A simple proof of Heymann's lemma

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Abstract. Heymann's lemma is proved by a simple induction argument.

The problem of pole assignment by state feedback in the system

$$x_{k+1} = Ax_k + Bu_k \;; \qquad (k = 0,1,\dots)$$

where A is an $n \times n$-matrix and B an $n \times m$-matrix, has been considered by many authors. The case $m = 1$ has been dealt with by Rissanen [3] in 1960. In 1964 Popov [2] showed the pole assignability for complex systems (more generally systems over an algebraically closed field). In 1967 Wonham gave a proof valid for real systems (or more generally for systems over an infinite field). Finally, in 1968, Heymann [1] gave a proof which is valid for systems over an arbitrary field. Heymann's proof depends on the following result.

LEMMA 1. *If* (A,b) *is controllable and* $b = Bv \neq 0$, *then there exists* F *such that* $(A + BF,b)$ *is controllable.*

By means of this result the multivariable problem can be reduced to the single variable problem.

It is the aim of this correspondence to give a simple proof of this lemma. The result follows immediately from

LEMMA 2. *If* (A,B) *is controllable and* $b = Bv \neq 0$, *then there exists* $u_1,\dots,u_{n-1}$ *such that the sequence defined by*

(1) $$x_1 := b\,. \quad x_{k+1} := Ax_k + Bu_k\,,$$

for $k = 1,\dots,n - 1$ *is independent.*

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Indeed, if Lemma 2 is shown we can choose u_n arbitrary and define F by $Fx_k = u_k$. Then it is easily seen that $(A + BF)^k b = x_k$, so that $(A + BF, b)$ is controllable.

PROOF OF LEMMA 2. We proceed stepwise. $x_1 \neq 0$ and hence independent. Suppose that $x_1, \ldots, x_k$ have been constructed according to (1) and are independent. Denote by $\mathcal{L}$ the linear space generated by $x_1, \ldots, x_k$. We have to choose u_k such that $x_{k+1} = Ax_k + Bu_k \notin \mathcal{L}$. If this is not possible, then

$$Ax_k + Bu \in \mathcal{L} \tag{2}$$

for all u. Choosing in particular $u = 0$ we find

$$Ax_k \in \mathcal{L} \tag{3}$$

and consequently, by the linearity of $\mathcal{L}$, $Bu \in \mathcal{L}$ for all u. That is, $\mathrm{im}B \subseteq \mathcal{L}$. Also, for $i < k$ we have

$$Ax_i = x_{i+1} - Bu_i \in \mathcal{L}$$

Hence $Ax_i \in \mathcal{L}$ for $i = 1, \ldots, k$, and, consequently, $\mathcal{L}$ is A-invariant. From the controllability of (A,B) it follows that $\mathcal{L}$ must be the whole state space, which implies that $k = n$. □

REMARK. In [1] and in [5, Lemma 2.2] proofs of Lemma 1 were given by constructing a particular sequence u_k satisfying the condition of Lemma 2. These constructions may suggest that such a special u_k is essential for the calculation of F, which is not the case as follows from the proof of Lemma 2. It also follows that the u_k's can be constructed recursively in the following sense: Once $u_1, \ldots, u_{\ell-1}$ have been chosen so as to render $x_1, \ldots, x_\ell$ independent, one can always continue the construction of the remaining u_k's.
The may be useful when it comes to an actual numerical compution of F.

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