A. K. Lerner (2013) "A Simple Proof of the A_2 Conjecture,"
International Mathematics Research Notices, Vol. 2013, No. 14, pp. 3159–3170
Advance Access Publication June 2, 2012
doi:10.1093/imrn/rns145

A Simple Proof of the A_2 Conjecture

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We give a simple proof of the A_2 conjecture proved recently by Hytönen. Our proof completely avoids the notion of the Haar shift operator, and is based only on the "local mean oscillation decomposition." Also our proof yields a simple proof of the "two-weight conjecture" as well.

1 Introduction

Let T be an L^2 bounded Calderón–Zygmund operator. We say that $w \in A_2$ if

$$\|w\|_{A_2} = \sup_{Q \subset \mathbb{R}^n} w(Q) w^{-1}(Q)/|Q|^2 < \infty.$$

In this note, we give a rather simple proof of the A_2 conjecture recently settled by Hytönen [7].

Theorem 1.1. For any $w \in A_2$,

$$\|T\|_{L^2(w)} \leq c(n, T)\|w\|_{A_2}. \tag{1.1}$$

□

Below is a partial list of important contributions to this result. First, (1.1) was proved for the following operators:

- Hardy–Littlewood maximal operator (Buckley [3], 1993);

Received February 17, 2012; Revised May 1, 2012; Accepted May 4, 2012
Communicated by Prof. Assaf Naor

- Beurling transform (Petermichl and Volberg [23], 2002);
- Hilbert transform (Petermichl [21], 2007);
- Riesz transform (Petermichl [22], 2008);
- Dyadic paraproduct (Beznosova [2], 2008);
- Haar shift (Lacey *et al.* [15], 2010).

After that, the following works appeared in very small intervals:

- a simplified proof for Haar shifts (Cruz-Uribe *et al.* [5, 6], 2010);
- the $L^2(w)$ bound for general T by $\|w\|_{A_2}\log(1+\|w\|_{A_2})$ (Pérez *et al.* [20], 2010);
- (1.1) in full generality (Hytönen [7], 2010);
- a simplification of the proof (Hytönen *et al.* [12], 2010);
- (1.1) for the maximal Calderón–Zygmund operator $T_\natural$ (Hytönen *et al.* [9], 2010).

The "Bellman function" proof of the A_2 conjecture in a geometrically doubling metric space was given by Nazarov *et al.* [18] (see also [19]).

All currently known proofs of (1.1) were based on the representation of T in terms of the Haar shift operators $\mathbb{S}_{\mathscr{D}}^{m,k}$. Such representations also have a long history; for general T it was found in [7]. The second key element of all known proofs was showing (1.1) for $\mathbb{S}_{\mathscr{D}}^{m,k}$ in place of T with the corresponding constant depending linearly (or polynomially) on the complexity. Observe that over the past year several different proofs of this step appeared (see, e.g., [14, 24]).

In a very recent work [17], we have proved that for any Banach function space $X(\mathbb{R}^n)$,

$$\|T_\natural f\|_X \le c(T,n)\sup_{\mathscr{D},\mathcal{S}}\|\mathcal{A}_{\mathscr{D},\mathcal{S}}|f|\|_X, \tag{1.2}$$

where

$$\mathcal{A}_{\mathscr{D},\mathcal{S}}f(x)=\sum_{j,k} f_{Q_j^k}\chi_{Q_j^k}(x)$$

(this operator is defined by means of a sparse family $\mathcal{S}=\{Q_j^k\}$ from a general dyadic grid $\mathscr{D}$; for these notions see Section 2 below).

Observe that for the operator $\mathcal{A}_{\mathscr{D},\mathcal{S}}f$ inequality (1.1) follows just in few lines by a very simple argument. This was first observed in [5, 6] (see also [17]). Hence, in the case when $X=L^2(w)$, inequality (1.2) easily implies the A_2 conjecture. Also, (1.2) yields the "two-weight conjecture" by Cruz-Uribe and Pérez; we refer to [17] for the details.

The proof of (1.2) in [17] still depended on the representation of T in terms of the Haar shift operators. In this note, we will show that this difficult step can be completely avoided. Our new proof of (1.2) is based only on the "local mean oscillation decomposition" proved by the author in [16]. It is interesting that we apply this decomposition twice. First, it is applied directly to $T_\natural$, and we obtain that $T_\natural$ is essentially pointwise dominated by the maximal operator M and a series of dyadic type operators $\mathcal{T}_m$. In order to handle $\mathcal{T}_m$, we apply the decomposition again to the adjoint operators $\mathcal{T}_m^\star$. After this step, we obtain a pointwise domination by the simplest dyadic operators $\mathcal{A}_{\mathscr{D},\mathcal{S}}$.

Note that all our estimates are actually pointwise, and they do not depend on a particular function space. This explains why we prefer to write (1.2) with a general Banach function space X.

2 Preliminaries

2.1 Calderón–Zygmund operators

By a Calderón–Zygmund operator in $\mathbb{R}^n$ we mean an L^2 bounded integral operator represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, \mathrm{d}y, \quad x \notin \operatorname{supp} f,$$

with kernel K satisfying the following growth and smoothness conditions:

(i) $|K(x, y)| \le \frac{c}{|x-y|^n}$ for all $x \neq y$;

(ii) there exists $0 < \delta \le 1$ such that

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \le c\frac{|x - x'|^\delta}{|x - y|^{n+\delta}},$$

whenever $|x - x'| < |x - y|/2$.

Given a Calderón–Zygmund operator T, define its maximal truncated version by

$$T_\natural f(x) = \sup_{0<\varepsilon<\nu} \left| \int_{\varepsilon<|y|<\nu} K(x, y) f(y)\, \mathrm{d}y \right|.$$

2.2 Dyadic grids

Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of the cubes

$$2^{-k}([0, 1)^n + j), \quad k \in \mathbb{Z}, \ j \in \mathbb{Z}^n.$$

Denote the standard grid by $\mathcal{D}$.

By a *general dyadic grid* $\mathscr{D}$, we mean a collection of cubes with the following properties: (i) for any $Q \in \mathscr{D}$ its sidelength ℓ_Q is of the form 2^k, $k \in \mathbb{Z}$; (ii) $Q \cap R \in \{Q, R, \emptyset\}$ for any $Q, R \in \mathscr{D}$; (iii) the cubes of a fixed sidelength 2^k form a partition of $\mathbb{R}^n$.

Given a cube Q_0, denote by $\mathcal{D}(Q_0)$ the set of all dyadic cubes with respect to Q_0, that is, the cubes from $\mathcal{D}(Q_0)$ are formed by repeated subdivision of Q_0 and each of its descendants into 2^n congruent subcubes. Observe that if $Q_0 \in \mathscr{D}$, then each cube from $\mathcal{D}(Q_0)$ will also belong to $\mathscr{D}$.

A well-known principle says that there are ξ_n general dyadic grids $\mathscr{D}_\alpha$ such that every cube $Q \subset \mathbb{R}^n$ is contained in some cube $Q' \in \mathscr{D}_\alpha$ such that $|Q'| \le c_n |Q|$. For $\xi_n = 3^n$ this is attributed in the literature to Christ and, independently, to Garnett and Jones. For $\xi_n = 2^n$, it can be found in a recent work by Hytönen and Pérez [11]. Very recently it was shown by Conde *et al.* [4] that one can take $\xi_n = n + 1$, and this number is optimal. For our purposes any of such variants is suitable. We will use the one from [11].

Proposition 2.1. There are 2^n dyadic grids $\mathscr{D}_\alpha$ such that for any cube $Q \subset \mathbb{R}^n$ there exists a cube $Q_\alpha \in \mathscr{D}_\alpha$ such that $Q \subset Q_\alpha$ and $\ell_{Q_\alpha} \le 6\ell_Q$. $\square$

The grids $\mathscr{D}_\alpha$ here are the following:

$$\mathscr{D}_\alpha = \{2^{-k}([0, 1)^n + j + \alpha)\}, \quad \alpha \in \{0, 1/3\}^n.$$

We outline briefly the proof. First, it is easy to see that it suffices to consider the one-dimensional case. Take an arbitrary interval $I \subset \mathbb{R}$. Fix $k_0 \in \mathbb{Z}$ such that $2^{-k_0-1} \le 3\ell_I < 2^{-k_0}$. If I does not contain any point $2^{-k_0} j$, $j \in \mathbb{Z}$, then I is contained in some $I' = [2^{-k_0} j, 2^{-k_0}(j+1))$ (since such intervals form a partition of $\mathbb{R}$), and $\ell_{I'} \le 6\ell_I$. On the other hand, if I contains some point $j_0 2^{-k_0}$, then I does not contain any point $2^{-k_0}(j + 1/3)$, $j \in \mathbb{Z}$ (since $\ell_I < 2^{-k_0}/3$), and therefore I is contained in some $I'' = [2^{-k_0}(j + 1/3), 2^{-k_0}(j + 4/3))$, and $\ell_{I''} \le 6\ell_I$.

2.3 Local mean oscillations

Given a measurable function f on $\mathbb{R}^n$ and a cube Q, the local mean oscillation of f on Q is defined by

$$\omega_\lambda(f; Q) = \inf_{c \in \mathbb{R}} ((f - c)\chi_Q)^*(\lambda |Q|) \quad (0 < \lambda < 1),$$

where f^* denotes the nonincreasing rearrangement of f.

By a median value of f over Q we mean a possibly nonunique, real number $m_f(Q)$ such that

$$\max(|\{x\in Q: f(x)>m_f(Q)\}|, |\{x\in Q: f(x)<m_f(Q)\}|)\le |Q|/2.$$

It is easy to see that the set of all median values of f is either one point or the closed interval. In the latter case, we will assume for the definiteness that $m_f(Q)$ is the *maximal* median value. Observe that it follows from the definitions that

$$|m_f(Q)|\le (f\chi_Q)^*(|Q|/2). \tag{2.1}$$

Given a cube Q_0, the dyadic local sharp maximal function $M^{\#,d}_{\lambda;\,Q_0} f$ is defined by

$$M^{\#,d}_{\lambda;\,Q_0} f(x)=\sup_{x\in Q'\in\mathcal{D}(Q_0)}\omega_\lambda(f;\,Q').$$

We say that $\{Q_j^k\}$ is a *sparse family* of cubes if: (i) the cubes Q_j^k are disjoint in j, with k fixed; (ii) if $\Omega_k=\cup_j Q_j^k$, then $\Omega_{k+1}\subset\Omega_k$; (iii) $|\Omega_{k+1}\cap Q_j^k|\le\frac{1}{2}|Q_j^k|$.

The following theorem was proved in [17] (its very similar version can be found in [16]).

Theorem 2.2. Let f be a measurable function on $\mathbb{R}^n$ and let Q_0 be a fixed cube. Then there exists a (possibly empty) sparse family of cubes $Q_j^k\in\mathcal{D}(Q_0)$ such that for a.e. $x\in Q_0$,

$$|f(x)-m_f(Q_0)|\le 4M^{\#,d}_{\frac{1}{2^{n+2}};\,Q_0} f(x)+2\sum_{k,j}\omega_{\frac{1}{2^{n+2}}}(f;\,Q_j^k)\chi_{Q_j^k}(x). \qquad \square$$

The following proposition is well known, and it can be found in a slightly different form in [**?**]. We give its proof here for the sake of the completeness. The proof is a classical argument used, for example, to show that T is bounded from L^∞ to BMO. Also the same argument is used to prove a good-λ inequality relating T and M.

Proposition 2.3. For any cube $Q\subset\mathbb{R}^n$,

$$\omega_\lambda(Tf;\,Q)\le c(T,\lambda,n)\sum_{m=0}^{\infty}\frac{1}{2^{m\delta}}\left(\frac{1}{|2^mQ|}\int_{2^mQ}|f(y)|\,dy\right) \tag{2.2}$$

and

$$\omega_\lambda(T_\natural f;\,Q)\le c(T,\lambda,n)\sum_{m=0}^{\infty}\frac{1}{2^{m\delta}}\left(\frac{1}{|2^mQ|}\int_{2^mQ}|f(y)|\,dy\right). \tag{2.3}$$

$\square$

Proof. Let $f_1 = f\chi_{2\sqrt{n}Q}$ and $f_2 = f - f_1$. If $x \in Q$ and x_0 is the center of Q, then by the kernel assumptions,

$$
\begin{aligned}
|T(f_2)(x) - T(f_2)(x_0)| &\le \int_{\mathbb{R}^n\setminus 2\sqrt{n}Q} |f(y)|\,|K(x,y) - K(x_0,y)|\,\mathrm{d}y \\
&\le c\ell_Q^{\delta}\int_{\mathbb{R}^n\setminus 2Q} \frac{|f(y)|}{|x-y|^{n+\delta}}\,\mathrm{d}y \le c\ell_Q^{\delta}\sum_{m=0}^{\infty}\frac{1}{(2^m\ell_Q)^{n+\delta}}\int_{2^{m+1}Q\setminus 2^m Q}|f(y)|\,\mathrm{d}y \\
&\le c\sum_{m=0}^{\infty}\frac{1}{2^{m\delta}}\left(\frac{1}{|2^mQ|}\int_{2^mQ}|f(y)|\,\mathrm{d}y\right).
\end{aligned}
$$

From this and from the weak type $(1,1)$ of T,

$$
\begin{aligned}
((Tf - T(f_2)(x_0))\chi_Q)^*(\lambda|Q|) &\le (T(f_1))^*(\lambda|Q|) + \|T(f_2) - T(f_2)(x_0)\|_{L^\infty(Q)} \\
&\le c\frac{1}{|Q|}\int_{2\sqrt{n}|Q|}|f(y)|\,\mathrm{d}y + c\sum_{m=0}^{\infty}\frac{1}{2^{m\delta}}\left(\frac{1}{|2^mQ|}\int_{2^mQ}|f(y)|\,\mathrm{d}y\right) \\
&\le c'\sum_{m=0}^{\infty}\frac{1}{2^{m\delta}}\left(\frac{1}{|2^mQ|}\int_{2^mQ}|f(y)|\,\mathrm{d}y\right),
\end{aligned}
$$

which proves (2.2).

The same inequalities hold for $T_\natural$ as well, which gives (2.3). The only trivial difference in the argument is that one needs to use the sublinearity of $T_\natural$ instead of the linearity of T. ■

3 Proof of (1.2)

Combining Proposition 2.3 and Theorem 2.2 with $Q_0 \in \mathcal{D}$, we get that there exists a sparse family $S = \{Q_j^k\} \in \mathcal{D}$ such that for a.e. $x \in Q_0$,

$$
|T_\natural f(x) - m_{Q_0}(T_\natural f)| \le c(n,T)\left(Mf(x) + \sum_{m=0}^{\infty}\frac{1}{2^{m\delta}}\mathcal{T}_{\mathcal{S},m}|f|(x)\right),
$$

where M is the Hardy–Littlewood maximal operator and

$$
\mathcal{T}_{\mathcal{S},m}f(x) = \sum_{j,k} f_{2^mQ_j^k}\chi_{Q_j^k}(x).
$$

If $f \in L^1$, then it follows from (2.1) that $|m_Q(T_\natural f)| \to 0$ as $|Q| \to \infty$. Therefore, letting Q_0 to anyone of 2^n quadrants and using Fatou's lemma, we obtain

$$\|T_\natural f\|_X \le c(n,T)\left(\|Mf\|_X + \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}} \sup_{\mathcal{S}\in\mathcal{D}} \|\mathcal{T}_{\mathcal{S},m}|f|\|_X\right)$$

(for the notion of the Banach function space X we refer to [1, Chapter 1]).

Hence, (1.2) will follow from

$$\|Mf\|_X \le c(n) \sup_{\mathscr{D},\mathcal{S}} \|\mathcal{A}_{\mathscr{D},\mathcal{S}} f\|_X \quad (f \ge 0) \tag{3.1}$$

and

$$\sup_{\mathcal{S}\in\mathcal{D}} \|\mathcal{T}_{\mathcal{S},m} f\|_X \le c(n)m \sup_{\mathscr{D},\mathcal{S}} \|\mathcal{A}_{\mathscr{D},\mathcal{S}} f\|_X \quad (f \ge 0). \tag{3.2}$$

Inequality (3.1) was proved in [17]; we give the proof here for the sake of the completeness. The proof is just a combination of Proposition 2.1 and the Calderón–Zygmund decomposition. First, by Proposition 2.1,

$$Mf(x) \le 6^n \sum_{\alpha=1}^{2^n} M^{\mathscr{D}_\alpha} f(x). \tag{3.3}$$

Second, by the Calderón-Zygmund decomposition, if

$$\Omega_k = \{x : M^d f(x) > 2^{(n+1)k}\} = \bigcup\nolimits_j Q_j^k \quad \text{and} \quad E_j^k = Q_j^k \setminus \Omega_{k+1},$$

then the family $\{Q_j^k\}$ is sparse and

$$M^d f(x) \le 2^{n+1} \sum_{k,j} f_{Q_j^k} \chi_{E_j^k}(x) \le 2^{n+1} \mathcal{A} f(x).$$

From this and from (3.3),

$$Mf(x) \le 2\cdot 12^n \sum_{\alpha=1}^{2^n} \mathcal{A}_{\mathscr{D}_\alpha,\mathcal{S}_\alpha} f(x), \tag{3.4}$$

where $\mathcal{S}_\alpha \in \mathscr{D}_\alpha$ depends on f. This implies (3.1) with $c(n) = 2\cdot 24^n$.

We now turn to the proof of (3.2). Fix a family $\mathcal{S} = \{Q_j^k\} \in \mathcal{D}$. Applying Proposition 2.1 again, we can decompose the cubes Q_j^k into 2^n disjoint families F_α such that for any $Q_j^k \in F_\alpha$ there exists a cube $Q_{j,\alpha}^k \in \mathscr{D}_\alpha$ such that $2^m Q_j^k \subset Q_{j,\alpha}^k$ and $\ell_{Q_{j,\alpha}^k} \le 6\ell_{2^m Q_j^k}$. Hence,

$$\mathcal{T}_{\mathcal{S},m} f(x) \le 6^n \sum_{\alpha=1}^{2^n} \sum_{j,k: Q_j^k \in F_\alpha} f_{Q_{j,\alpha}^k} \chi_{Q_j^k}(x).$$

Set

$$\mathcal{A}_{m,\alpha} f(x) = \sum_{j,k} f_{Q_{j,\alpha}^k} \chi_{Q_j^k}(x).$$

We have that (3.2) will follow from

$$\|\mathcal{A}_{m,\alpha} f\|_X \le c(n) m \sup_{\mathscr{D},\mathcal{S}} \|\mathcal{A}_{\mathscr{D},\mathcal{S}} f\|_X \quad (f \ge 0). \tag{3.5}$$

Consider the formal adjoint to $\mathcal{A}_{m,\alpha}$:

$$\mathcal{A}_{m,\alpha}^\star f = \sum_{j,k} \left(\frac{1}{|Q_{j,\alpha}^k|} \int_{Q_j^k} f \right) \chi_{Q_{j,\alpha}^k}(x).$$

Proposition 3.1. For any $m \in \mathbb{N}$,

$$\|\mathcal{A}_{m,\alpha}^\star f\|_{L^2} = \|\mathcal{A}_{m,\alpha} f\|_{L^2} \le 8 \|f\|_{L^2}. \qquad \square$$

Proof. Set $E_j^k = Q_j^k \setminus \Omega_{k+1}$. Observe that the sets E_j^k are pairwise disjoint and $|Q_j^k| \le 2|E_j^k|$. From this,

$$\begin{aligned}\int_{\mathbb{R}^n} (\mathcal{A}_{m,\alpha} f) g \, dx = \sum_{k,j} f_{Q_{j,\alpha}^k} g_{Q_j^k} |Q_j^k| &\le 2 \sum_{k,j} \int_{E_j^k} (M^{\mathscr{D}_\alpha} f)(M^d g) \, dx \\ &\le 2 \int_{\mathbb{R}^n} (M^{\mathscr{D}_\alpha} f)(M^d g) \, dx.\end{aligned}$$

From this, using Hölder's inequality, the L^2 boundedness of M^d and duality, we get the L^2 bound for $\mathcal{A}_{m,\alpha}$. ■

Lemma 3.2. For any $m \in \mathbb{N}$,

$$\|\mathcal{A}_{m,\alpha}^\star f\|_{L^{1,\infty}} \le c(n) m \|f\|_{L^1}. \qquad \square$$

Proof. Set $\Omega = \{x : Mf(x) > \alpha\}$ and let $\Omega = \cup_l Q_l$ be a Whitney decomposition such that $3Q_l \subset \Omega$, where $Q_l \in \mathcal{D}$ (see, e.g., [1, p. 348]). Set also

$$b_l = (f - f_{Q_l})\chi_{Q_l}, \quad b = \sum_l b_l$$

and $g = f - b$. We have

$$|\{x : |\mathcal{A}^{\star}_{m,\alpha} f(x)| > \alpha\}| \le |\Omega| + |\{x : |\mathcal{A}^{\star}_{m,\alpha} g(x)| > \alpha/2\}| + |\{x \in \Omega^c : |\mathcal{A}^{\star}_{m,\alpha} b(x)| > \alpha/2\}|. \tag{3.6}$$

Further, $|\Omega| \le \frac{c(n)}{\alpha}\|f\|_{L_1}$, and, by the L^2 boundedness of $\mathcal{A}^{\star}_{m,\alpha}$,

$$|\{x : |\mathcal{A}^{\star}_{m,\alpha} g(x)| > \alpha/2\}| \le \frac{4}{\alpha^2}\|\mathcal{A}^{\star}_{m,\alpha} g\|^2_{L^2} \le \frac{c}{\alpha^2}\|g\|^2_{L^2} \le \frac{c}{\alpha}\|g\|_{L^1} \le \frac{c}{\alpha}\|f\|_{L^1}$$

(we have used here that $g \le c\alpha$).

It remains therefore to estimate the term in (3.6). For $x \in \Omega^c$, consider

$$\mathcal{A}^{\star}_{m,\alpha} b(x) = \sum_l \sum_{k,j} \left(\frac{1}{|Q^k_{j,\alpha}|}\int_{Q^k_j} b_l\right)\chi_{Q^k_{j,\alpha}}(x).$$

The second sum is taken over those cubes Q^k_j for which $Q^k_j \cap Q_l \ne \emptyset$. If $Q_l \subseteq Q^k_j$, then $(b_l)_{Q^k_j} = 0$. Therefore, one can assume that $Q^k_j \subset Q_l$. On the other hand, $Q^k_{j,\alpha} \cap \Omega^c \ne \emptyset$. Since $3Q_l \subset \Omega$, we have that $Q_l \subset 3Q^k_{j,\alpha}$. Hence

$$\ell_{Q_l} \le 3\ell_{Q^k_{j,\alpha}} \le 18 \cdot 2^m \ell_{Q^k_j}.$$

The family of all dyadic cubes Q for which $Q \subset Q_l$ and $\ell_{Q_l} \le 18 \cdot 2^m \ell_Q$ can be decomposed into $m+4$ families of disjoint cubes of equal length. Therefore,

$$\sum_{k,j: Q^k_j \subset Q_l \subset 3Q^k_{j,\alpha}} \chi_{Q^k_j} \le (m+4)\chi_{Q_l}.$$

From this, we obtain

$$\begin{aligned} |\{x \in \Omega^c : |\mathcal{A}^{\star}_{m,\alpha} b(x)| > \alpha/2\}| &\le \frac{2}{\alpha}\|\mathcal{A}^{\star}_{m,\alpha} b\|_{L^1(\Omega^c)} \\ &\le \frac{2}{\alpha}\sum_l \sum_{k,j: Q^k_j \subset Q_l \subset 3Q^k_{j,\alpha}} \int_{Q^k_j} |b_l|\, dx \end{aligned}$$

$$\leq \frac{2(m+4)}{\alpha}\sum_l \int_{Q_l}|b_l|\,\mathrm{d}x$$

$$\leq \frac{4(m+4)}{\alpha}\|f\|_{L^1}.$$

The proof is complete. ■

Lemma 3.3. For any cube $Q \in \mathscr{D}_\alpha$,

$$\omega_{\lambda_n}(\mathcal{A}^\star_{m,\alpha} f;\, Q) \leq c(n) m f_Q \quad (\lambda_n = 1/2^{n+2}).$$ □

Proof. For $x \in Q$,

$$\sum_{k,j:Q\subseteq Q^k_{j,\alpha}} \left(\frac{1}{|Q^k_{j,\alpha}|}\int_{Q^k_j} f\right)\chi_{Q^k_{j,\alpha}}(x) = \sum_{k,j:Q\subseteq Q^k_{j,\alpha}} \left(\frac{1}{|Q^k_{j,\alpha}|}\int_{Q^k_j} f\right) \equiv c.$$

Hence

$$|\mathcal{A}^\star_{m,\alpha} f(x) - c|\chi_Q(x) = \sum_{k,j:Q^k_{j,\alpha}\subset Q} \left(\frac{1}{|Q^k_{j,\alpha}|}\int_{Q^k_j} f\right)\chi_{Q^k_{j,\alpha}}(x) \leq \mathcal{A}^\star_{m,\alpha}(f\chi_Q)(x).$$

From this and from Lemma 3.2,

$$\inf_c((\mathcal{A}^\star_{m,\alpha} f - c)\chi_Q)^*(\lambda_n|Q|) \leq (\mathcal{A}^\star_{m,\alpha}(f\chi_Q))^*(\lambda_n|Q|) \leq c(n) m f_Q,$$

which completes the proof. ■

We are now ready to prove (3.5). One can assume that the sum defining $\mathcal{A}_{m,\alpha}$ is finite. Then $m_{\mathcal{A}^\star_{m,\alpha} f}(Q) = 0$ for Q big enough. Hence, By Lemma 3.3 and Theorem 2.2, for a.e. $x \in Q$ (where $Q \in \mathscr{D}_\alpha$),

$$\mathcal{A}^\star_{m,\alpha} f(x) \leq c(n) m (Mf(x) + \mathcal{A}_{\mathcal{S}_\alpha,\mathscr{D}_\alpha} f(x)).$$

From this and from (3.4), for any $g \geq 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} (\mathcal{A}_{m,\alpha} f) g \, \mathrm{d}x &= \int_{\mathbb{R}^n} f(\mathcal{A}^{\star}_{m,\alpha} g) \, \mathrm{d}x \\ &\leq c_n m \sum_{\alpha=1}^{2^n+1} \int_{\mathbb{R}^n} f(\mathcal{A}_{\mathscr{D}_\alpha, \mathcal{S}_\alpha} g) \, \mathrm{d}x \\ &= c_n m \sum_{\alpha=1}^{2^n+1} \int_{\mathbb{R}^n} (\mathcal{A}_{\mathscr{D}_\alpha, \mathcal{S}_\alpha} f) g \, \mathrm{d}x \leq c'_n m \sup_{\mathscr{D}, \mathcal{S}} \|\mathcal{A}_{\mathscr{D}, \mathcal{S}} f\|_X \|g\|_{X'}.\end{aligned}$$

Taking here the supremum over g with $\|g\|_{X'} = 1$ completes the proof.

Added in proof. We have just learned that Hytönen *et al.* [10] have also found a proof of the A_2 conjecture avoiding a representation of T in terms of Haar shifts. The first step in this proof is the same: the "local mean oscillation decomposition" combined with Proposition 2.3 which reduces the problem to operators $\mathcal{A}_{m,\alpha}$. In order to handle $\mathcal{A}_{m,\alpha}$, the authors use the result from [8] where it was observed that this operator can be viewed as a positive Haar shift operator of complexity m. As we have mentioned previously, our proof avoids completely the notion of the Haar shift operator, and to bound $\mathcal{A}_{m,\alpha}$ we apply the decomposition again (as it is shown starting with Lemma 3.2). ■

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