordinates, and vanish in isotropic turbulence. Furthermore, because  $II \geq 0$  the phase space becomes semi-infinite in the variable  $II$ . The evolution equations for  $II(\tau)$  and  $III(\tau)$  will now be derived. Multiplying (25) by  $b_{ij}$  and  $b_{ik}b_{kj}$ , respectively, we obtain the following equations

$$\frac{dII}{d\tau} = -2\{ (\alpha_1 - 2)II + \alpha_2 III \} \quad (27)$$

$$\frac{dIII}{d\tau} = -3\{ (\alpha_1 - 2)III + \alpha_2 [\text{tr}(\mathbf{b}^4) - \frac{II^2}{3}] \} \quad (28)$$

In (28),  $\text{tr}(\mathbf{b}^4)$  denotes the trace of the tensor  $\mathbf{b}^4$ . Using the Cayley-Hamilton theorem, it follows that

$$\mathbf{b}^4 = \frac{1}{2}II\mathbf{b}^2 + \frac{III}{3}\mathbf{b} \quad (29)$$

By taking the trace of (29), and substituting the result into (28), we obtain

$$\frac{dIII}{d\tau} = -3\{ (\alpha_1 - 2)III + \alpha_2 \frac{II^2}{6} \} \quad (30)$$

Equations (27) and (30) constitute a coupled nonlinear system of ordinary differential equations for  $II(\tau)$  and  $III(\tau)$ .

Both the Rotta model (14) and the Lumley model (16) choose  $\alpha_2 = 0$ . When  $\alpha_2 = 0$ , (27) and (30) can be combined and integrated to yield the general solution

$$(II)^{1/2} = c(III)^{1/3} \quad (31)$$

where  $c$  is a constant of integration determined by initial conditions. In terms of the variables

$$\begin{aligned} \xi &= (III)^{1/3} \\ \eta &= (II)^{1/2} \end{aligned} \quad (32)$$

which were first introduced by Choi and Lumley <sup>4</sup>, (31) takes the linear form

$$\frac{\eta}{\eta_0} = \frac{\xi}{\xi_0} \quad (33)$$

Thus, these alternative variables have the advantage that in the  $\xi - \eta$  phase space both the linear and quasi-linear models show *straight line* trajectories. Some of the experimental data that will be shown later exhibit *curved* trajectories in the  $\xi - \eta$  phase space which constitutes conclusive evidence in support of the need for nonlinear models.

#### 4. Fixed point analysis in phase space

A fixed point analysis of the quadratic model in the phase space of the invariants of  $b_{ij}$  will now be presented. The fixed points of (27) and (30) (where, for the quadratic model,  $\alpha_1 = C_1$  and  $\alpha_2 = -C_2$ ) are the equilibrium solutions in the limit as  $\tau \rightarrow \infty$  which are denoted by  $(II_\infty, III_\infty)$ . If the turbulence is to return to isotropy

$$II_\infty = III_\infty = 0 \quad (34)$$

Therefore, since physical and numerical experiments indicate that any initial anisotropy which is realizable tends to relax to zero when the mean velocity gradients are removed, it is essential that nonlinear models for the slow pressure-strain correlation have *no stable fixed points* other than  $(0, 0)$ .

Since the linear model (for which  $C_2 = 0$ ) gives rise to the following system of evolution equations

$$\begin{aligned} \frac{dII}{d\tau} &= -2(C_1 - 2)II \\ \frac{dIII}{d\tau} &= -3(C_1 - 2)III \end{aligned} \quad (35)$$

it is a simple matter to show that  $(0, 0)$  is the only fixed point, and that it is a stable node provided that  $C_1 > 2$ .

We now examine the existence and stability of the fixed points predicted by our quadratic model (22). The quadratic model leads to the following equations for  $II(\tau)$

and  $III(\tau)$ :

$$\begin{aligned}\frac{dII}{d\tau} &= -2(C_1 - 2)II + 2C_2III \\ \frac{dIII}{d\tau} &= -3(C_1 - 2)III + \frac{C_2}{2}II^2\end{aligned}\quad (36)$$

The fixed points of the system (36) are solutions of the nonlinear algebraic equations obtained by setting  $dII/d\tau = dIII/d\tau = 0$ , and are given by

$$II_\infty = 0 \quad , \quad III_\infty = 0 \quad (37)$$

$$II_\infty = 6\beta^2 \quad , \quad III_\infty = 6\beta^3 \quad (38)$$

where  $\beta = (C_1 - 2)/C_2$ . In order to examine the stability of these fixed points we introduce the new variables  $(z_1, z_2)$  defined by

$$z_1 = II - II_\infty, \quad z_2 = III - III_\infty \quad (39)$$

If we substitute (39) into (36), and retain only the terms that are linear in  $z_1$  and  $z_2$ , we obtain the following linear system of ordinary differential equations

$$\frac{dz_i}{d\tau} = A_{ij}z_j \quad (40)$$

where  $A_{ij}$  is the following matrix

$$\begin{bmatrix} -2(C_1 - 2) & 2C_2 \\ C_2II_\infty & -3(C_1 - 2) \end{bmatrix}$$

The eigenvalues of  $A_{ij}$  determine the stability of the fixed points. A fixed point is stable if all the eigenvalues of  $A_{ij}$  are negative. The fixed point  $(0, 0)$  has eigenvalues  $A^{(\alpha)}$  given by

$$A^{(1)} = -2(C_1 - 2), \quad A^{(2)} = -3(C_1 - 2) \quad (41)$$

and, hence, is a stable node provided that

$$C_1 > 2 \quad (42)$$

On the other hand, the fixed point  $(6\beta^2, 6\beta^3)$  has the eigenvalues

$$A^{(1)} = C_1 - 2, \quad A^{(2)} = -6(C_1 - 2) \quad (43)$$

Since the eigenvalues are of opposite sign no matter what value is taken for  $C_1$ , the fixed point  $(6\beta^2, 6\beta^3)$  is a saddle which is *unstable*. It is thus clear that  $III_\infty = II_\infty = 0$  is the only stable equilibrium solution and the quadratic model predicts a return to isotropy for all initial conditions on  $b_{ij}$ .

## 5. Constraints on the model constants

The quadratic model has two constants  $C_1$  and  $C_2$  which, for the purpose of stability, need only satisfy the constraint  $C_1 > 2$ . We will now explore whether there are additional physical constraints on the range of allowable values of  $C_1$  and  $C_2$ . Choi and Lumley<sup>4</sup> and Le Penven, Gence and Comte-Bellot<sup>8</sup> have analyzed available sets of experimental data (approximately ten in number) on the relaxation of anisotropic turbulence. They conclude from the data that a characteristic nondimensional time rate for the relaxation (defined by  $-\frac{1}{II} \frac{dII}{dr}$ ) *decreases* with increasing values of  $III$ . An examination of (36) shows that in order for the model to reproduce this trend we must choose

$$C_2 > 0 \quad (44)$$

Lumley<sup>2</sup> uses the concept of realizability, which requires that model predictions satisfy all moment inequalities; for example, the model is required to yield non-negative component energies. Lumley's method leads to various equations relating the model constants. Pope<sup>9</sup> suggests that a related but weaker mathematical statement, which constitutes a sufficient condition to guarantee non-negative component energies, should be used. The application of both approaches in the context of the quadratic model for  $\Phi_{ij}$  will now be discussed.

The transport equations for  $b_{ij}$  and  $\overline{u_i u_j}$  in the present problem are

$$\dot{b}_{ij} = -\frac{\epsilon}{q^2}(\Phi_{ij} - 2b_{ij}) \quad (45)$$

$$\overline{\dot{u}_i u_j} = -\epsilon \Phi_{ij} - \frac{2}{3} \epsilon \delta_{ij} \quad (46)$$

The three eigenvalues of  $\overline{u_i u_j}$  are the three components in the principal axes of  $\overline{u_i u_j}$ ; these components are  $\overline{u_1^2}$ ,  $\overline{u_2^2}$  and  $\overline{u_3^2}$ . Consider one of the eigenvalues, for example  $\overline{u_1^2}$ . Realizability requires that

$$\overline{u_1^2} \geq 0 \quad (47)$$

Using the Lumley<sup>2</sup> approach to satisfy realizability leads to the requirement that

$$\overline{\dot{u}_1^2} = 0 \quad \text{when} \quad \overline{u_1^2} = 0 \quad (48)$$

whereas the Pope<sup>9</sup> approach leads to the requirement that

$$\overline{\dot{u}_1^2} \geq 0 \quad \text{when} \quad \overline{u_1^2} = 0 \quad (49)$$

Both approaches guarantee that the realizability condition (47) is satisfied. However (49) can prevent  $\overline{u_1^2}$  from ever becoming zero (which is the physically realizable limit of two-dimensional turbulence), and hence is more conservative than (48) in this regard. But since (49) does guarantee positive component energies, we will be content for the moment to consider it since it places less restrictions on  $C_1$  and  $C_2$ .

Since (49) is to be satisfied, using (46) we require that

$$-\epsilon \phi_{11} - \frac{2}{3} \epsilon \geq 0 \quad \text{when} \quad \overline{u_1^2} = 0. \quad (50)$$

This implies that

$$\phi_{11} \leq -\frac{2}{3} \quad \text{when} \quad \overline{u_1^2} = 0 \quad (51)$$

Substituting the quadratic model for  $\Phi_{ij}$  from (22) into (51) leads to the inequality

$$C_1 b_{11} - C_2 (b_{11}^2 - \frac{II}{3}) \leq -\frac{2}{3} \quad \text{when} \quad \overline{u_1^2} = 0 \quad (52)$$

Since  $b_{11} = -1/3$  when  $\overline{u_1^2} = 0$ , (52) becomes

$$C_1 - 2 \geq C_2 (II - \frac{1}{3}) \quad (53)$$

From the constraint (42), it is clear that the left-hand-side of (53) is positive. Since from (44)  $C_2 > 0$ , the maximum value of the right-hand-side of (53) occurs when  $II$  is at its

maximum value. Therefore a sufficient condition for satisfying (53) is obtained when  $II$  is equal to its maximum value  $II_{maz}$ . It can be easily shown that

$$II_{maz} = \frac{2}{3} \quad (54)$$

By substituting the maximum value of  $II$ , which is given by (54), into (53), we obtain the following constraint

$$C_2 \leq 3(C_1 - 2) \quad (55)$$

For a given value of  $C_1$ , (55) determines the maximum allowable value of  $C_2$ .

## 6. Model performance

The coupled system of nonlinear ordinary differential equations (27) and (30) were numerically integrated by a Runge-Kutta method to obtain  $II(\tau)$  and  $III(\tau)$ . The initial conditions corresponding to those of the previously conducted experiments were used. The phase plots in  $\xi - \eta$  space and the transient plots of  $II(\tau)$  were the yardsticks by which the model results and experimental data were compared.

A numerical optimization showed that for a given value of  $C_1$  it was best to choose the maximum value of  $C_2$  allowed by (55), that is, to set

$$C_2 = 3(C_1 - 2). \quad (56)$$

Due to (56),  $C_2$  is uniquely determined by  $C_1$  making the model one with a *single* free constant  $C_1$ . It is interesting to note that (56) is similar to the constraint that is obtained by using Lumley's method of enforcing realizability. Thus it appears that Lumley's method works better for this problem. The following choice of the one independent model constant

$$C_1 = 3.4 \quad (57)$$

seems to best reflect the experimental data.

Results on the performance of the linear Rotta model, quasi-linear Lumley model, and the present quadratic model are compared with four sets of experimental data in Figs. 1-8.

Figs. 1-2 consider the plane distortion experiment of Choi and Lumley<sup>4</sup>. The experimental data in Fig. 1 shows strong non-linearity in the  $\xi - \eta$  phase plane (as demonstrated by a curved trajectory), which is reproduced by the quadratic model. However, the trajectories in  $\xi - \eta$  phase space of the Rotta and Lumley models are always straight lines, and are therefore in serious disagreement with this experimental data. Fig. 2 shows the temporal decay of  $II$ . It is seen that the present quadratic model is superior to both the Rotta model and the Lumley model with regard to capturing the temporal behavior of  $II$ . Figs. 3-4 pertain to the axisymmetric expansion experiment of Choi and Lumley<sup>4</sup>. The experimental phase space trajectory seems to be fairly linear as shown in Fig. 3, and the present quadratic model also shows almost linear behavior. Apparently, the quadratic model not only shows nonlinear phase trajectories in  $\xi - \eta$  phase space when called for, but also gives rise to approximately linear trajectories in the appropriate region of phase space where nonlinear effects are small. It is seen in Fig. 4 that the results of both the Rotta model and the quadratic model on the temporal behavior of  $II$  are in reasonable agreement with the experimental data; however, the Lumley model predicts a decay rate of  $II$  which is much more rapid than the experimentally observed result.

Le Penven, et al.<sup>8</sup> used two different plane contractions to generate anisotropic turbulence with positive  $III$  in one case and negative  $III$  in the other case. Their experimental data on the relaxation of turbulence with positive  $III$  is shown in Figs. 5-6. Their data of Fig. 5 indicates a moderately curved trajectory in phase space, and the quadratic model yields a prediction that is in qualitatively good agreement with the data. Fig. 6 shows that the quadratic model does a better job than the Lumley model in predicting the temporal decay of  $II$ , and that the Rotta model is slightly better than the quadratic model with regard to predicting the temporal behavior of  $II$ . In Figs. 7-8, we consider the experimental data of Le Penven, et al.<sup>8</sup> pertaining to the case with negative  $III$ . The phase portrait of Fig. 7 shows that their experimental data exhibits mild curvature in the  $\xi - \eta$  phase space, which is reproduced by the quadratic model. Fig. 8, which depicts the temporal evolution of  $II$ , shows that the Rotta model grossly underpredicts the rate of decay of  $II$ . The Lumley model is seen to be somewhat better than the proposed quadratic model in



capturing the rapid decay of  $II$  in this experiment. Nevertheless, it is clear from the results presented in this section that the quadratic model does the best overall job in predicting the experimental trends.

## 7. Conclusion

A quadratic generalization of the linear Rotta model has been developed for the slow pressure-strain correlation of turbulence. This simple nonlinear model, which has only one independent constant, was shown to constitute an improvement on both the Rotta model and the quasi-linear Lumley model in predicting the trends observed in return to isotropy experiments. Furthermore, the quadratic model was shown to satisfy realizability and to have no stable non-trivial equilibrium solutions for the anisotropy tensor in relaxational flows— a crucial constraint which guarantees that the model predicts a return to isotropy, independent of the initial conditions. The most notable feature of this new quadratic model is its ability to capture the essence of the return to isotropy problem with a nonlinear structure that is substantially simpler than previously proposed nonlinear generalizations of the Rotta model.

Future research can be directed on two fronts: the development of a low turbulence Reynolds number correction of the model and the addition of successively higher order (for example, cubic) nonlinear terms. Insofar as the former research direction is concerned, the direct numerical simulations of Lee and Reynolds<sup>10</sup> have indicated that history dependent effects of the Reynolds stress and dissipation rate anisotropy tensors are important in low Reynolds number turbulent flows. These are complicating features that are not likely to be described by simple nonlinear models with an algebraic structure. In regard to the addition of higher degree nonlinearities, we feel that it is best to first provide a systematic test of a quadratic model since second-order closures that are quadratic in the anisotropy tensor can satisfy constraints such as material frame-indifference in the limit of two dimensional turbulence<sup>11,12</sup> and realizability. The satisfaction of these constraints, which are violated by simple second-order closures such as the Launder, Reece and Rodi model<sup>6</sup>, allow for the description of more complex turbulent flows within the framework of a workable theory. A

systematic program for the development of improved second-order closure models that are quadratically nonlinear is currently underway and will be the subject of a future paper.

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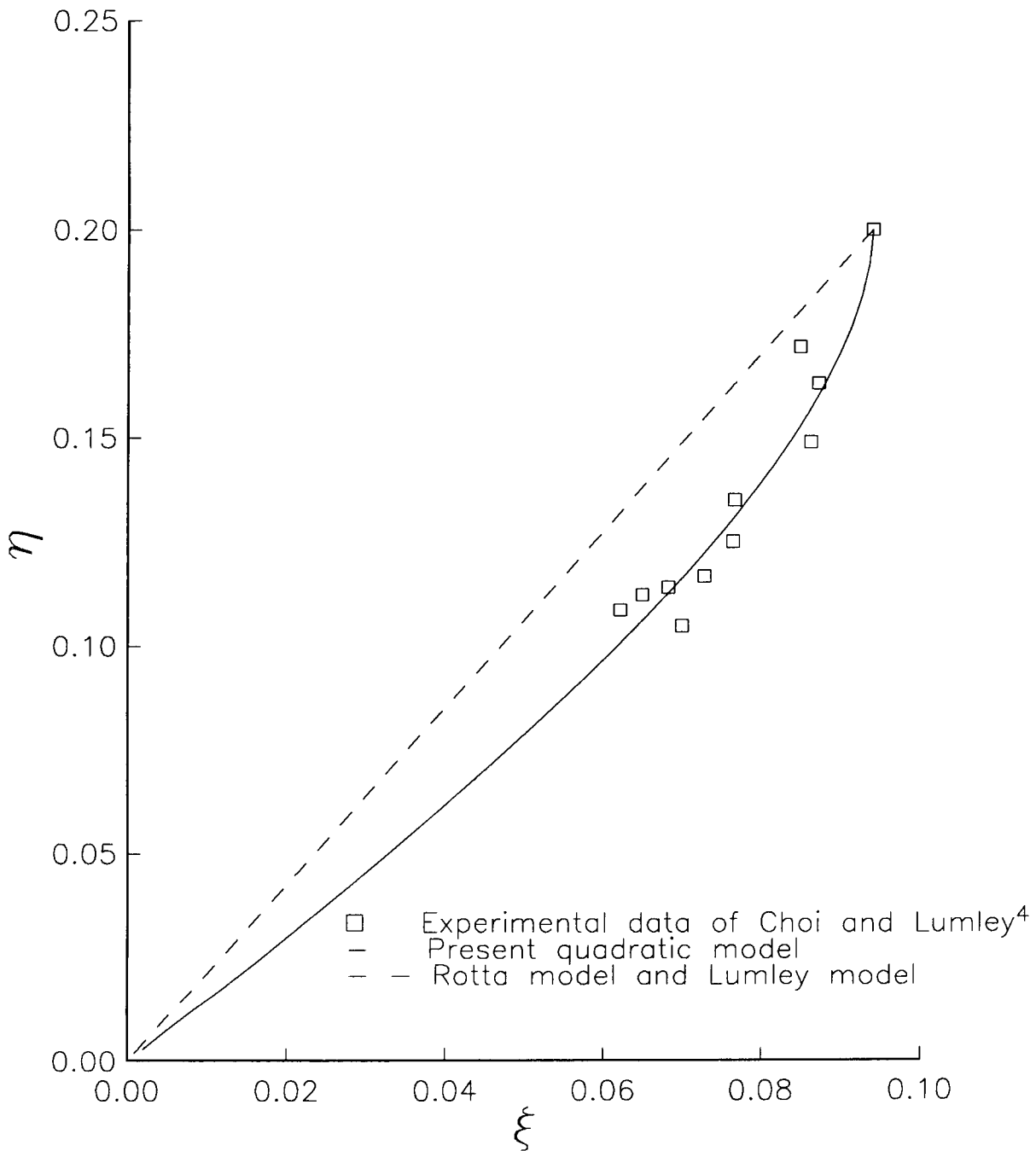
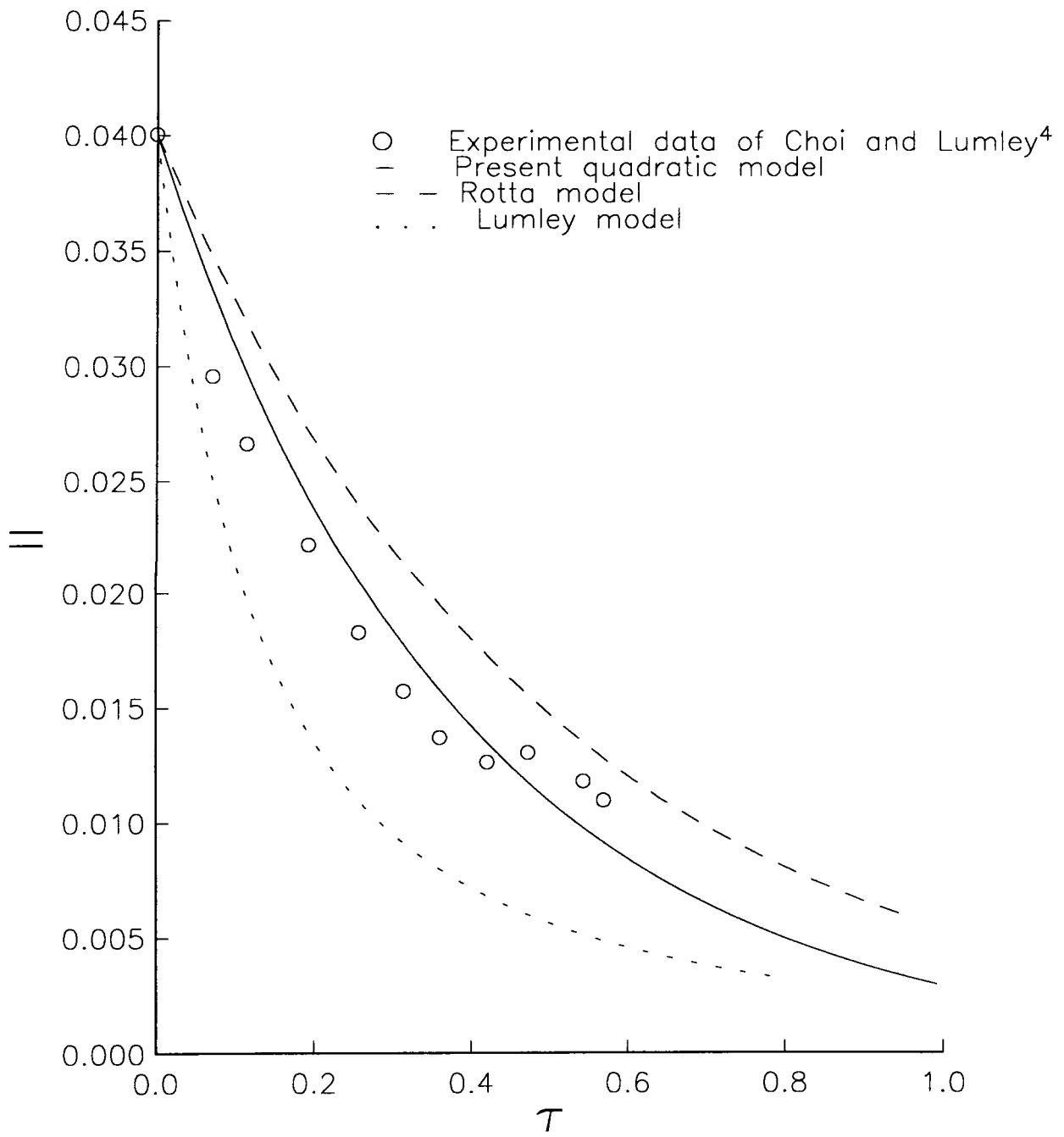
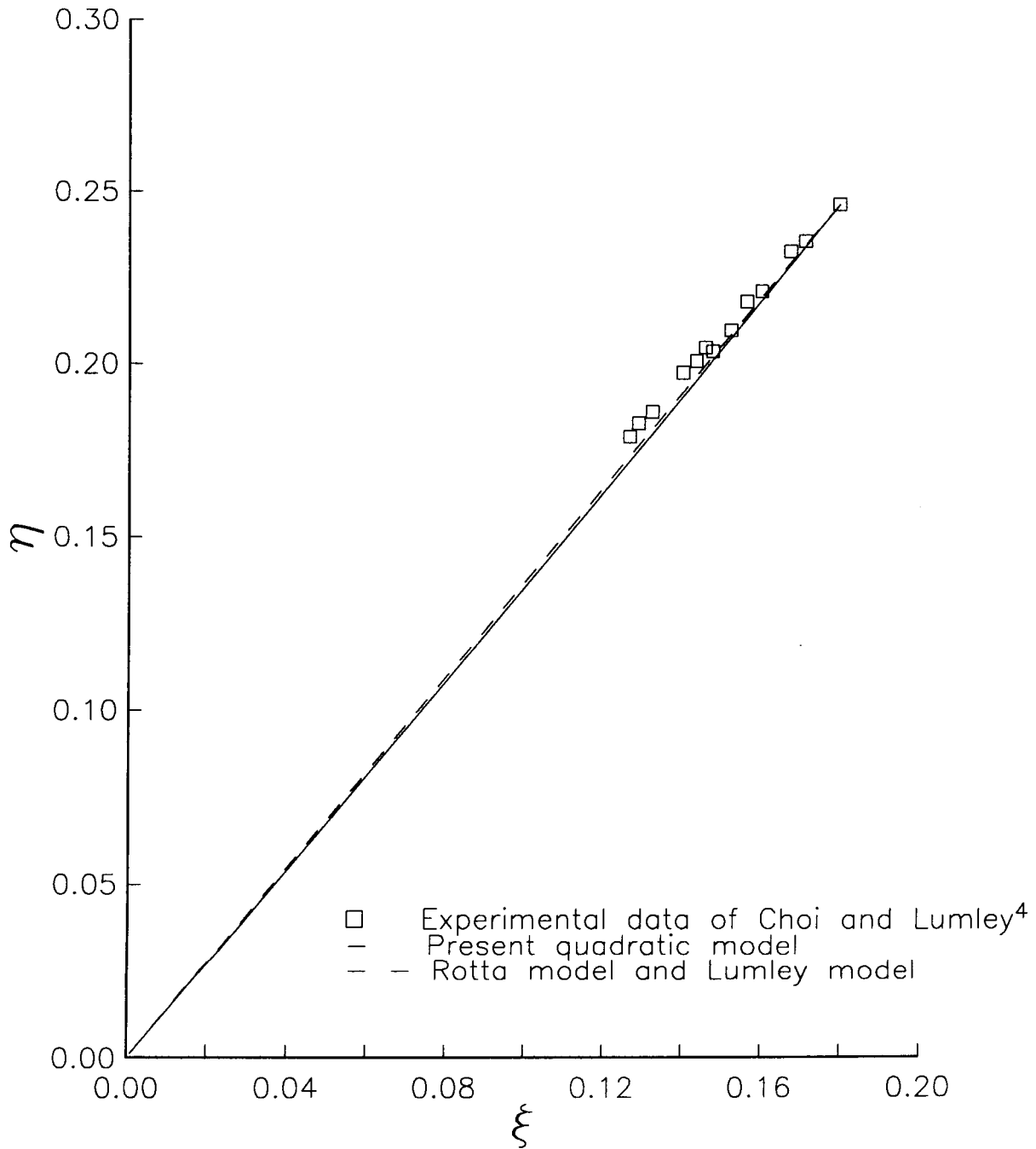


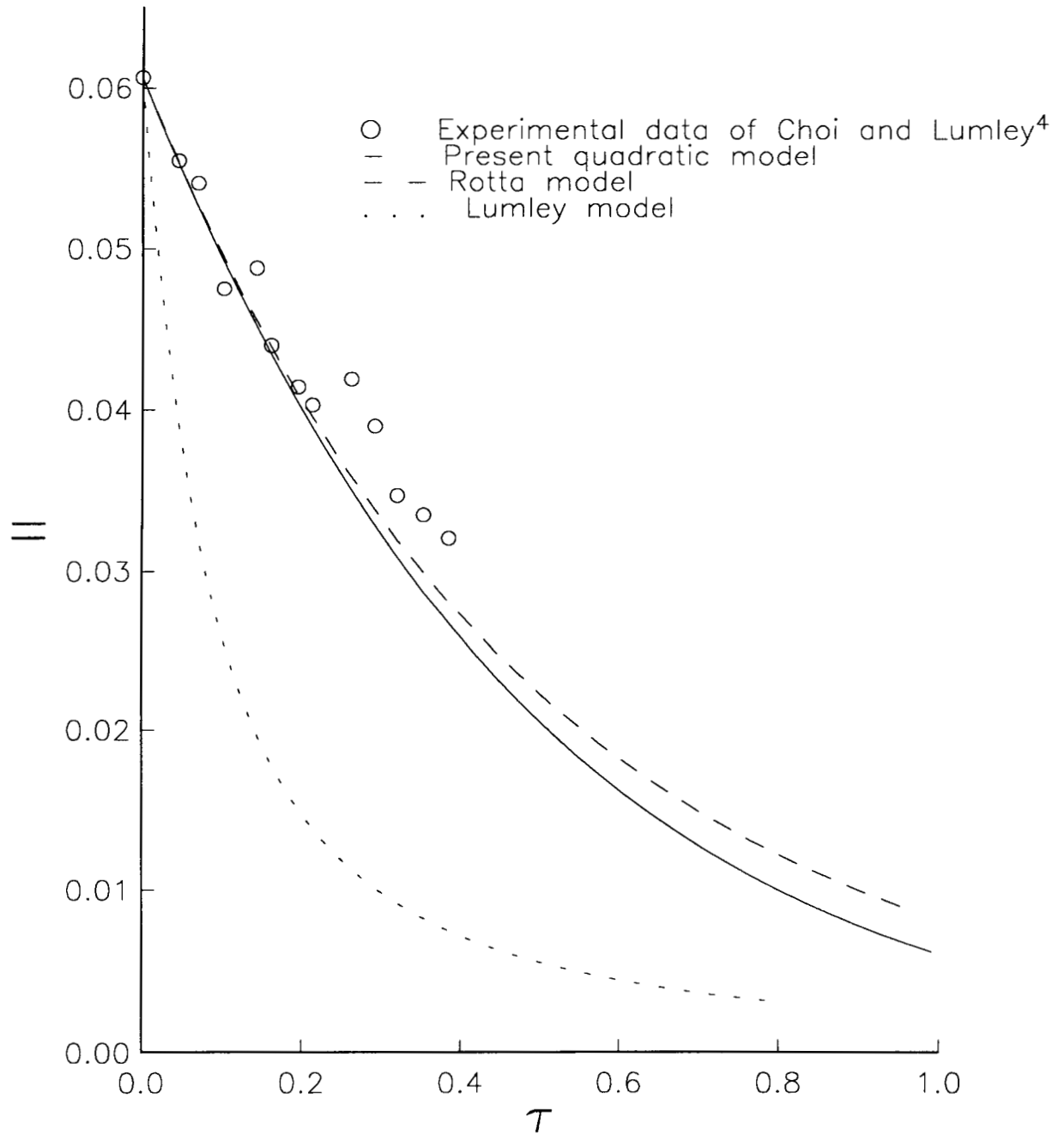
Figure 1. Phase space comparison of the model predictions with the plane distortion experiment of Choi and Lumley<sup>4</sup>.



**Figure 2.** Temporal decay of the second invariant of the anisotropy tensor: comparison of the model predictions with the plane distortion experiment of Choi and Lumley<sup>4</sup>.



**Figure 3. Phase space comparison of the model predictions with the axisymmetric expansion experiment of Choi and Lumley<sup>4</sup>.**



**Figure 4.** Temporal decay of the second invariant of the anisotropy tensor: comparison of the model predictions with the axisymmetric expansion experiment of Choi and Lumley<sup>4</sup>.



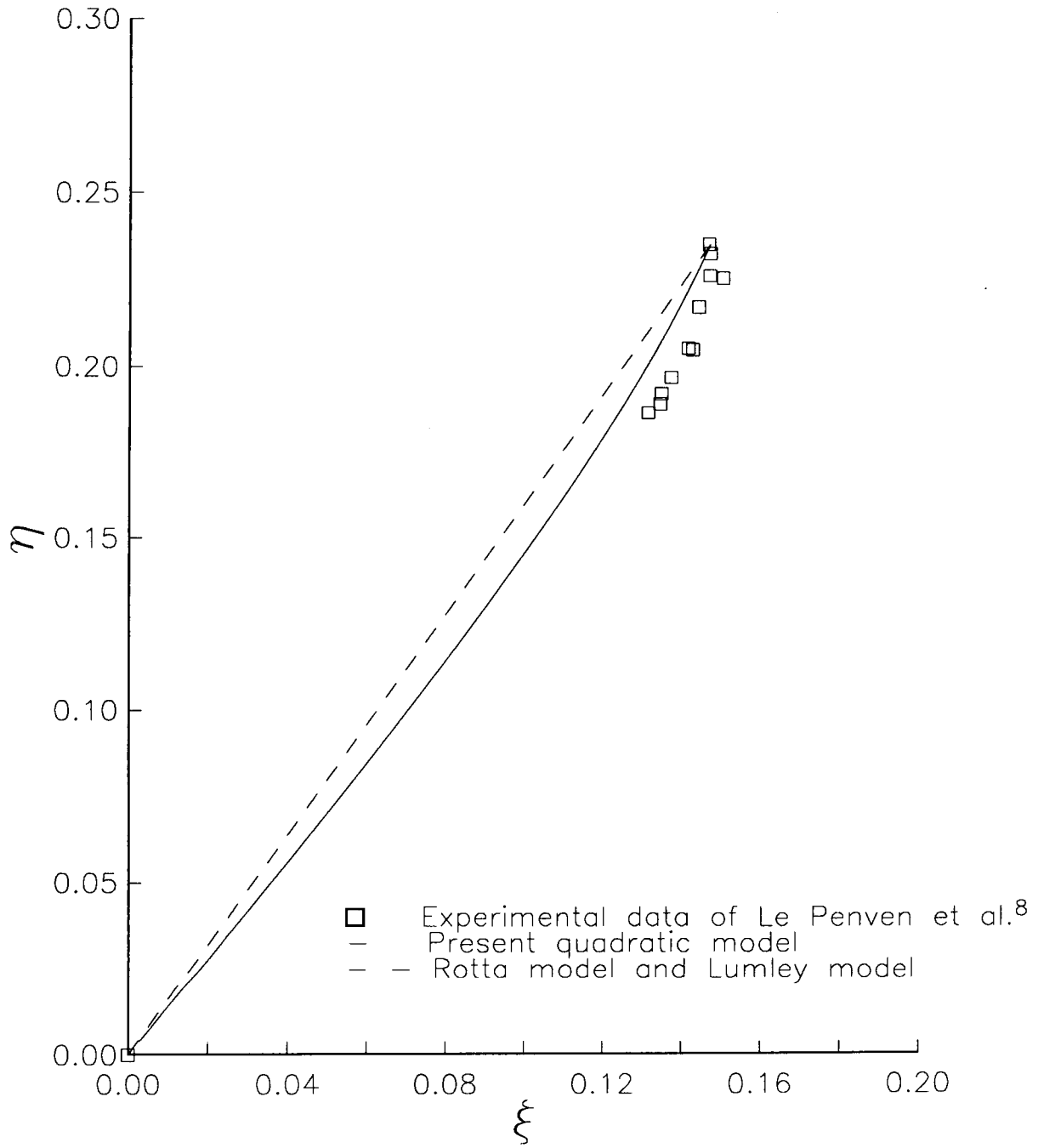


Figure 5. Phase space comparison of the model predictions with the plane contraction experiment of Le Penven et al.<sup>8</sup> for  $III(0) > 0$ .

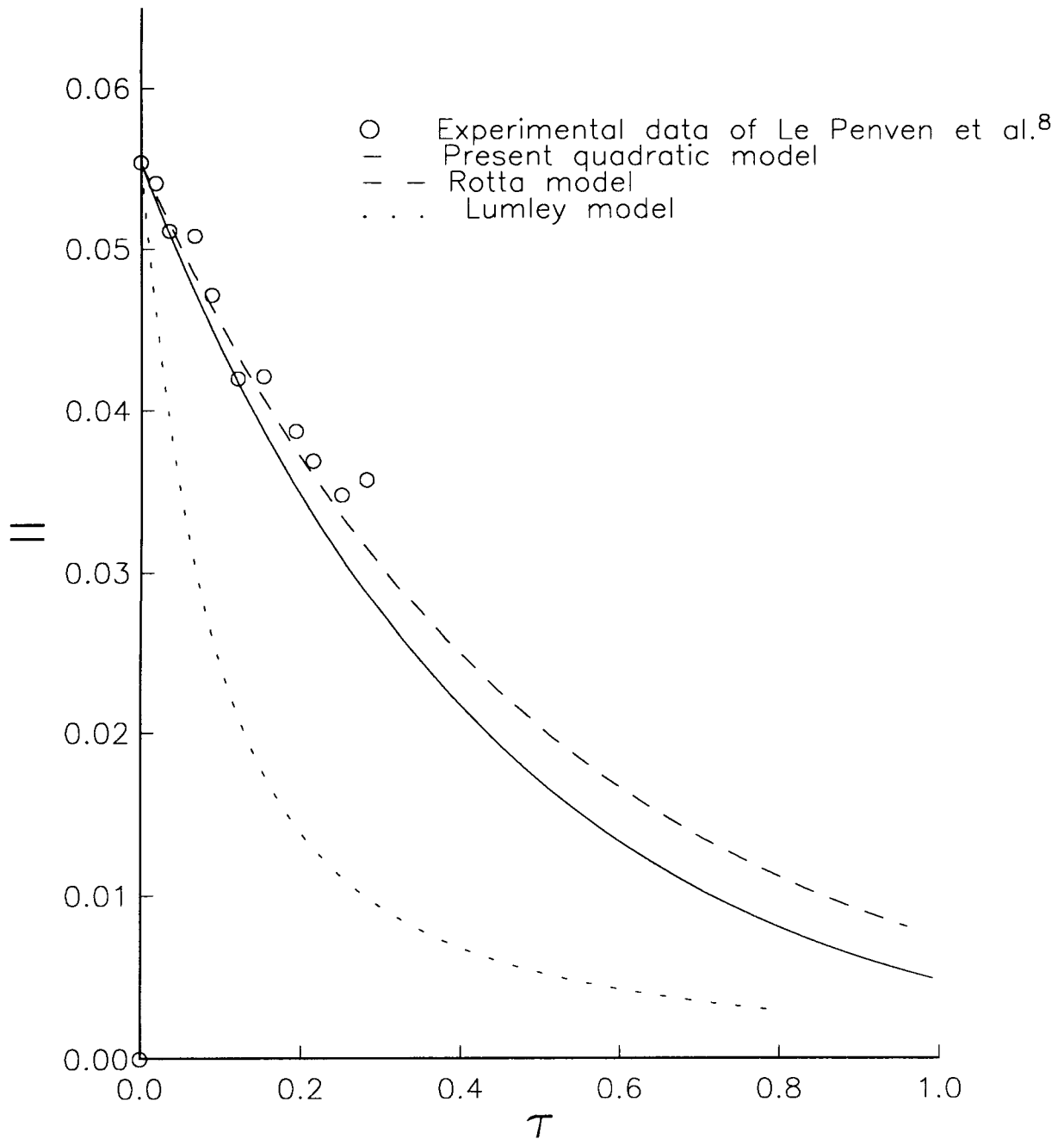


Figure 6. Temporal decay of the second invariant of the anisotropy tensor: comparison of the model predictions with the plane contraction experiment of Le Penven et al.<sup>8</sup> for  $III(0) > 0$ .

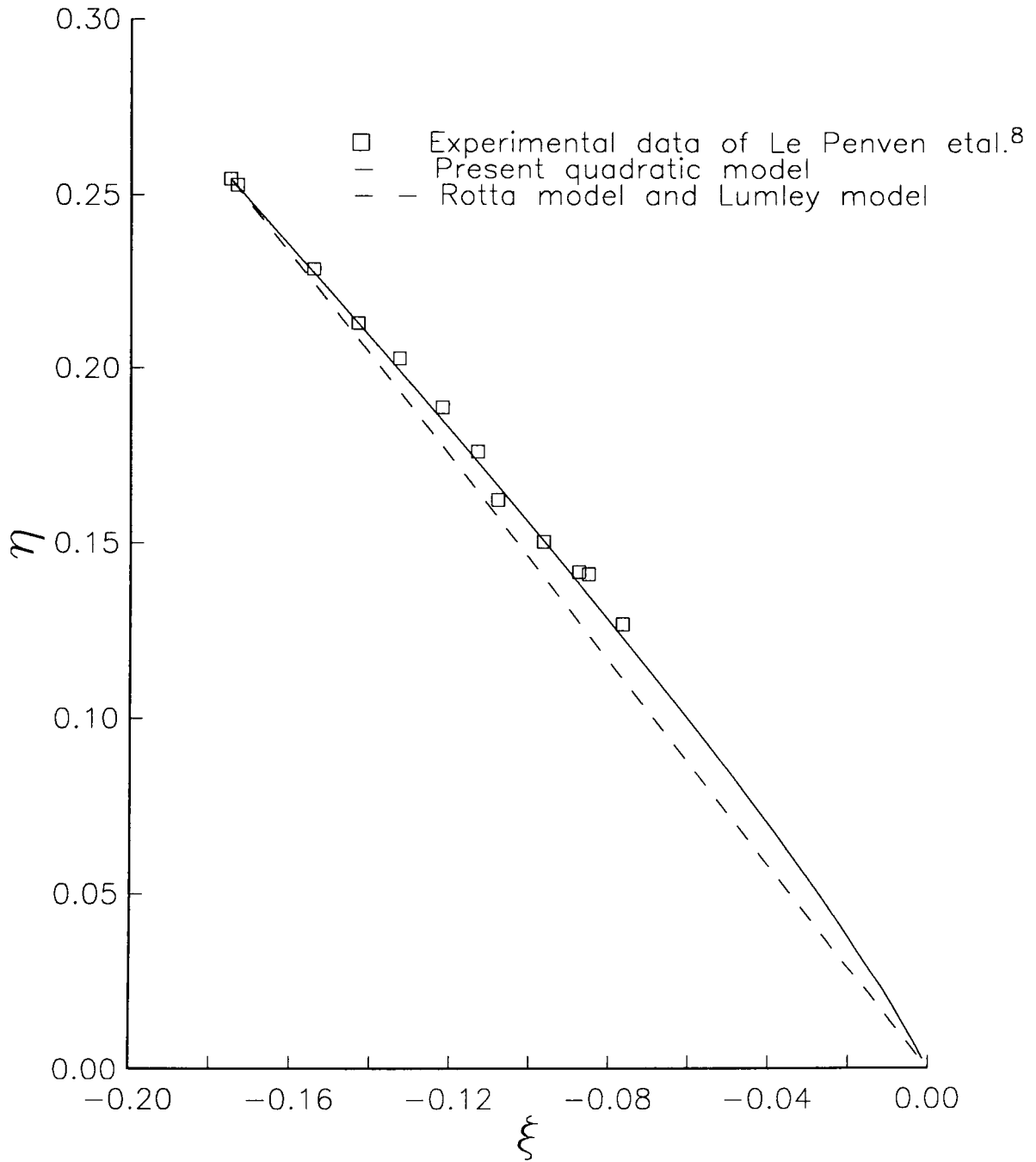


Figure 7. Phase space comparison of the model predictions with the plane contraction experiment of Le Penven et al.<sup>8</sup> for  $III(0) < 0$ .

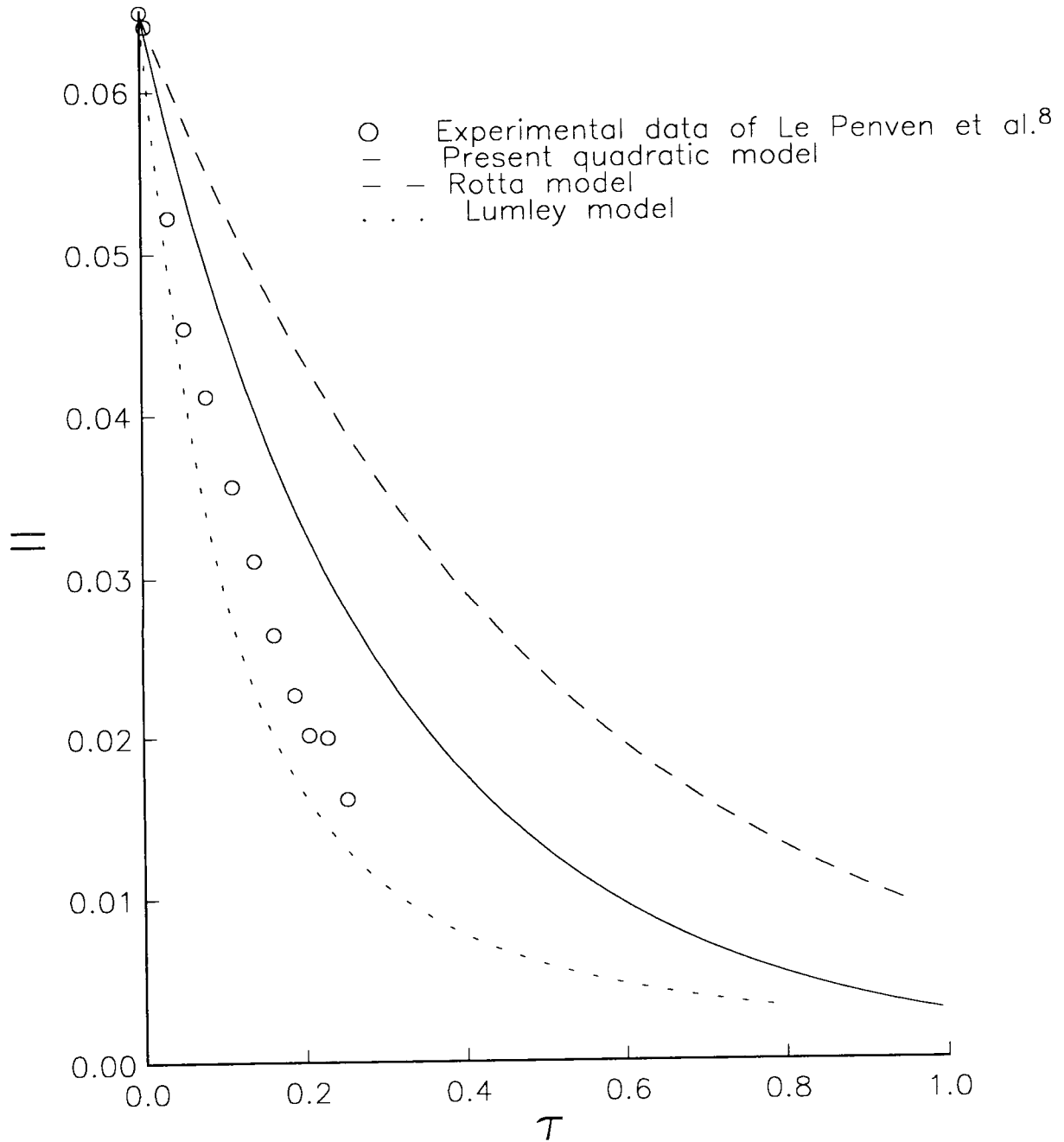


Figure 8. Temporal decay of the second invariant of the anisotropy tensor: comparison of the model predictions with the plane contraction experiment of Le Penven et al.<sup>8</sup> for  $III(0) < 0$ .



# Report Documentation Page

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16. Abstract  A quadratic nonlinear generalization of the linear Rotta model for the slow pressure-stain correlation of turbulence is developed. The model is shown to satisfy realizability and to give rise to no stable non-trivial equilibrium solutions for the anisotropy tensor in the case of vanishing mean velocity gradients. The absence of stable non-trivial equilibrium solutions is a necessary condition to ensure that the model predicts a return to isotropy for all relaxational turbulent flows. Both the phase space dynamics and the temporal behavior of the model are examined and compared against experimental data for the return to isotropy problem. It is demonstrated that the quadratic model successfully captures the experimental trends which clearly exhibit nonlinear behavior. Direct comparisons are also made with the predictions of the Rotta model and the Lumley model.					
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