

A Simple Proof of the Aztec Diamond Theorem

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Abstract

Based on a bijection between domino tilings of an Aztec diamond and non-intersecting lattice paths, a simple proof of the Aztec diamond theorem is given by means of Hankel determinants of the large and small Schröder numbers.

Keywords: Aztec diamond, domino tilings, Hankel matrices, Schröder numbers, lattice paths

1 Introduction

The *Aztec diamond* of order n , denoted by AD_n , is defined as the union of all the unit squares with integral corners (x, y) satisfying $|x| + |y| \leq n + 1$. A *domino* is simply a 1-by-2 or 2-by-1 rectangle with integral corners. A *domino tiling* of a region R is a set of non-overlapping dominoes the union of which is R . Figure 1 shows the Aztec diamond of order 3 and a domino tiling. The Aztec diamond theorem, first proved by Elkies *et al.* in [4], states that the number a_n of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$. They give four proofs, relating the tilings in turn to alternating sign matrices,

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monotone triangles, representations of general linear groups, and domino shuffling. Other approaches to this theorem appear in [2, 3, 6]. Ciucu [3] derives the recurrence relation $a_n = 2^n a_{n-1}$ by means of perfect matchings of cellular graphs. Kuo [6] develops a method, called graphical condensation, to derive the recurrence relation $a_n a_{n-2} = 2a_{n-1}^2$, for $n \geq 3$. Recently, Brualdi and Kirkland [2] give a proof by considering a matrix of order $n(n+1)$ the determinant of which gives a_n . Their proof is reduced to the computation of the determinant of a Hankel matrix of order n that involves large Schröder numbers. In this note we give a proof by means of Hankel determinants of the large and small Schröder numbers based on a bijection between the domino tilings of an Aztec diamond and non-intersecting lattice paths.

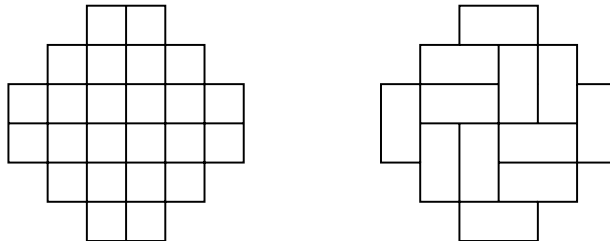


Figure 1: The AD_3 and a domino tiling

The *large Schröder numbers* $\{r_n\}_{n \geq 0} := \{1, 2, 6, 22, 90, 394, 1806, \dots\}$ and the *small Schröder numbers* $\{s_n\}_{n \geq 0} := \{1, 1, 3, 11, 45, 197, 903, \dots\}$ are registered in Sloane's On-Line Encyclopedia of Integer Sequences [7], namely A006318 and A001003, respectively. Among many other combinatorial structures, the n th large Schröder number r_n counts the number of lattice paths in the plane $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to $(2n, 0)$ using *up* steps $(1, 1)$, *down* steps $(1, -1)$, and *level* steps $(2, 0)$ that never pass below the x -axis. Such a path is called a *large Schröder path* of length n (or a *large n -Schröder path* for short). Let U , D , and L denote an up, down, and level step, respectively. Note that the terms of $\{r_n\}_{n \geq 1}$ are twice of those in $\{s_n\}_{n \geq 1}$. It turns out that the n th small Schröder number s_n counts the number of large n -Schröder paths without level steps on the x -axis, for $n \geq 1$. Such a path is called a *small n -Schröder path*. Refer to [8, Exercise 6.39] for more information.

Our proof relies on the determinants of the following *Hankel matrices* of the large and small Schröder numbers

$$H_n^{(1)} := \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ r_2 & r_3 & \cdots & r_{n+1} \\ \vdots & \vdots & & \vdots \\ r_n & r_{n+1} & \cdots & r_{2n-1} \end{bmatrix}, \quad G_n^{(1)} := \begin{bmatrix} s_1 & s_2 & \cdots & s_n \\ s_2 & s_3 & \cdots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n-1} \end{bmatrix}.$$

Making use of a method of Gessel and Viennot [5], we associate the determinants of $H_n^{(1)}$ and $G_n^{(1)}$ with the numbers of n -tuples of non-intersecting large and small Schröder paths, respectively. Note that $H_n^{(1)} = 2G_n^{(1)}$. This relation bridges the recurrence relation (2)

that leads to the result $\det(H_n^{(1)}) = 2^{n(n+1)/2}$ as well as the number of the required n -tuples of non-intersecting large Schröder paths (see Proposition 2.1). Our proof of the Aztec diamond theorem is completed by a bijection between domino tilings of an Aztec diamond and non-intersecting large Schröder paths (see Proposition 2.2).

We remark that Brualdi and Kirkland [2] use an algebraic method, relying on a J -fraction expansion of generating functions, to evaluate the determinant of a Hankel matrix of large Schröder numbers. Here we use a combinatorial approach that simplifies the evaluation of the Hankel determinants of large and small Schröder numbers significantly.

2 A proof of the Aztec diamond theorem

Let Π_n (resp. Ω_n) denote the set of n -tuples (π_1, \dots, π_n) of large Schröder paths (resp. small Schröder paths) satisfying the following two conditions.

(A1) Each path π_i goes from $(-2i + 1, 0)$ to $(2i - 1, 0)$, for $1 \leq i \leq n$.

(A2) Any two paths π_i and π_j do not intersect.

There is an immediate bijection ϕ between Π_{n-1} and Ω_n , for $n \geq 2$, which carries $(\pi_1, \dots, \pi_{n-1}) \in \Pi_{n-1}$ into $\phi((\pi_1, \dots, \pi_{n-1})) = (\omega_1, \dots, \omega_n) \in \Omega_n$, where $\omega_1 = \text{UD}$ and $\omega_i = \text{UU}\pi_{i-1}\text{DD}$ (i.e., ω_i is obtained from π_{i-1} with 2 up steps attached in the beginning and 2 down steps attached in the end, and then rises above the x -axis), for $2 \leq i \leq n$. For example, on the left of Figure 2 is a triple $(\pi_1, \pi_2, \pi_3) \in \Pi_3$. The corresponding quadruple $(\omega_1, \omega_2, \omega_3, \omega_4) \in \Omega_4$ is shown on the right. Hence, for $n \geq 2$, we have

$$|\Pi_{n-1}| = |\Omega_n|. \tag{1}$$

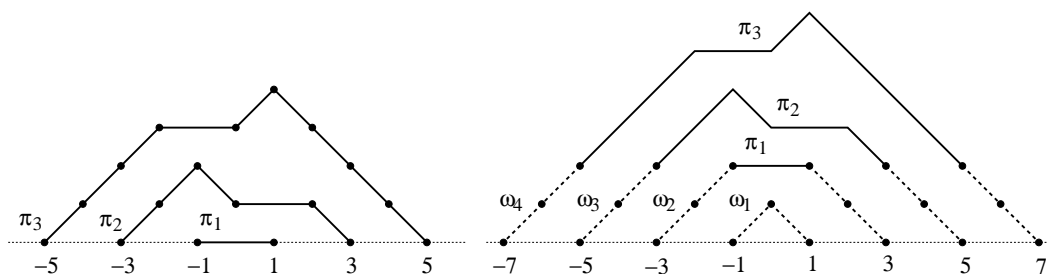


Figure 2: A triple $(\pi_1, \pi_2, \pi_3) \in \Pi_3$ and the corresponding quadruple $(\omega_1, \omega_2, \omega_3, \omega_4) \in \Omega_4$

For a permutation $\sigma = z_1 z_2 \cdots z_n$ of $\{1, \dots, n\}$, the *sign* of σ , denoted by $\text{sgn}(\sigma)$, is defined by $\text{sgn}(\sigma) := (-1)^{\text{inv}(\sigma)}$, where $\text{inv}(\sigma) := \text{Card}\{(z_i, z_j) \mid i < j \text{ and } z_i > z_j\}$ is the number of *inversions* of σ .

Using the technique of a sign-reversing involution over a signed set, we prove that the cardinalities of Π_n and Ω_n coincide with the determinants of $H_n^{(1)}$ and $G_n^{(1)}$, respectively. Following the same steps as [9, Theorem 5.1], a proof is given here for completeness.

Proposition 2.1 *For $n \geq 1$, we have*

(i) $|\Pi_n| = \det(H_n^{(1)}) = 2^{n(n+1)/2}$, and

(ii) $|\Omega_n| = \det(G_n^{(1)}) = 2^{n(n-1)/2}$.

Proof: For $1 \leq i \leq n$, let A_i denote the point $(-2i + 1, 0)$ and let B_i denote the point $(2i - 1, 0)$. Let h_{ij} denote the (i, j) -entry of $H_n^{(1)}$. Note that $h_{ij} = r_{i+j-1}$ is equal to the number of large Schröder paths from A_i to B_j . Let P be the set of ordered pairs $(\sigma, (\tau_1, \dots, \tau_n))$, where σ is a permutation of $\{1, \dots, n\}$, and (τ_1, \dots, τ_n) is an n -tuple of large Schröder paths such that τ_i goes from A_i to $B_{\sigma(i)}$. According to the sign of σ , the ordered pairs in P are partitioned into P^+ and P^- . Then

$$\det(H_n^{(1)}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n h_{i, \sigma(i)} = |P^+| - |P^-|.$$

We show that there exists a sign-reversing involution φ on P , in which case $\det(H_n^{(1)})$ is equal to the number of fixed points of φ . Let $(\sigma, (\tau_1, \dots, \tau_n)) \in P$ be such a pair that at least two paths of (τ_1, \dots, τ_n) intersect. Choose the first pair $i < j$ in lexicographical order such that τ_i intersects τ_j . Construct new paths τ'_i and τ'_j by switching the tails after the last point of intersection of τ_i and τ_j . Now τ'_i goes from A_i to $B_{\sigma(j)}$ and τ'_j goes from A_j to $B_{\sigma(i)}$. Since $\sigma \circ (ij)$ carries i into $\sigma(j)$, j into $\sigma(i)$, and k into $\sigma(k)$, for $k \neq i, j$, we define

$$\varphi((\sigma, (\tau_1, \dots, \tau_n))) = (\sigma \circ (ij), (\tau_1, \dots, \tau'_i, \dots, \tau'_j, \dots, \tau_n)).$$

Clearly, φ is sign-reversing. Since this first intersecting pair $i < j$ of paths is not affected by φ , φ is an involution. The fixed points of φ are the pairs $(\sigma, (\tau_1, \dots, \tau_n)) \in P$, where τ_1, \dots, τ_n do not intersect. It follows that τ_i goes from A_i to B_i , for $1 \leq i \leq n$ (i.e., σ is the identity) and $(\tau_1, \dots, \tau_n) \in \Pi_n$. Hence $\det(H_n^{(1)}) = |\Pi_n|$. By the same argument, we have $\det(G_n^{(1)}) = |\Omega_n|$. It follows from (1) and the relation $H_n^{(1)} = 2G_n^{(1)}$ that

$$|\Pi_n| = \det(H_n^{(1)}) = 2^n \cdot \det(G_n^{(1)}) = 2^n |\Omega_n| = 2^n |\Pi_{n-1}|. \tag{2}$$

Note that $|\Pi_1| = 2$, and hence, by induction, assertions (i) and (ii) follow. □

To prove the Aztec diamond theorem, we shall establish a bijection between Π_n and the set of domino tilings of AD_n based on an idea, due to D. Randall, mentioned in [8, Solution of Exercise 6.49].

Proposition 2.2 *There is a bijection between the set of domino tilings of the Aztec diamond of order n and the set of n -tuples (π_1, \dots, π_n) of large Schröder paths satisfying conditions (A1) and (A2).*

Proof: Given a tiling T of AD_n , we associate T with an n -tuple (τ_1, \dots, τ_n) of non-intersecting paths as follows. Let the rows of AD_n be indexed by $1, 2, \dots, 2n$ from bottom to top. For each i , ($1 \leq i \leq n$) we define a path τ_i from the center of the left-hand edge of the i th row to the center of the right-hand edge of the i th row. Namely, each step of the path is from the center of a domino edge (where a domino is regarded as having six edges of unit length) to the center of another edge of the some domino D , such that the step is symmetric with respect to the center of D . One can check that for each tiling there is a unique such an n -tuple (τ_1, \dots, τ_n) of paths, moreover, any two paths τ_i, τ_j of which do not intersect. Conversely, any such n -tuple of paths corresponds to a unique domino tiling of AD_n .

Let Λ_n denote the set of such n -tuples (τ_1, \dots, τ_n) of non-intersecting paths associated with domino tilings of AD_n . We shall establish a bijection ψ between the set of domino tilings of AD_n to Π_n with Λ_n as the intermediate stage. Given a tiling T of AD_n , let $(\tau_1, \dots, \tau_n) \in \Lambda_n$ be the n -tuple of paths associated with T . The mapping ψ is defined by carrying T into $\psi(T) = (\pi_1, \dots, \pi_n)$, where $\pi_i = \mathbf{U}_1 \cdots \mathbf{U}_{i-1} \tau_i \mathbf{D}_{i-1} \cdots \mathbf{D}_1$ (i.e., the large Schröder path π_i is obtained from τ_i with $i - 1$ up steps attached in the beginning of τ_i and with $i - 1$ down steps attached in the end, and then rises above the x -axis), for $1 \leq i \leq n$. One can verify that π_1, \dots, π_n satisfy conditions (A1) and (A2), and hence $\psi(T) \in \Pi_n$.

To find ψ^{-1} , given $(\pi_1, \dots, \pi_n) \in \Pi_n$, we can recover an n -tuple $(\tau_1, \dots, \tau_n) \in \Lambda_n$ of non-intersecting paths from (π_1, \dots, π_n) by a reverse procedure. Then we retrieve the required domino tiling $\psi^{-1}((\pi_1, \dots, \pi_n))$ of AD_n from (τ_1, \dots, τ_n) . \square

For example, on the left of Figure 3 is a tiling T of AD_3 and the associated triple (τ_1, τ_2, τ_3) of non-intersecting paths. On the right of Figure 3 is the corresponding triple $\psi(T) = (\pi_1, \pi_2, \pi_3) \in \Pi_3$ of large Schröder paths.

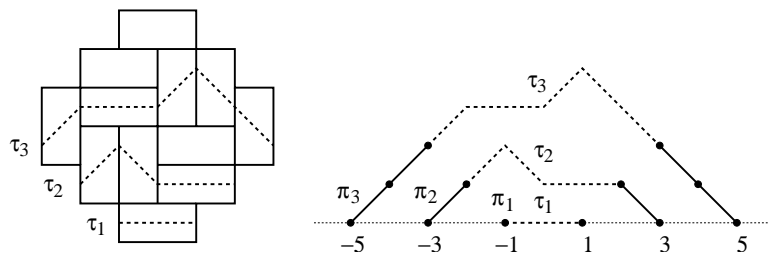


Figure 3: A tiling of AD_3 and the corresponding triple of non-intersecting Schröder paths

By Propositions 2.1 and 2.2, we deduce the Aztec diamond theorem anew.

Theorem 2.3 (Aztec diamond theorem) *The number of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$.*

Remark: The proof of Proposition 2.1 relies on the recurrence relation $\Pi_n = 2^n \Pi_{n-1}$ essentially, which is derived by means of the determinants of the Hankel matrices $H_n^{(1)}$ and $G_n^{(1)}$. We are interested to hear a purely combinatorial proof of this recurrence relation.

In a similar manner we derive the determinants of the Hankel matrices of large and small Schröder paths of the forms

$$H_n^{(0)} := \begin{bmatrix} r_0 & r_1 & \cdots & r_{n-1} \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & & \vdots \\ r_{n-1} & r_n & \cdots & r_{2n-2} \end{bmatrix}, \quad G_n^{(0)} := \begin{bmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{bmatrix}.$$

Let Π_n^* (resp. Ω_n^*) be the set of n -tuples $(\mu_0, \mu_1, \dots, \mu_{n-1})$ of large Schröder paths (resp. small Schröder paths) satisfying the following two conditions.

- (B1) Each path μ_i goes from $(-2i, 0)$ to $(2i, 0)$, for $0 \leq i \leq n-1$.
- (B2) Any two paths μ_i and μ_j do not intersect.

Note that μ_0 degenerates into a single point and that Π_n^* and Ω_n^* are identical since for any $(\mu_0, \mu_1, \dots, \mu_{n-1}) \in \Pi_n^*$ all of the paths μ_i have no level steps on the x -axis. Moreover, for $n \geq 2$, there is a bijection ρ between Π_{n-1} and Π_n^* that carries $(\pi_1, \dots, \pi_{n-1}) \in \Pi_{n-1}$ into $\rho((\pi_1, \dots, \pi_{n-1})) = (\mu_0, \mu_1, \dots, \mu_{n-1}) \in \Pi_n^*$, where μ_0 is the origin and $\mu_i = \cup \pi_i \mathbf{D}$, for $1 \leq i \leq n-1$. Hence, for $n \geq 2$, we have

$$|\Pi_n^*| = |\Pi_{n-1}|. \tag{3}$$

For example, on the left of Figure 4 is a triple $(\pi_1, \pi_2, \pi_3) \in \Pi_3$ of non-intersecting large Schröder paths. The corresponding quadruple $(\mu_0, \mu_1, \mu_2, \mu_3) \in \Pi_4^*$ is shown on the right.

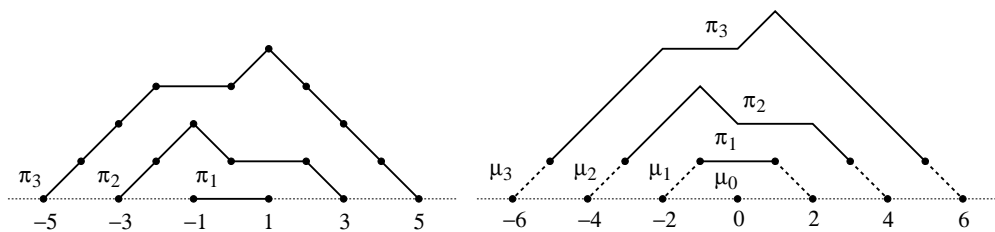


Figure 4: A triple $(\pi_1, \pi_2, \pi_3) \in \Pi_3$ and the corresponding quadruple $(\mu_0, \mu_1, \mu_2, \mu_3) \in \Pi_4^*$

By a similar argument to that of Proposition 2.1, we have $\det(H_n^{(0)}) = |\Pi_n^*| = |\Omega_n^*| = \det(G_n^{(0)})$. Hence, by (3) and Proposition 2.1(i), we have the following result.

Proposition 2.4 For $n \geq 1$, $\det(H_n^{(0)}) = \det(G_n^{(0)}) = 2^{n(n-1)/2}$.

Hankel matrices $H_n^{(0)}$ and $H_n^{(1)}$ may be associated with any given sequence of real numbers. As noted by Aigner in [1, Section 1(D)] that the sequence of determinants

$$\det(H_1^{(0)}), \det(H_1^{(1)}), \det(H_2^{(0)}), \det(H_2^{(1)}), \dots$$

uniquely determines the original number sequence provided that $\det(H_n^{(0)}) \neq 0$ and $\det(H_n^{(1)}) \neq 0$, for all $n \geq 1$, we have a characterization of large and small Schröder numbers.

Corollary 2.5 *The following results hold.*

- (i) *The large Schröder numbers $\{r_n\}_{n \geq 0}$ are the unique sequence with the Hankel determinants $\det(H_n^{(0)}) = 2^{n(n-1)/2}$ and $\det(H_n^{(1)}) = 2^{n(n+1)/2}$, for all $n \geq 1$.*
- (ii) *The small Schröder numbers $\{s_n\}_{n \geq 0}$ are the unique sequence with the Hankel determinants $\det(G_n^{(0)}) = \det(G_n^{(1)}) = 2^{n(n-1)/2}$, for all $n \geq 1$.*

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