

## A simple proof of a Kramers' type law for self-stabilizing diffusions\*

Julian Tugaut<sup>†</sup>

### Abstract

We provide a new proof of a Kramers' type law for self-stabilizing diffusion. These diffusions correspond to the hydrodynamical limit of a mean-field system of particles and may be seen as the probabilistic interpretation of the granular media equation. We use the same hypotheses as the ones used in the work "Large deviations and a Kramers' type law for self-stabilizing diffusions" by Herrmann, Imkeller and Peithmann in which the authors obtain a first proof of the statement.

**Keywords:** self-stabilizing diffusion; exit time; large deviations; coupling method.

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## 1 Introduction

In the remarkable work "Large deviations and a Kramers' type law for self-stabilizing diffusions", Herrmann, Imkeller and Peithmann establish large deviation results and solve the exit problem of the so-called self-stabilizing diffusion. This consists of the following model.

$$X_t^\epsilon = X_0 + \int_0^t V(X_s^\epsilon) ds - \int_0^t \int_{\mathbb{R}^d} \Phi(X_s^\epsilon - x) du_s^\epsilon(x) ds + \sqrt{\epsilon} W_t. \quad (1.1)$$

In this equation,  $V$  and  $\Phi$  denote vector fields on  $\mathbb{R}^d$ ;  $(W_t)_{t \geq 0}$  is a  $d$ -dimensional Wiener process;  $du_t^\epsilon$  denotes the law of the random variable  $X_t^\epsilon$  and  $X_0$  is a deterministic real. We remark that the own law of the process intervenes in the drift which explains the term *self-stabilizing*.

Equation (1.1) corresponds to the hydrodynamical limit of a mean-field system of particles.

$$Z_t^{\epsilon, i, N} = X_0 + \sqrt{\epsilon} W_t^i + \int_0^t V(Z_s^{\epsilon, i, N}) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t \Phi(Z_s^{\epsilon, i, N} - Z_s^{\epsilon, j, N}) ds,$$

for all  $1 \leq i \leq N$ . Here, the  $W^i$  are independent Brownian motions and  $W^1 = W$ . See [4].

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<sup>†</sup>Université Jean Monnet, France. E-mail: tugaut@math.cnrs.fr

In [3], the authors consider an open domain  $\mathcal{D}$  satisfying some hypotheses and they study the limit as  $\epsilon$  goes to 0 of  $\epsilon \log \{\mathbb{E} [\tau_{\mathcal{D}}(\epsilon)]\}$  where  $\tau_{\mathcal{D}}(\epsilon)$  is defined as the first exit time of  $X^\epsilon$  from the domain  $\mathcal{D}$ :

$$\tau_{\mathcal{D}}(\epsilon) := \inf \{t \geq 0 \mid X_t^\epsilon \notin \mathcal{D}\} .$$

More precisely, they obtain the limit

$$\lim_{\epsilon \rightarrow 0} \mathbb{P} \left\{ e^{\frac{1}{\epsilon}(\overline{Q_\infty} - \xi)} < \tau_{\mathcal{D}}(\epsilon) < e^{\frac{1}{\epsilon}(\overline{Q_\infty} + \xi)} \right\} = 1 ,$$

for any  $\xi > 0$ . Here,  $\overline{Q_\infty}$  denotes the exit cost of the domain  $\mathcal{D}$ :

$$\overline{Q_\infty} := \inf_{z \in \partial \mathcal{D}} \inf_{T > 0} \inf_{\varphi \in H_z^1} \frac{1}{2} \int_0^T \|\dot{\varphi}_t - V(\varphi_t) + \Phi(\varphi_t - x_{stable})\|^2 dt .$$

The set  $H_z^1$  denotes the space of absolutely continuous functions  $f$  such that  $f(0) = x_{stable}$  and  $f(T) = z$ . And,  $x_{stable}$  is the unique point at which the vector field  $V$  is equal to 0.

In a gradient case, we simply have  $\overline{Q_\infty} := 2 \inf_{z \in \partial \mathcal{D}} (H(z) - H(x_{stable}))$ , where the potential  $H$  is defined by  $\nabla H(x) = V(x) - \Phi(x - x_{stable})$ .

To obtain their result, they reconstruct Freidlin-Wentzell theory to the self-stabilizing diffusion. More precisely, they establish a large deviation principle with the good rate function

$$I_{[0;T]}^y(\varphi) := \frac{1}{2} \int_0^T \|\dot{\varphi}_t - V(\varphi_t) + \Phi(\varphi_t - x_{stable})\|^2 ,$$

if  $\varphi(0) = y$  and if  $\varphi$  is absolutely continuous. Otherwise,  $I_{[0;T]}^y(\varphi) := +\infty$ . Here  $x_{stable}$  is the unique point which anneals  $V$ . To obtain this large deviation principle, Herrmann, Imkeller and Peithmann prove that the time homogeneous diffusion

$$Y_t^{\infty,y} = y + \sqrt{\epsilon} W_t + \int_0^t \left[ V(Y_s^{\infty,y}) - \Phi(Y_s^{\infty,y} - x_{stable}) \right] ds$$

is an exponentially good approximation of the McKean-Vlasov diffusion.

More recently, we published a work ([5]) which proves the same result (albeit only in the gradient case) by using a different method. We solve the exit problem of the first particle  $Z^{\epsilon,1,N}$  and we use a coupling method between  $Z^{\epsilon,1,N}$  and  $X^\epsilon$ . The proof is more natural and intuitive but is more technical.

In the present paper, we do not use large deviation principle but a coupling method between the time-homogeneous diffusion and the McKean-Vlasov diffusion so that the results on the exit-time of  $Y^{\infty,y}$  can be used for the exit-time of solution of Equation (1.1).

The aim of this paper is to provide a much simpler method than the ones of [3, 5] to obtain the result. Moreover, in [3, 5], the confining and interacting potentials (in the gradient case) are assumed to be convex.

In [3], the authors need some uniform (with respect to the time) convergence of the drift. In [5], the convexity is required in order to ensure the stability of some domains. We expect some refinement in our work in the sense that we do not use directly the global convexity. Indeed, let us note that the present work aims at solving the exit problem of general time-inhomogeneous diffusions provided that long-time convergence (with a rate of convergence which does not depend on the diffusion coefficient) holds together with a local convexity (at least on the domain from which we search the exit).

For a complete review of Freidlin-Wentzell theory, see [1, 2].

First, we give the assumptions of the paper, that are the same as the ones in [3] (see pages 1383 and 1406). Then, we remind the reader of the main result of the paper that

concerns the exit time. We have chosen not to deal with the exit location because we do not have any improvement on this question. In a third section, we provide the proof of the theorem.

## 2 Assumptions and notations

In this work, we take the same hypotheses as the ones in [3].

### Assumption (A):

**(A-1)** The coefficients  $V$  and  $\Phi$  are locally Lipschitz, that is, for each  $R > 0$  there exists  $K_R > 0$  such that  $\|V(x) - V(y)\| + \|\Phi(x) - \Phi(y)\| \leq K_R \|x - y\|$ , for  $x, y \in B_R(0) := \{z \in \mathbb{R}^d : \|z\| < R\}$ .

**(A-2)** The interaction function  $\Phi$  is rotationally invariant, that is, there exists a function  $\phi$  from  $[0; +\infty[$  to  $[0; +\infty[$  such that  $\Phi(x) = \frac{x}{\|x\|} \phi(\|x\|)$ ,  $x \neq 0$ .

**(A-3)** The function  $\phi$  is convex and  $\phi(0) = 0$ .

**(A-4)** The function  $\Phi$  grows at most polynomially: there exists  $K > 0$  and  $r \in \mathbb{N}$  such that  $\|\Phi(x) - \Phi(y)\| \leq \|x - y\| (K + \|x\|^r + \|y\|^r)$ ,  $x, y \in \mathbb{R}^d$ .

**(A-5)** The function  $V$  is continuously differentiable.

**(A-6)** Let  $DV(x)$  denote the Jacobian of  $V$ . We assume that there exists  $K_V > 0$  such that  $\langle h; DV(x)h \rangle \leq -K_V$ , for  $h \in \mathbb{R}^d$  such that  $\|h\| = 1$  and  $x \in \mathbb{R}^d$ .

**(A-7)** We assume that the unique point at which the vector field is equal to 0 is  $x_{stable}$ .

Under these assumptions, there exists a positive integer  $n_0$  such that for any compact  $\mathcal{K}$ , we have  $\sup_{x \in \mathcal{K}} |\Phi * \mu(x) - \Phi(x - x_{stable})| \leq K(M_1, \dots, M_{n_0})$ ,  $K$  being a continuous function such that  $K(0, \dots, 0) = 0$  and

$$M_p := \int_{\mathbb{R}^d} \|y - x_{stable}\|^p \mu(dy).$$

We now present the definition of what we denote as “stable by”.

**Definition 2.1.** Let  $k$  be any positive integer. Let  $\mathcal{G}$  be a subset of  $\mathbb{R}^k$  and let  $U$  be a vector field from  $\mathbb{R}^k$  to  $\mathbb{R}^k$  which satisfies the same assumptions as in  $V$  in Assumptions (A). For all  $x \in \mathbb{R}^k$ , we consider the dynamical system  $\psi_t(x) = x + \int_0^t U(\psi_s(x)) ds$ . We say that the domain  $\mathcal{G}$  is stable by  $U$  if the orbit  $\{\psi_t(x); t \in \mathbb{R}_+\}$  is included in  $\mathcal{G}$  for all  $x \in \mathcal{G}$ .

**Hypothesis 2.2.** We consider the dynamical system  $\varphi_t = X_0 + \int_0^t V(\varphi_s) ds$  where  $X_0$  is introduced in (1.1). The orbit  $\{\varphi_t; t > 0\}$  is included in  $\mathcal{D}$ .

**Hypothesis 2.3.** The open domain  $\mathcal{D}$  is stable by  $V - \Phi(\cdot - x_{stable})$ .

**Definition 2.4.**  $\mathbb{B}_\kappa^\infty$  denotes the set of all the probability measures  $\mu$  on  $\mathbb{R}^d$  satisfying  $\int_{\mathbb{R}^d} \|x - x_{stable}\|^{2r} \mu(dx) \leq \kappa^{2r}$ .

We now give the main result of the current work.

**Theorem:** We consider vector fields  $V$  and  $\Phi$  which satisfy the set of assumptions (A). Under Hypotheses 2.2–2.3, for all  $\xi > 0$ , we have the limit:

$$\lim_{\epsilon \rightarrow 0} \mathbb{P} \left\{ e^{\frac{1}{\epsilon}(\overline{Q}_\infty - \xi)} < \tau_{\mathcal{D}}(\epsilon) < e^{\frac{1}{\epsilon}(\overline{Q}_\infty + \xi)} \right\} = 1.$$

Let us notice that Herrmann, Imkeller and Peithmann assume a stronger hypothesis than Hypothesis 2.3. Indeed, in our work, the domain  $\mathcal{D}$  is not assumed to be stable by  $V$ .

## 3 Proof of Theorem

For the sake of the reading, we assume  $x_{stable} = 0$  in this section. Immediately, we have  $V(0) = 0$ . Let us note that there is no loss of generality.

### 3.1 Control of the moments

We now establish an important result about all the moments of  $X^\epsilon$ . Indeed, since these moments intervene in the drift, the asymptotic behaviour (deterministic) of the law  $u_t^\epsilon$  is related to the asymptotic behaviour (probabilistic) of the trajectories.

**Proposition 3.1. 1.** *The 2nth moment is uniformly bounded:*

$$\sup_{t \in \mathbb{R}_+} \mathbb{E} \left\{ \|X_t^\epsilon\|^{2n} \right\} \leq \max \left\{ \|X_0\|^{2n} ; \left( \frac{2n-1}{2K_V} \right)^n \epsilon^n \right\}. \quad (3.1)$$

2. *Let  $\kappa$  be a positive real. We introduce the deterministic time*

$$T_\kappa^n(\epsilon) := \min \left\{ t \geq 0 \mid \mathbb{E} \left\{ \|X_t^\epsilon\|^{2n} \right\} \leq \kappa^{2n} \right\}.$$

*For  $\epsilon < \frac{\kappa^2 K_V}{2n-1}$ , we have the inequality:  $T_\kappa^n(\epsilon) \leq \frac{1}{nK_V \kappa^{2n}} \|X_0\|^{2n}$ .*

3. *Moreover, for all  $t \geq T_\kappa^n(\epsilon)$ ,  $\mathbb{E} \left\{ \|X_t^\epsilon\|^{2n} \right\} \leq \kappa^{2n}$ , still for  $\epsilon < \frac{\kappa^2 K_V}{2n-1}$ .*

*Proof.* We put  $\xi_\epsilon(t) := \mathbb{E} \left\{ \|X_t^\epsilon\|^{2n} \right\}$ . We apply the Itô formula, we integrate, we take the expectation then we take the derivative. We obtain:

$$\begin{aligned} \xi'_\epsilon(t) &= 2n\mathbb{E} \left\{ \|X_t^\epsilon\|^{2n-2} \langle X_t^\epsilon ; V(X_t^\epsilon) \rangle \right\} - 2n\mathbb{E} \left\{ \|X_t^\epsilon\|^{2n-2} \langle X_t^\epsilon ; \Phi * u_t^\epsilon(X_t^\epsilon) \rangle \right\} \\ &\quad + n(2n-1)\epsilon \mathbb{E} \left\{ \|X_t^\epsilon\|^{2n-2} \right\} =: 2n(a_\epsilon(t) + b_\epsilon(t)) + c_\epsilon(t). \end{aligned}$$

By definition, the second term  $b_\epsilon(t)$  can be written as

$$b_\epsilon(t) = -\mathbb{E} \left[ \|X_t^\epsilon\|^{2n-2} \langle X_t^\epsilon ; \Phi(X_t^\epsilon - Y_t^\epsilon) \rangle \right]$$

where  $Y^\epsilon$  is a solution of (1.1) independent from  $X^\epsilon$ . We can exchange  $X^\epsilon$  and  $Y^\epsilon$ . Thereby, by using the assumptions, we get:

$$\begin{aligned} b_\epsilon(t) &= \mathbb{E} \left\{ \frac{\phi(\|X_t^\epsilon - Y_t^\epsilon\|)}{\|X_t^\epsilon - Y_t^\epsilon\|} \left\langle \|X_t^\epsilon\|^{2n-2} X_t^\epsilon ; X_t^\epsilon - Y_t^\epsilon \right\rangle \right\} \\ &= \frac{1}{2} \mathbb{E} \left\{ \frac{\phi(\|X_t^\epsilon - Y_t^\epsilon\|)}{\|X_t^\epsilon - Y_t^\epsilon\|} \left\langle X_t^\epsilon \|X_t^\epsilon\|^{2n-2} - Y_t^\epsilon \|Y_t^\epsilon\|^{2n-2} ; X_t^\epsilon - Y_t^\epsilon \right\rangle \right\}. \end{aligned}$$

This last term is nonnegative. Indeed, the Cauchy-Schwarz inequality implies

$$\left\langle x \|x\|^{2n-2} - y \|y\|^{2n-2} ; x - y \right\rangle \geq \left( \|x\|^{2n-1} - \|y\|^{2n-1} \right) (\|x\| - \|y\|) \geq 0$$

for all  $x, y \in \mathbb{R}^d$ . Therefore, we obtain  $b_\epsilon(t) \geq 0$ .

Moreover, the convexity assumption implies

$$a_\epsilon(t) = \mathbb{E} \left\{ \|X_t^\epsilon\|^{2n-2} \langle X_t^\epsilon ; V(X_t^\epsilon) \rangle \right\} \leq -K_V \mathbb{E} \left\{ \|X_t^\epsilon\|^{2n} \right\} = -K_V \xi_\epsilon(t).$$

Hence, by using Jensen inequality, we deduce  $c_\epsilon(t) \leq n(2n-1)\epsilon \xi_\epsilon(t)^{1-\frac{1}{2n}}$ . By combining results on  $a_\epsilon(t)$ ,  $b_\epsilon(t)$  and  $c_\epsilon(t)$ , we obtain

$$\xi'_\epsilon(t) \leq -2nK_V \xi_\epsilon(t)^{1-\frac{1}{n}} \left\{ \xi_\epsilon(t)^{\frac{1}{n}} - \frac{(2n-1)\epsilon}{2K_V} \right\}. \quad (3.2)$$

The statements of the lemma are obvious consequences of Inequality (3.2).  $\square$

This means that the self-stabilizing process tends to be trapped in a ball with center  $0 = x_{stable}$ .

**3.2 Probability of exiting before  $T_\kappa(\epsilon)$**

In this paragraph, we give the following result.

**Proposition 3.2.** *We have:  $\lim_{\epsilon \rightarrow 0} \mathbb{P}(\tau_{\mathcal{D}}(\epsilon) < T_\kappa(\epsilon)) = 0$  for any  $\kappa > 0$ .*

We skip the proof but the ideas are the following. For all  $\delta > 0$ , we introduce  $\tau_\delta(\epsilon) := \inf \{t > 0 : \|X_t^\epsilon - \varphi_t\| > \delta\}$ , where we remind the reader that  $\varphi_t = X_0 + \int_0^t V(\varphi_s) ds$ . Thus, for any  $T > 0$ , the following limit is an easy and classical result:  $\lim_{\epsilon \rightarrow 0} \mathbb{P}(\tau_\delta(\epsilon) < T) = 0$ . However, here, we consider the interval  $[0; T_\kappa(\epsilon)]$  which depends on  $\epsilon$ . But, we have uniformly bounded  $T_\kappa(\epsilon)$ . Indeed, we have  $\mathbb{P}(\tau_\delta(\epsilon) < T_\kappa(\epsilon)) \leq \mathbb{P}\left(\tau_\delta(\epsilon) < \frac{1}{nK_V\kappa^{2n}} \|X_0\|^{2n}\right)$ , which goes to 0 as the noise elapses.

Due to Hypothesis 2.2, we have  $\{\varphi_t : t > 0\} \subset \mathcal{D}$ . Consequently, for any  $\kappa > 0$ , we obtain the limit  $\lim_{\epsilon \rightarrow 0} \mathbb{P}(\tau_{\mathcal{D}}(\epsilon) < T_\kappa(\epsilon)) = 0$ .

**3.3 Coupling result**

Let  $\mathcal{K}$  be a compact domain which contains the open set  $\mathcal{D}$ .

We have proven that the diffusion does not exit the domain  $\mathcal{D}$  before the time  $T_\kappa(\epsilon)$  with large probability when  $\epsilon$  is small. Now, we study the exit of the diffusion from the domain after the time  $T_\kappa(\epsilon)$ . To do so, we use the inequality:  $\sup_{t \geq T_\kappa(\epsilon)} \mathbb{E} \left\{ \|X_t^\epsilon\|^{2n} \right\} \leq \kappa^{2n}$ , which holds for any  $\kappa > 0$  (as soon as  $\epsilon < \frac{\kappa^2 K_V}{2n-1}$ ), we deduce that the drift  $V - \Phi * u_t^\epsilon$  is close to the vector field  $V - \Phi * \delta_0 = V - \Phi$ . Consequently, we consider the following diffusion defined for  $t \geq T_\kappa(\epsilon)$ :

$$Y_t^\epsilon = X_{T_\kappa(\epsilon)} + \sqrt{\epsilon} (W_t - W_{T_\kappa(\epsilon)}) + \int_{T_\kappa(\epsilon)}^t V(Y_s^\epsilon) ds - \int_{T_\kappa(\epsilon)}^t \Phi(Y_s^\epsilon) ds,$$

if  $X_{T_\kappa(\epsilon)} \in \mathcal{K}$  and  $Y_t^\epsilon := X_t^\epsilon$  otherwise. We introduce the two exit times:  $\tau_{\mathcal{K}}(\epsilon) := \inf \{t > T_\kappa(\epsilon) : X_t^\epsilon \notin \mathcal{K}\}$  and  $\tau'_{\mathcal{K},\kappa}(\epsilon) := \inf \{t > T_\kappa(\epsilon) : Y_t^\epsilon \notin \mathcal{K}\}$ .

We introduce the stopping time:  $\mathcal{T}_{\mathcal{K},\kappa}(\epsilon) := \min \{\tau_{\mathcal{K}}(\epsilon); \tau'_{\mathcal{K},\kappa}(\epsilon)\}$ . The following result tells us that the two diffusions are close on  $[T_\kappa(\epsilon); \mathcal{T}_{\mathcal{K},\kappa}(\epsilon)]$ .

**Theorem 3.3.** *There exists  $\kappa_0$  such that for all  $\kappa < \kappa_0$ , there exists  $\epsilon_0(\kappa) > 0$  such that  $\mathbb{P} \left\{ \sup_{T_\kappa(\epsilon) \leq t \leq \mathcal{T}_{\mathcal{K},\kappa}(\epsilon)} \|X_t^\epsilon - Y_t^\epsilon\| \geq r(\kappa) \right\} \leq r(\kappa)$  for all  $\epsilon < \epsilon_0(\kappa)$ . Here,  $r$  is a positive, continuous and increasing function such that  $r(0) = 0$ .*

*Proof. Step 1.* We introduce the vector fields  $H_\infty(x) := V(x) - \Phi(x)$  and  $H_t(x) := V(x) - \Phi * u_t^\epsilon(x)$ . The assumptions on  $V$  and  $\Phi$  imply  $DH_t(x) \leq -K_V < 0$ . From now on, we put  $\xi_\epsilon(t) := \|X_t^\epsilon - Y_t^\epsilon\|$ . If  $X_{T_\kappa}^\epsilon, Y_{T_\kappa}^\epsilon \in \mathcal{K}$  then, for all  $T_\kappa \leq t \leq \mathcal{T}_{\mathcal{K},\kappa}(\epsilon)$ , we have:

$$\begin{aligned} \frac{d}{dt} (\xi_\epsilon(t))^2 &= -2 \langle H_t(X_t^\epsilon) - H_\infty(Y_t^\epsilon); X_t^\epsilon - Y_t^\epsilon \rangle \\ &= -2 \langle H_t(X_t^\epsilon) - H_t(Y_t^\epsilon); X_t^\epsilon - Y_t^\epsilon \rangle \\ &\quad - 2 \langle \Phi * u_t^\epsilon(Y_t^\epsilon) - \Phi(Y_t^\epsilon); X_t^\epsilon - Y_t^\epsilon \rangle \\ &\leq -2K_V (\xi_\epsilon(t))^2 + 2\xi_\epsilon(t) f_{\mathcal{K}}(\kappa), \end{aligned} \tag{3.3}$$

where we set  $f_{\mathcal{K}}(\kappa) := \sup_{\mu_1 \in \mathbb{B}_\kappa^\infty} \sup_{x \in \mathcal{K}} \|\Phi * \mu_1(x) - \Phi(x)\| =: K_V r(\kappa)^{\frac{3}{2}}$ . Inequality (3.3) directly implies  $\sup_{T_\kappa \leq t \leq \mathcal{T}_{\mathcal{K},\kappa}(\epsilon)} \|X_t^\epsilon - Y_t^\epsilon\|^2 \leq r(\kappa)^3$  which yields  $\mathbb{E} \left\{ \sup_{T_\kappa(\epsilon) \leq t \leq \mathcal{T}_{\mathcal{K},\kappa}(\epsilon)} \|X_t^\epsilon - Y_t^\epsilon\|^2 \right\} \leq r(\kappa)^3$ . The claim thus follows from the Markov inequality.  $\square$

**3.4 Proof**

**Step 1.** Let  $\kappa > 0$ . We can easily prove (by proceeding like in [5]) that there exist two families of domains  $(\mathcal{D}_{i,\kappa})_{\kappa>0}$  and  $(\mathcal{D}_{e,\kappa})_{\kappa>0}$  such that

- $\mathcal{D}_{i,\kappa} \subset \mathcal{D} \subset \mathcal{D}_{e,\kappa}$ .
- $\mathcal{D}_{i,\kappa}$  and  $\mathcal{D}_{e,\kappa}$  are stable by  $V - \Phi$ . The terminology “stable by” has been introduced in Definition 2.1.
- $\sup_{z \in \partial \mathcal{D}_{i,\kappa}} d(z; \mathcal{D}^c) + \sup_{z \in \partial \mathcal{D}_{e,\kappa}} d(z; \mathcal{D})$  tends to 0 when  $\kappa$  goes to 0.
- $\inf_{z \in \partial \mathcal{D}_{i,\kappa}} d(z; \mathcal{D}^c) = \inf_{z \in \partial \mathcal{D}_{e,\kappa}} d(z; \mathcal{D}) = r(\kappa)$ .

**Step 2.** By  $\tau'_{i,\kappa}(\epsilon)$  (resp.  $\tau'_{e,\kappa}(\epsilon)$ ), we denote the first exit time of  $Y^\epsilon$  from  $\mathcal{D}_{i,\kappa}$  (resp.  $\mathcal{D}_{e,\kappa}$ ).

**Step 3.** We prove here the upper bound:

$$\begin{aligned} \mathbb{P} \left\{ \tau_{\mathcal{D}}(\epsilon) \geq e^{\frac{\overline{Q_\infty + \xi}}{\epsilon}} \right\} &= \mathbb{P} \left\{ \tau_{\mathcal{D}}(\epsilon) \geq e^{\frac{\overline{Q_\infty + \xi}}{\epsilon}} ; \tau'_{e,\kappa}(\epsilon) \geq e^{\frac{\overline{Q_\infty + \xi}}{\epsilon}} \right\} \\ &\quad + \mathbb{P} \left\{ \tau_{\mathcal{D}}(\epsilon) \geq e^{\frac{\overline{Q_\infty + \xi}}{\epsilon}} ; \tau'_{e,\kappa}(\epsilon) < e^{\frac{\overline{Q_\infty + \xi}}{\epsilon}} \right\} \\ &\leq \mathbb{P} \left\{ \tau_{e,\kappa}(\epsilon) \geq e^{\frac{\overline{Q_\infty + \xi}}{\epsilon}} \right\} \\ &\quad + \mathbb{P} \left\{ \tau_{\mathcal{D}}(\epsilon) \geq e^{\frac{\overline{Q_\infty + \xi}}{\epsilon}} ; \tau'_{e,\kappa}(\epsilon) < e^{\frac{\overline{Q_\infty + \xi}}{\epsilon}} \right\} \\ &=: a_\kappa(\epsilon) + b_\kappa(\epsilon). \end{aligned}$$

**Step 3.1.** By classical results in Freidlin-Wentzell theory, there exists  $\kappa_1 > 0$  such that for all  $0 < \kappa < \kappa_1$ , we have:  $\lim_{\epsilon \rightarrow 0} \mathbb{P} \left\{ \tau'_{e,\kappa}(\epsilon) < \exp \left[ \frac{1}{\epsilon} (\overline{Q_\infty} + \xi) \right] \right\} = 0$ . Therefore, the first term  $a_\kappa(\epsilon)$  tends to 0 as  $\epsilon$  goes to 0.

**Step 3.2.** Let us look at the second term  $b_\kappa(\epsilon)$ . For  $\kappa$  sufficiently small, we have  $\mathcal{D}_{e,\kappa} \subset \mathcal{K}$ . Consequently, we have:

$$\begin{aligned} &\mathbb{P} \left\{ \tau_{\mathcal{D}}(\epsilon) \geq e^{\frac{\overline{Q_\infty + \xi}}{\epsilon}} ; \tau'_{e,\kappa}(\epsilon) \leq e^{\frac{\overline{Q_\infty + \xi}}{\epsilon}} \right\} \\ &\leq \mathbb{P} \left\{ \left\| X_{\tau'_{e,\kappa}(\epsilon)}^\epsilon - Y_{\tau'_{e,\kappa}(\epsilon)}^\epsilon \right\| \geq r(\kappa) \right\} \leq \mathbb{P} \left\{ \sup_{T_\kappa(\epsilon) \leq t \leq \mathcal{T}_{\mathcal{K},\kappa}(\epsilon)} \|X_t^\epsilon - Y_t^\epsilon\| \geq r(\kappa) \right\}. \end{aligned}$$

According to Theorem 3.3, there exists  $\epsilon_0 > 0$  such that the previous term is less than  $r(\kappa)$  for all  $\epsilon < \epsilon_0$ .

**Step 3.3.** Let  $\xi$  be a positive real. By taking  $\kappa$  arbitrarily small, we obtain the upper bound  $\lim_{\epsilon \rightarrow 0} \mathbb{P} \left\{ \tau_{\mathcal{D}}(\epsilon) \geq \exp \left[ \frac{\overline{Q_\infty + \xi}}{\epsilon} \right] \right\} = 0$ .

**Step 3.** Analogous arguments show that  $\lim_{\epsilon \rightarrow 0} \mathbb{P} \left\{ T_\kappa(\epsilon) \leq \tau(\epsilon) \leq e^{\frac{\overline{Q_\infty - \xi}}{\epsilon}} \right\} = 0$ . However, we have  $\lim_{\epsilon \rightarrow 0} \mathbb{P} \left\{ \tau(\epsilon) \leq T_\kappa(\epsilon) \right\} = 0$ . This ends the proof.  $\square$

**Remark 3.4.** We could use the same technics to provide the exit time of the first particle in the mean-field system of particles. The only difference is that we would need to use the first time that the empirical measure exits from the ball  $\mathbb{B}_\kappa^\infty$ ; which is close to the arguments in [5].

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