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A Simple Treatment of Constraint Forces and Constraint Moments in the Dynamics of Rigid Bodies

Oliver M. O'Reilly · Arun R. Srinivasa

November 6, 2014

Abstract In this expository article, a simple concise treatment of Lagrange's prescription for constraint forces and constraint moments in the dynamics of rigid bodies is presented. The treatment is suited to both Newton-Euler and Lagrangian treatments of rigid body dynamics and is illuminated with a range of examples from classical mechanics and orthopedic biomechanics.

Keywords Lagrange's equations · Constraints · Constraints forces · Constraint moments

1 Introduction

Consider formulating the equations of motion for the classical problem of a thin circular disk shown in Figure 1 that is sliding with a point X_P in contact with a smooth horizontal surface. One approach to formulating the equations of motion is to use Lagrange's equations. To proceed, one picks a coordinate system (denoted by $q^1, \ldots, q^6$) to describe the rotation and translation of the disk. An astute choice is to select the Cartesian coordinates x and y of the center of mass, a set of 3-1-3 Euler angles ψ, θ, and ϕ, and the vertical coordinate z_P of the point of contact X_P. The coordinate z_P is related to the vertical coordinate z of the center of mass by the simple relation

$$q^6 = z_P = z - R\sin(\theta). \tag{1}$$

After choosing the coordinate system, we can calculate the kinetic T and potential U energies of the disk. For the disk in motion on the plane, the coordinate q^6 is constrained to be zero. As a result, the equations of motion have a Lagrange multiplier λ on the right hand side:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\Gamma}\right) - \frac{\partial T}{\partial q^\Gamma} + \frac{\partial U}{\partial q^\Gamma} = \Phi_\Gamma, \qquad (\Gamma = 1, \ldots, 6), \tag{2}$$

where the generalized constraint force is

$$\Phi_\Gamma = \lambda \delta^6_\Gamma, \tag{3}$$

and δ^6_Γ is the Kronecker delta.

The equations of motion (2) supplemented with (3) and the constraint $q^6 = 0$ provide 5 differential equations for $q^1, \ldots q^5$ and an equation for λ. The latter equation is

$$\frac{d}{dt}\left(mR\dot{\theta}\cos(\theta)\right) + mg = \lambda. \tag{4}$$

It is important to note that λ depends on the motion of the disk and is determined by the equations of motion. From a historical perspective, the introduction of the multiplier λ dates to Lagrange's first treatment of his equations of motion for holonomically constrained rigid bodies in the 1780s (cf. [15]).

In the centuries following Lagrange's work, his implementation of the Lagrange multiplier to accommodate constraints was extended to include constraints of

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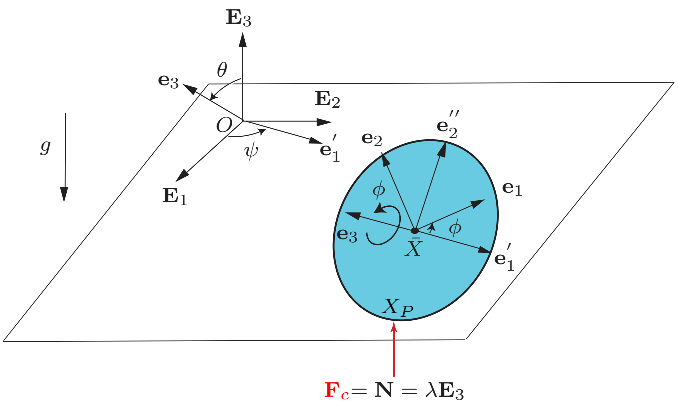

Fig. 1: *A circular disk of radius R moving on a horizontal plane. The position vector of the instantaneous point of contact X_P of the disk and the plane relative to the center of mass $\bar{X}$ of the disk is always along $\mathbf{e}_2''$. A set of 3-1-3 Euler angles are used to parameterize the rotation of the disk.*

the form

$$\sum_{\Gamma=1}^{6} \mathsf{A}_\Gamma \dot{q}^\Gamma + e_C = 0. \quad (5)$$

Here, the seven functions A_Γ and e_C depend on $q^1, \ldots, q^6$ and t. The prescription for the generalized constraint forces, which are denoted by $\Phi_1, \ldots, \Phi_6$, associated with (5) is (see, e.g., [1,28,32]):

$$\Phi_1 = \lambda \mathsf{A}_1, \ldots, \Phi_6 = \lambda \mathsf{A}_6. \quad (6)$$

The (determinate system of) equations of motion for the system are then (2) supplemented by (5) where the generalized constraint forces are prescribed by (6). It should be clear that (6) subsumes the case of the sliding disk above because the constraint $q^6 = 0$ can be written as $\dot{q}^6 = 0$ and (6) then leads to $\Phi_1 = \ldots = \Phi_5 = 0$ and $\Phi_6 = \lambda$.

From a pedagogical point of view, it is natural to ask if a physical interpretation can be given to λ in (6)? The answer to this question is yes. For instance, for the sliding disk considered previously, $\lambda \mathbf{E}_3$ is the normal force $\mathbf{N}$ acting at the point X_P. Indeed with this insight it is easy to see that (4) is simply a balance of the vertical inertia of the disk, gravity and the normal force acting at the instantaneous point of contact X_P. However, a physical interpretation of (6) is not always obvious and is challenging for many instructors to explain. In addition, because of the predominant use of Lagrange's equations of motion and its progeny[1] in establishing equations of motion for constrained mechanical systems, little emphasis is placed in textbooks on physical interpretations of the generalized constraint forces. Consequently, the benefits of establishing the equations of motion using the methods of analytical mechanics often come at the expense of a lack of detail and transparency on the physical nature of the constraint forces and constraint moments acting on the rigid body.[2]

The purpose of the present paper is to provide a treatment of generalized constraint forces that readily establishes their connection to the constraint forces $\mathbf{F}_c$ and constraints moments $\mathbf{M}_c$ acting on a rigid body. While $\mathbf{F}_c$ and $\mathbf{M}_c$ ensure that the rigid body's motion satisfies the constraints, they also constitute an additional set of unknowns that must be solved along with the motion of the rigid body. The treatment is a development of our earlier works [4,21,22,26] and uses tools that are employed to represent conservative moments, moments in anatomical joints, and constraints on the rotational motion of a rigid body.

After reviewing some background and notation, we start with a series of examples of constrained rigid bodies and examine the nature of the constraint forces and constraint moments acting on them. The series of examples motivates a prescription for $\mathbf{F}_c$ and $\mathbf{M}_c$. We then show how this prescription is identical to the prescription featuring Lagrange multipliers that dominates analytical mechanics. The paper closes with a discussion of systems of rigid bodies and suggested avenues for students to explore.

2 Background and Notation

The motion of a material point X on a rigid body can be conveniently described by decomposing the motion into the translation of the center of mass $\bar{X}$ and the rotation of the body about the center of mass:

$$\mathbf{x} = \mathbf{Q}\left(\mathbf{X} - \bar{\mathbf{X}}\right) + \bar{\mathbf{x}}. \quad (7)$$

In this equation, $\mathbf{Q} = \mathbf{Q}(t)$ is the rotation tensor of the rigid body, and $\mathbf{x}$ is the position vector of X and $\bar{\mathbf{x}}$ is the position vector of the center of mass in the present configuration of the rigid body at time t. The vectors $\mathbf{X}$ and $\bar{\mathbf{X}}$ are the respective position vectors of X and $\bar{X}$ in a fixed reference configuration of the rigid body. In many problems, the reference configuration is chosen

[1] For example, the Boltzmann-Hamel equations, the Gibbs-Appell equations or Kane's equations [1,13,27,33].

[2] Notable discussions on constraint forces in systems of particles in analytical mechanics include Gantmacher [9] and Planck [29], however, apart from [12,22,30], discussions of constraint (or reaction) moments are notably absent from textbooks.

to be the present configuration at time $t = t_0$ and, in this case $\mathbf{Q}(t_0)$ is the identity tensor.

The equation (7) can be differentiated to yield an equation relating the velocities of any pair of material points X_A and X_C, say, on a rigid body:

$$\mathbf{v}_A - \mathbf{v}_C = \boldsymbol{\omega} \times (\mathbf{x}_A - \mathbf{x}_C), \tag{8}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body. This vector is the axial vector of $\dot{\mathbf{Q}}\mathbf{Q}^T$: $\boldsymbol{\omega} \times \mathbf{a} = \dot{\mathbf{Q}}\mathbf{Q}^T\mathbf{a}$ for any vector $\mathbf{a}$.

As can be seen from Shuster's authoritative review [31], the rotation tensor $\mathbf{Q}$ has several representations. One of the most useful representations is found by defining a fixed Cartesian basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and a corotational (body-fixed) basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ such that $\mathbf{e}_i(t) = \mathbf{Q}(t)\mathbf{E}_i$. For several examples in this paper we will also use a set of 3-1-3 Euler angles (ψ, θ, ϕ) to parameterize $\mathbf{Q}$ (see Figure 1). The vector $\boldsymbol{\omega}$ then has the representation

$$\boldsymbol{\omega} = \dot{\psi}\mathbf{g}_1 + \dot{\theta}\mathbf{g}_2 + \dot{\phi}\mathbf{g}_3, \tag{9}$$

where the three unit vectors

$$\begin{aligned}\mathbf{g}_1 &= \mathbf{E}_3, \\ \mathbf{g}_2 &= \mathbf{e}_1^{'} = \cos(\psi)\mathbf{E}_1 + \sin(\psi)\mathbf{E}_2, \\ \mathbf{g}_3 &= \mathbf{e}_3.\end{aligned} \tag{10}$$

constitute a basis which is known as the Euler basis (cf. Figure 1).

While the Euler basis vectors are readily related to the axes of rotation that are used to construct the Euler angle representation for $\mathbf{Q}$, they form a non-orthogonal basis. In particular, $\mathbf{g}_1 \cdot \mathbf{g}_3 = \cos(\theta)$. As a result, $\boldsymbol{\omega} \cdot \mathbf{g}_3 = \dot{\psi}\cos(\theta) + \dot{\phi}$. In rigid body dynamics, it is useful to have a basis that we can readily use to isolate the speeds $\dot{\psi}$, $\dot{\theta}$, and $\dot{\phi}$ by taking the inner product of $\boldsymbol{\omega}$ with these basis vectors. Such a basis is known as the dual Euler basis [22,26]. Given the 3-1-3 Euler basis vectors, it is straightforward to compute the corresponding dual Euler basis vectors $\mathbf{g}^1$, $\mathbf{g}^2$ and $\mathbf{g}^3$:

$$\begin{aligned}\mathbf{g}^1 &= \frac{1}{g}(\mathbf{g}_2 \times \mathbf{g}_3) \\ &= \operatorname{cosec}(\theta)\cos(\phi)\mathbf{e}_2 + \operatorname{cosec}(\theta)\sin(\phi)\mathbf{e}_1, \\ \mathbf{g}^2 &= \mathbf{g}_2 = \cos(\phi)\mathbf{e}_1 - \sin(\phi)\mathbf{e}_2, \\ \mathbf{g}^3 &= \frac{1}{g}(\mathbf{g}_1 \times \mathbf{g}_2) \\ &= \mathbf{e}_3 - \cot(\theta)(\cos(\phi)\mathbf{e}_2 + \sin(\phi)\mathbf{e}_1),\end{aligned} \tag{11}$$

where $g = (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 = -\sin(\theta)$. The dual Euler basis is not defined when $\sin(\theta) = 0$. For this pair $(\theta = 0, \pi)$ of singular values of θ, the Euler basis fails to be a basis. We leave it as an exercise to verify that

$$\boldsymbol{\omega} \cdot \mathbf{g}^1 = \dot{\psi}, \qquad \boldsymbol{\omega} \cdot \mathbf{g}^2 = \dot{\theta}, \qquad \boldsymbol{\omega} \cdot \mathbf{g}^3 = \dot{\phi}. \tag{12}$$

As a result of these identities, the dual Euler basis vectors find applications with constraint moments, conservative moments and for moments in a joint coordinate system [6,21,24,25].

The equations of motion of a rigid body of mass m can be determined from the balances of linear and angular momentum:

$$\mathbf{F} = m\dot{\bar{\mathbf{v}}}, \qquad \mathbf{M} = \dot{\mathbf{H}}. \tag{13}$$

Here, $\mathbf{H}$ is the angular momentum of the rigid body relative to its center of mass $\bar{X}$, $\mathbf{F}$ is the resultant force acting on the rigid body, and $\mathbf{M}$ is the resultant moment relative to $\bar{X}$ acting on the rigid body. When the body is subject to N constraints, the balance laws are supplemented by the N constraints and N prescriptions for the constraint forces and constraint moments. As emphasized in a marvelous discussion by Planck [29, Chapter VI], the resulting system of $6 + N$ equations should form a determinate system *both* for the 6 unknowns $\bar{\mathbf{x}}$ and $\mathbf{Q}$ and the constraint forces and constraint moments.

3 Illustrative Examples

To set the stage for our discussion of constraint forces and moments, we examine a range of problems from rigid body dynamics. We start with a familiar example from an undergraduate dynamics course, and then turn to examples of rolling and sliding rigid bodies and a more complex example of a whirling pendulum.

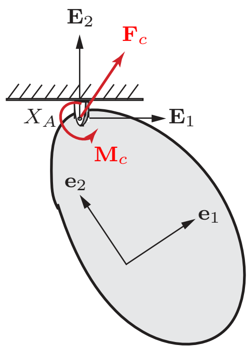

Fig. 2: *A rigid body rotating in a plane about a fixed point X_A. The constraint force $\mathbf{F}_c$ and constraint moment $\mathbf{M}_c$ which ensure that the point X_A remains fixed and the axis of rotation is constrained to be $\mathbf{E}_3$ by a revolute joint at X_A, respectively, are also shown.*

3.1 Planar Motion of a Body About a Fixed Point

For our first example, we consider a body which is connected by a revolute joint to the ground at a fixed material point X_A (cf. Figure 2). The body is free to perform planar rotations about an axis of rotation $\mathbf{E}_3$. However, because X_A is fixed and the body cannot rotate about the planar directions $\mathbf{E}_1$ and $\mathbf{E}_2$, its motion is subject to five constraints. The most convenient method to express these constraints uses the velocity vector of the point A and the angular velocity vector of the rigid body:

$$\mathbf{E}_1 \cdot \mathbf{v}_A = 0, \qquad \mathbf{E}_2 \cdot \mathbf{v}_A = 0, \qquad \mathbf{E}_3 \cdot \mathbf{v}_A = 0,$$

$$\mathbf{E}_1 \cdot \boldsymbol{\omega} = 0, \qquad \mathbf{E}_2 \cdot \boldsymbol{\omega} = 0. \tag{14}$$

To make sure that X_A stays fixed, a reaction force $\mathbf{R}_A$ acts at this point. This reaction force has three independent components: each one restricting the translation of X_A in a given direction. The body also cannot rotate about any axis other than $\mathbf{E}_3$. Hence, the joint needs to provides a reaction moment that prevents rotation in the $\mathbf{E}_1$ and $\mathbf{E}_2$ directions.

We refer to $\mathbf{R}_A$ as a constraint force, $\mathbf{F}_c = \mathbf{R}_A$, and the reaction moment provided by the joint as a constraint moment $\mathbf{M}_c$. In addition to the reaction force's three independent components, the reaction moment should have two independent components which model the resistance of the revolute joint to rotation in the $\mathbf{E}_1 - \mathbf{E}_2$ plane. In conclusion, we prescribe

$$\mathbf{F}_c = \sum_{k=1}^{3} \lambda_k \mathbf{E}_k \text{ acting at the point } X_A,$$

$$\mathbf{M}_c = \lambda_4 \mathbf{E}_1 + \lambda_5 \mathbf{E}_2. \tag{15}$$

Here, the components $\lambda_1, \ldots, \lambda_5$ are found from the balances of linear and angular momentum. For example, in an undergraduate dynamics course it is common to use $\mathbf{F} = m\mathbf{a}$ to determine the joint reaction force $\mathbf{F}_c$ (i.e., λ_1, λ_2, and λ_3). The $\mathbf{E}_3$ component of the balance of angular momentum relative to the point X_A, $\mathbf{M}_A = \dot{\mathbf{H}}_A$, is also used to determine the differential equation governing the motion of the rigid body. Less standard in textbooks is to use the $\mathbf{E}_1$ and $\mathbf{E}_2$ components of $\mathbf{M}_A = \dot{\mathbf{H}}_A$ to determine λ_4 and λ_5.[3]

It is useful at this stage in our discussion to point out that the prescription (15) assumes that the revolute joint is free of Coulomb friction. The reader might also note the similarities between the constraints (14) and the expressions (15) for $\mathbf{F}_c$ and $\mathbf{M}_c$. For example, observe the fact that there are 5 constraints and 5 quantities $\lambda_1, \ldots, \lambda_5$. This feature will appear in all the forthcoming examples.

[3] An example of such a calculation can be found in [23].

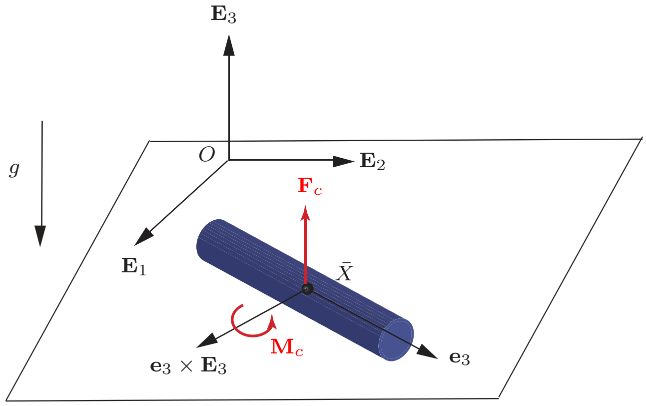

Fig. 3: *A cylinder sliding on a smooth horizontal surface. The constraint force* $\mathbf{F}_c$ *and constraint moment* $\mathbf{M}_c$ *which ensure that the center of mass* $\bar{X}$ *moves in the horizontal plane and the cylinder does not rotate into the plane, respectively, are also shown.*

3.2 Sliding Cylinder

As a second example, consider a cylinder sliding on a smooth horizontal surface shown in Figure 3. The normal to the surface is $\mathbf{E}_3$ and the axis of symmetry of the cylinder is $\mathbf{e}_3$. Clearly, the center of mass $\bar{X}$ of the cylinder can only move horizontally and the cylinder cannot rotate into the plane. Following [21,22], these two constraints can be expressed as

$$\mathbf{E}_3 \cdot \bar{\mathbf{v}} = 0, \qquad (\mathbf{e}_3 \times \mathbf{E}_3) \cdot \boldsymbol{\omega} = 0. \tag{16}$$

In the second constraint, the vector $\mathbf{e}_3 \times \mathbf{E}_3$ has the property that it is always normal to the axis of symmetry of the cylinder and lies in the horizontal plane.

Along the line of contact of the cylinder and the surface a normal force field will be present. This force field ensures that the motion of the cylinder satisfies (16) and appears in the equations of motion of the cylinder as a resultant force (or normal force) and a resultant moment. The resultant force will be in the $\mathbf{E}_3$ direction and act at the center of mass and the moment will be normal to the contact line and the axis of the cylinder. We summarize these observations, by noting that the resultant force is a constraint force $\mathbf{F}_c$ acting at $\bar{X}$ and the moment is a constraint moment $\mathbf{M}_c$ where

$$\mathbf{F}_c = \lambda_1 \mathbf{E}_3 \text{ acting at the point } \bar{X},$$

$$\mathbf{M}_c = \lambda_2 (\mathbf{e}_3 \times \mathbf{E}_3). \tag{17}$$

The quantities λ_1 and λ_2 must be determined from the balance laws.

We can use the balance laws or, equivalently, Lagrange's equations of motion, to solve for the motion of the cylinder and the unknowns λ_1 and λ_2. Paralleling the methods in [21,22], we use a set of 3-1-3 Euler angles to parameterize $\mathbf{Q}$ and a set of Cartesian coordinates to parameterize the motion of the center of mass. The integrable constraints on the rigid body can be expressed as $\theta = \frac{\pi}{2}$ and $\bar{\mathbf{x}} \cdot \mathbf{E}_3 = 0$. Leaving the establishment of the results as an exercise, we would find

$$\mathbf{F}_c = mg\mathbf{E}_3,$$
$$\mathbf{M}_c = I_a\dot{\psi}\dot{\phi}\left(\cos(\phi)\mathbf{e}_1 - \sin(\phi)\mathbf{e}_2\right), \tag{18}$$

where I_a is the moment of inertia of the cylinder about the $\mathbf{e}_3$ axis.

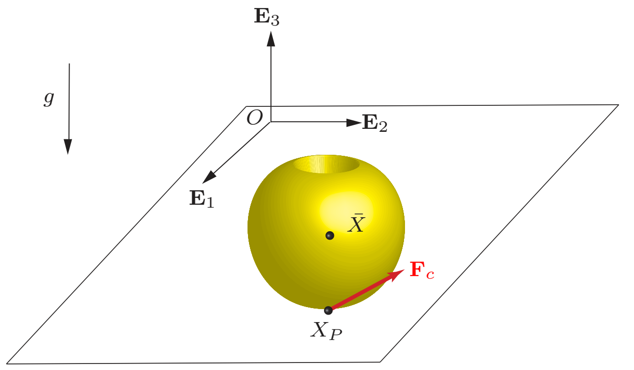

Fig. 4: *A rigid body rolling without slipping on a rough horizontal plane. The constraint force* $\mathbf{F}_c$ *acting at the instantaneous point of contact* X_P *(where* $\mathbf{v}_P = \mathbf{0}$*) can be decomposed into a normal force* $\mathbf{N}$ *and friction force* $\mathbf{F}_f$*:* $\mathbf{F}_c = \mathbf{N} + \mathbf{F}_f$*.*

3.3 Rolling and Sliding Rigid Bodies

As a third example, consider a rigid body in motion with a single instantaneous point X_P of contact with a surface (see, e.g., Figure 4). If the body is sliding, then there is a single constraint:

$$\mathbf{n} \cdot \mathbf{v}_P = f(t), \tag{19}$$

where $\mathbf{n}$ is the normal vector to the surface at X_P and $f(t)$ is a prescribed function of time which vanishes if the surface is fixed. The vector $\mathbf{n}$ depends on the location of the rigid body and time (if the surface is moving). For example for the sliding disk shown in Figure 1, $\mathbf{n} = \mathbf{E}_3$ and $f = 0$. At the point of contact, a normal force $\mathbf{N}$ exerts a force on the sliding body. This normal force is a constraint force and so we prescribe

$$\mathbf{F}_c = \mathbf{N} = \lambda\mathbf{n} \text{ acting at the point } X_P, \qquad \mathbf{M}_c = \mathbf{0}. \tag{20}$$

Note that we are assuming that the constraint at X_P is enforced purely by a force acting at that point. If we assumed dynamic friction were also present at the point, then a friction force would be added to $\mathbf{F}_c$ and a frictional moment would be added to $\mathbf{M}_c$. For an example of this instance see the discussions of the dynamics of Euler's disk in [14,16,17].

For a rigid body rolling on the surface mentioned previously, one has three constraints:

$$\mathbf{E}_k \cdot \mathbf{v}_P - \mathbf{v}_s \cdot \mathbf{E}_k = 0, \qquad (k = 1, 2, 3). \tag{21}$$

Here, $\mathbf{v}_s$ is the velocity vector of the point on the surface that is coincident with X_P. Rolling contact is maintained by static Coulomb friction $\mathbf{F}_f$ while a normal force $\mathbf{N}$ ensures that the rolling body doesn't penetrate the surface. The resultant of these forces is the constraint force acting on the rigid body:

$$\mathbf{F}_c = \mathbf{N} + \mathbf{F}_f = \sum_{k=1}^{3} \lambda_k \mathbf{E}_k \text{ acting at the point } X_P. \tag{22}$$

The three unknowns λ_1, λ_2, and λ_3 reflect the fact that normal force has one unknown component and the friction force has two independent components. It may be helpful to recall that part of the solution procedure when solving for the motion of a rolling rigid body involves solving the unknowns $\mathbf{F}_f$ and $\mathbf{N}$.

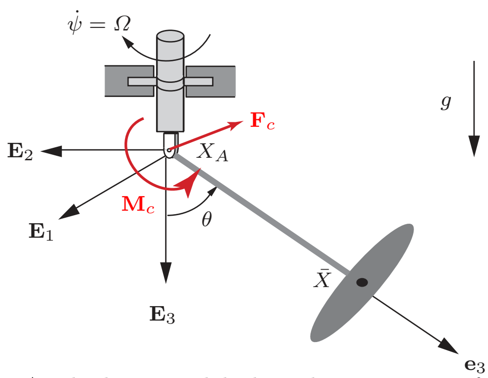

Fig. 5: *A whirling rigid body. The constraint force* $\mathbf{F}_c$ *and constraint moment* $\mathbf{M}_c$ *are also shown.*

3.4 A Whirling Rigid Body

Consider the rigid body in Figure 5 which is attached to a spinning shaft by a revolute joint at X_A. The shaft is subject to a prescribed angular speed $\Omega(t)$ and the rigid body is subject to 5 constraints:

$$\mathbf{E}_k \cdot \mathbf{v}_A = 0, \qquad (k = 1, 2, 3),$$

$$\mathbf{g}^1 \cdot \boldsymbol{\omega} - \Omega = 0, \qquad \mathbf{g}^3 \cdot \boldsymbol{\omega} = 0, \tag{23}$$

where we use a set of 3-1-3 Euler angles (ψ, θ, ϕ) to parameterize the rotation of the rigid body. Note the use of the dual Euler basis to express the constraints in compact form. Expressions for these basis vectors are recorded in (11).

To make sure that X_A stays fixed, a reaction force $\mathbf{R}_A$ acts at this point. As in our first example, this reaction force has three independent components: each one restricting the translation of X_A in a given direction. The body is only free to rotate in the $\mathbf{g}_2$ direction and so we need constraint moments in directions perpendicular to $\mathbf{g}_2$ to ensure this freedom is preserved. Because $\mathbf{g}^1 \cdot \mathbf{g}_2 = 0$ and $\mathbf{g}^3 \cdot \mathbf{g}_2 = 0$, we thus prescribe

$$\mathbf{F}_c = \sum_{k=1}^{3} \lambda_k \mathbf{E}_k, \text{ acting at the point } X_A,$$
$$\mathbf{M}_c = \lambda_4 \mathbf{g}^1 + \lambda_5 \mathbf{g}^3. \tag{24}$$

The constraint moment $\lambda_4 \mathbf{g}^1$ can be interpreted as the motor torque needed to drive the shaft at its prescribed speed. The power required to achieve this is given by $\lambda_4 \mathbf{g}^1 \cdot \boldsymbol{\omega} = \lambda_4 \Omega$. We also note that $\mathbf{g}^1$ is never parallel to $\mathbf{g}_1$.

4 Constraints and Lagrange's Prescription

Traditionally prescriptions for $\mathbf{F}_c$ and $\mathbf{M}_c$ are either couched in terms of physical arguments as discussed in the four examples discussed in the previous section or feature generalized constraint forces and a virtual work assumption (as in the sliding disk example discussed in the introduction). An alternative formulation of Lagrange's prescription can be obtained by combining features of both prescriptions as follows.

Suppose that a constraint $\pi_C = 0$ on the motion of a rigid body can be expressed in the form

$$\pi_C = \mathbf{f}_C \cdot \mathbf{v}_C + \mathbf{h}_C \cdot \boldsymbol{\omega} + e_C, \tag{25}$$

where $\mathbf{v}_C$ is the velocity vector of a material point X_C on the body, and the functions $\mathbf{f}_C$, $\mathbf{h}_C$, and e_C depend on $\mathbf{Q}$, $\bar{\mathbf{x}}$ and t. Then, we define Lagrange's prescription for $\mathbf{F}_c$ and $\mathbf{M}_c$ as

$$\mathbf{F}_c = \lambda \frac{\partial \pi_C}{\partial \mathbf{v}_C} = \lambda \mathbf{f}_C \text{ acting at the point } X_C,$$
$$\mathbf{M}_c = \lambda \frac{\partial \pi_C}{\partial \boldsymbol{\omega}} = \lambda \mathbf{h}_C, \tag{26}$$

where λ is a function which is determined by the equations of motion.

The prescription (26) can be generalized in an obvious manner to systems of constraints and in Section 5 we will show how the prescription (26) is equivalent to traditional prescriptions for generalized constraint forces. In anticipation of this equivalence result we used λ in (26). As evidenced from the examples discussed in Section 3, for each constraint we have a single λ. This ensures that the system of equations governing the motion of the body are sufficient to determine both the constrained motion of the body and the constraint force $\mathbf{F}_c$ and constraint moment $\mathbf{M}_c$.

Table 1: *Summary of the constraints and constraint forces* $\mathbf{F}_c$ *and constraint moments* $\mathbf{M}_c$ *for the examples from Section 3.*

	Constraints	Constraint Forces Constraint Moments
Pinjointed Rigid Body	$\mathbf{E}_k \cdot \mathbf{v}_A = 0$ $\mathbf{E}_1 \cdot \boldsymbol{\omega} = 0$ $\mathbf{E}_2 \cdot \boldsymbol{\omega} = 0$	$\mathbf{F}_c = \sum_{k=1}^{3} \lambda_k \mathbf{E}_k$ $\mathbf{M}_c = \lambda_4 \mathbf{E}_1 + \lambda_5 \mathbf{E}_2$ $\mathbf{F}_c$ acts at X_A
Sliding Rigid Body	$\mathbf{n} \cdot \mathbf{v}_P = 0$	$\mathbf{F}_c = \lambda \mathbf{n}$ $\mathbf{M}_c = \mathbf{0}$ $\mathbf{F}_c$ acts at X_P
Rolling Rigid Body	$\mathbf{E}_k \cdot \mathbf{v}_P = \mathbf{v}_s \cdot \mathbf{E}_k$	$\mathbf{F}_c = \sum_{k=1}^{3} \lambda_k \mathbf{E}_k$ $\mathbf{M}_c = \mathbf{0}$ $\mathbf{F}_c$ acts at X_P
Sliding Cylinder	$\mathbf{E}_3 \cdot \bar{\mathbf{v}} = 0$ $(\mathbf{e}_3 \times \mathbf{E}_3) \cdot \boldsymbol{\omega} = 0$	$\mathbf{F}_c = \lambda_1 \mathbf{E}_3$ $\mathbf{M}_c = \lambda_2 \mathbf{e}_3 \times \mathbf{E}_3$ $\mathbf{F}_c$ acts at $\bar{X}$
Whirling Rigid Body	$\mathbf{E}_k \cdot \mathbf{v}_A = 0$ $\mathbf{g}^1 \cdot \boldsymbol{\omega} = \Omega$ $\mathbf{g}^3 \cdot \boldsymbol{\omega} = 0$	$\mathbf{F}_c = \sum_{k=1}^{3} \lambda_k \mathbf{E}_k$ $\mathbf{M}_c = \lambda_4 \mathbf{g}^1 + \lambda_5 \mathbf{g}^3$ $\mathbf{F}_c$ acts at X_A

4.1 A Review of the Examples

To see if the prescription (26) makes physical sense, we return to the examples discussed in Section 3. The constraints and the constraint forces and constraint moments for the examples are summarized in Table 1. It is straightforward to conclude from the results summarized in this table that the prescriptions for $\mathbf{F}_c$ and $\mathbf{M}_c$ are completely compatible with Lagrange's prescription (26).

4.2 Power of the Constraint Force and Constraint Moment

The combined mechanical power $\mathcal{P}$ of $\mathbf{F}_c$ and $\mathbf{M}_c$ can be computed:

$$\begin{aligned} \mathcal{P} &= \mathbf{F}_c \cdot \mathbf{v}_C + \mathbf{M}_c \cdot \boldsymbol{\omega} \\ &= \lambda \left(\mathbf{f}_C \cdot \mathbf{v}_C + \mathbf{h}_C \cdot \boldsymbol{\omega} \right) \\ &= -\lambda e_C. \end{aligned} \tag{27}$$

Hence, if $e_C \neq 0$, we anticipate that $\mathcal{P}$ will be nonzero. In this case, the combined effects of $\mathbf{F}_c$ and $\mathbf{M}_c$

will produce work and change the total energy of the rigid body. An example of this instance occurs in the whirling rigid body discussed in Section 3.4.

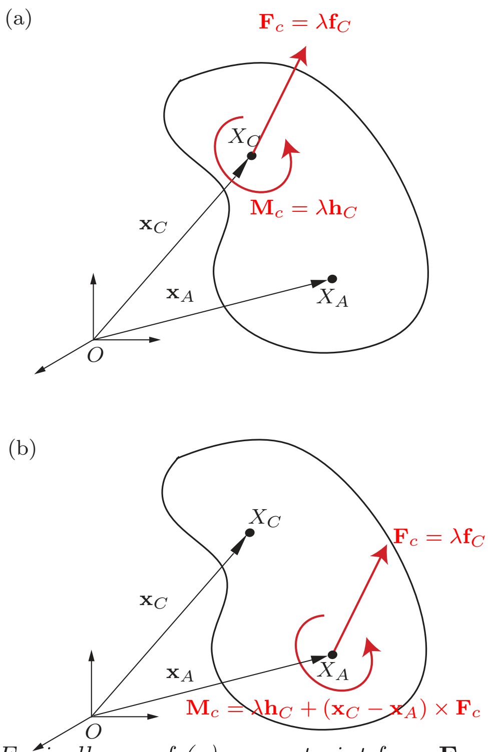

Fig. 6: *Equipollence of (a) a constraint force* $\mathbf{F}_c$ *acting at* X_C *and a constraint moment* $\mathbf{M}_c = \lambda \mathbf{h}_C$ *and (b) the same force acting at* X_A *and a different constraint moment* $\mathbf{M}_c = \lambda \mathbf{h}_C + (\mathbf{x}_C - \mathbf{x}_A) \times \mathbf{F}_c$.

4.3 Choosing a Different Material Point

Consider the sliding disk shown in Figure 1 and discussed in Sections 1 and 3.3. Earlier, we wrote the constraint that the disk is sliding as $\mathbf{v}_P \cdot \mathbf{E}_3 = 0$. Invoking Lagrange's prescription, we can then prescribe $\mathbf{F}_c = \lambda \mathbf{E}_3$ acting at X_P and $\mathbf{M}_c = \mathbf{0}$. It is natural to ask what would happen if we wrote the constraint using a different material point. For example, suppose we wrote the constraint using the center of mass:

$$\mathbf{E}_3 \cdot \bar{\mathbf{v}} + \left(-R\mathbf{e}_2'' \times \mathbf{E}_3\right) \cdot \boldsymbol{\omega} = 0. \tag{28}$$

Using Lagrange's prescription with this constraint, we find

$$\begin{aligned} \mathbf{F}_c &= \lambda \mathbf{E}_3 \text{ acting at } \bar{X}, \\ \mathbf{M}_c &= \lambda \left(-R\mathbf{e}_2'' \times \mathbf{E}_3\right). \end{aligned} \tag{29}$$

With some insight, we conclude that this force-moment pair is equipollent to a force $\mathbf{F}_c = \lambda \mathbf{E}_3$ acting at X_P. As a result, one can work with either $\mathbf{v}_P \cdot \mathbf{E}_3 = 0$ or (28) and arrive at equivalent constraint force and constraint moment prescriptions.

Based on the previous example, we are lead to the suspicion that Lagrange's prescription doesn't depend on the choice of material point X_C that we choose to formulate the constraint. To see that this is true in general let us choose to express the constraint function (25) in terms of another material point, say X_A (cf. Figure 6). The transformation of the function π_C given by (25) to the equivalent constraint function π_A,

$$\pi_A = \mathbf{f}_A \cdot \mathbf{v}_A + \mathbf{h}_A \cdot \boldsymbol{\omega} + e_A, \tag{30}$$

where $\pi_A = 0$ is equivalent to $\pi_C = 0$, can be achieved using the identity

$$\mathbf{v}_C = \mathbf{v}_A + \boldsymbol{\omega} \times (\mathbf{x}_C - \mathbf{x}_A). \tag{31}$$

Substituting for $\mathbf{v}_C$ in (25) then yields the correspondences

$$\begin{aligned} \mathbf{f}_A &= \mathbf{f}_C, \\ \mathbf{h}_A &= \mathbf{h}_C + (\mathbf{x}_C - \mathbf{x}_A) \times \mathbf{f}_C, \\ e_A &= e_C. \end{aligned} \tag{32}$$

Using (32) it is straightforward to see that the constraint force ($\lambda \mathbf{f}_A$ acting at X_A) and constraint moment ($\lambda \mathbf{h}_A$) provided by Lagrange's prescription using (30) are equipollent to (26):

$$\begin{gathered} \left\{ \begin{array}{c} \mathbf{F}_c = \lambda \mathbf{f}_C \text{ acting at } X_C \\ \mathbf{M}_c = \lambda \mathbf{h}_C \end{array} \right\} \Leftrightarrow \\ \left\{ \begin{array}{c} \mathbf{F}_c = \lambda \mathbf{f}_A = \lambda \mathbf{f}_C \text{ acting at } X_A \\ \mathbf{M}_c = \lambda \mathbf{h}_A = \lambda \mathbf{h}_C + (\mathbf{x}_C - \mathbf{x}_A) \times \mathbf{F}_c \end{array} \right\}. \end{gathered} \tag{33}$$

As shown schematically in Figure 6, the difference in the constraint moments can be attributed entirely to the difference in the point of application of the force $\mathbf{F}_c$ for the two cases.

5 Lagrange's Prescription and Lagrange's Equations of Motion

Lagrange's equations of motion for a rigid body can be written in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\Gamma}\right) - \frac{\partial T}{\partial q^\Gamma} = Q_\Gamma, \qquad (\Gamma = 1, \ldots, 6), \tag{34}$$

where $T = T\left(q^1, \ldots, q^6, \dot{q}^1, \ldots, \dot{q}^6\right)$ is the kinetic energy of the rigid body, Q_Γ are the generalized forces, and $q^1, \ldots, q^6$ are the coordinates used to parameterize the motion $(\bar{\mathbf{x}}, \mathbf{Q})$ of the rigid body. For example, q^1, q^2, and q^3 could be a set of Cartesian coordinates for $\bar{\mathbf{x}}$ and q^4, q^5, and q^6 might be a set of Euler angles used to parameterize $\mathbf{Q}$.

If the resultant force acting on the rigid body is $\mathbf{F}$ and the resultant moment (relative to the center of mass) of the rigid body is $\mathbf{M}$, then the following well-known identification for Q_Γ holds:[4]

$$Q_\Gamma = \mathbf{F} \cdot \frac{\partial \bar{\mathbf{v}}}{\partial \dot{q}^\Gamma} + \mathbf{M} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^\Gamma}. \tag{35}$$

For future purposes, it is illuminating to consider a force $\mathbf{F}_C$ acting at a point C on a rigid body with $\mathbf{x}_C = \bar{\mathbf{x}} + \boldsymbol{\pi}_C$. It is straightforward to show that $\mathbf{F}_C$'s contribution to Q_Γ can be expressed in two equivalent manners:

$$\mathbf{F}_C \cdot \frac{\partial \bar{\mathbf{v}}}{\partial \dot{q}^\Gamma} + (\boldsymbol{\pi}_C \times \mathbf{F}_C) \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^\Gamma} = \mathbf{F}_C \cdot \frac{\partial \mathbf{v}_C}{\partial \dot{q}^\Gamma} \tag{36}$$

where the following pair of identities are used to establish (36) from (35):

$$\mathbf{v}_C = \bar{\mathbf{v}} + \boldsymbol{\omega} \times \boldsymbol{\pi}_C,$$
$$\frac{\partial \mathbf{v}_C}{\partial \dot{q}^\Gamma} = \frac{\partial \bar{\mathbf{v}}}{\partial \dot{q}^\Gamma} - \boldsymbol{\pi}_C \times \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^\Gamma}. \tag{37}$$

The key to establishing $(37)_2$ from the $(37)_1$ is to note that the relative position vector $\boldsymbol{\pi}_C$ depends on the coordinates q^Γ and not their velocities $\dot{q}^\Gamma$.

Consider a constraint $\pi_C = 0$ (cf. (25)) on the motion of a rigid body. Using the coordinates and (25), the constraint $\pi_C = 0$ can be expressed in the equivalent form:

$$\sum_{\Gamma=1}^{6} \mathsf{A}_\Gamma \dot{q}^\Gamma + e_C = 0. \tag{38}$$

Here, the six functions A_Γ depend on $q^1, \ldots, q^6$ and t:

$$\mathsf{A}_\Gamma = \mathbf{f}_C \cdot \frac{\partial \mathbf{v}_C}{\partial \dot{q}^\Gamma} + \mathbf{h}_C \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^\Gamma}. \tag{39}$$

As mentioned in the introduction, the prescription for the generalized constraint force Φ_Γ associated with (38) is well known (see, e.g., [1,32]):

$$\Phi_\Gamma = \tilde{\lambda} \mathsf{A}_\Gamma \tag{40}$$

where $\tilde{\lambda}$ is a Lagrange multiplier. Lagrange's prescription for the same constraint is given by (26): $\mathbf{F}_c = \lambda \mathbf{f}_C$ acting at X_C and $\mathbf{M}_c = \lambda \mathbf{h}_C$. To help show that (26) and (40) are equivalent, we observe with the help of (35) and (36) that the contributions of $\mathbf{F}_c = \lambda \mathbf{f}_C$ acting at X_C and $\mathbf{M}_c = \lambda \mathbf{h}_C$ to Q_Γ are

$$\begin{aligned} \mathbf{F}_c \cdot \frac{\partial \mathbf{v}_C}{\partial \dot{q}^\Gamma} + \mathbf{M}_c \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^\Gamma} &= \lambda \mathbf{f}_C \cdot \frac{\partial \mathbf{v}_C}{\partial \dot{q}^\Gamma} + \lambda \mathbf{h}_C \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^\Gamma} \\ &= \lambda \mathsf{A}_\Gamma. \end{aligned} \tag{41}$$

[4] This identification ensures that Lagrange's equations of motion (34) are equivalent to the Newton-Euler balance laws $\mathbf{F} = m\dot{\bar{\mathbf{v}}}$ and $\mathbf{M} = \dot{\mathbf{H}}$ where $\mathbf{H}$ is the angular momentum of the rigid body of mass m relative to $\bar{X}$ (see, e.g., [1,3,7,22] for further details on the equivalence).

Identifying $\lambda = \tilde{\lambda}$, we conclude that the prescription (40), which is used in most textbooks on analytical dynamics, is equivalent to Lagrange's prescription (26).[5]

The beauty of Lagrange's prescription when applied to integrable constraints is best appreciated by examining the contributions of $\mathbf{F}_c$ and $\mathbf{M}_c$ to the generalized forces. To see this we consider an integrable constraint of the form (25) and suppose that the coordinates are chosen so that the constraint can be simply expressed as

$$\dot{q}^6 + e_C(t) = 0. \tag{42}$$

That is,

$$\dot{q}^6 = \mathbf{f}_C \cdot \mathbf{v}_C + \mathbf{h}_C \cdot \boldsymbol{\omega}. \tag{43}$$

We next assume that the associated constraint force $\mathbf{F}_c$ and constraint moment are prescribed by Lagrange's prescription (26). With the help of (35) and (36), we find that $\mathbf{F}_c$ and $\mathbf{M}_c$ make the following contribution to Q_Γ:

$$\begin{aligned} \mathbf{F}_c \cdot \frac{\partial \mathbf{v}_C}{\partial \dot{q}^\Gamma} + \mathbf{M}_c \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^\Gamma} &= \lambda \mathbf{f}_C \cdot \frac{\partial \mathbf{v}_C}{\partial \dot{q}^\Gamma} + \lambda \mathbf{h}_C \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^\Gamma} \\ &= \lambda \frac{\partial}{\partial \dot{q}^\Gamma} \left(\mathbf{f}_C \cdot \mathbf{v}_C + \mathbf{h}_C \cdot \boldsymbol{\omega} \right) \\ &= \lambda \frac{\partial \dot{q}^6}{\partial \dot{q}^\Gamma} \\ &= \lambda \delta^6_\Gamma. \end{aligned} \tag{44}$$

Hence, $\mathbf{M}_c$ and $\mathbf{F}_c$ will only contribute to the sixth member of Lagrange equations. The remaining five Lagrange's equations can be used to determine differential equations for the generalized coordinates $q^1, \ldots, q^5$.

The result (44) justifies the traditional approach employed with Lagrange's equations of motion for holonomically constrained rigid bodies. In these works, the integrable constraints are imposed during the computation of the kinetic energy to yield a constrained kinetic energy. Assuming that Lagrange's prescription can be used to prescribe $\mathbf{F}_c$ and $\mathbf{M}_c$, then this constrained kinetic energy suffices to determine the equations of motion governing the generalized coordinates of the rigid body.

6 Closing Remarks

The treatment of Lagrange's prescription we have presented can be readily extended to situations featuring two or more rigid bodies. Some guidance on this matter can be found by following the developments of [26],

[5] With relation to other treatments of constraint forces, it is easy to observe from (27) and (39) that the combined virtual work of $\mathbf{F}_c$ and $\mathbf{M}_c$ will be zero as expected.

however instead of writing the constraint in terms of the motions of the centers of mass of the individual bodies, one can express the constraints in terms of the motions of the material points on both bodies featuring in the constraint. For example, if two bodies are connected by a ball-and-socket joint at a point X_{C_1} of one body and X_{C_2} of the second body, then the three constraints on the system of two rigid bodies can be expressed as

$$\mathbf{E}_k \cdot \mathbf{v}_{C_1} - \mathbf{E}_k \cdot \mathbf{v}_{C_2} = 0, \qquad (k = 1, 2, 3)\,, \tag{45}$$

and Lagrange's prescription will yield a pair of constraint forces and constraint moments

$$\begin{aligned} \mathbf{F}_{c_1} &= \sum_{k=1}^{3} \lambda_k \mathbf{E}_k \text{ acting at } X_{C_1} \text{ on the first body,} \\ \mathbf{M}_{c_1} &= 0, \\ \mathbf{F}_{c_2} &= -\sum_{k=1}^{3} \lambda_k \mathbf{E}_k \text{ acting at } X_{C_1} \text{ on the second body,} \\ \mathbf{M}_{c_2} &= 0. \end{aligned} \tag{46}$$

Note that the constraint force $\mathbf{F}_{c_1}$ is equal and opposite to the constraint force acting on the second body: $\mathbf{F}_{c_2} = -\mathbf{F}_{c_1}$. As noted in [5, Ch. 7], [8], [19], and [26], among others, this is the evidence for how Lagrange's prescription is intimately related to Newton's third law in this case. It might be of interest to some readers that Noll [20] has also remarked on this coincidence for conservative force fields.

In addition to mechanisms, one of the most useful areas of application of Lagrange's prescription is to multibody models for anatomical joints. For example, consider the knee joint shown in Figure 7. This joint governs the motion of the tibia relative to the femur and features the surface of the femur moving on the tibial plateau. The constraints on the relative motion are approximately similar to those experienced by a cone sliding on a horizontal plane. Modulo relabeling of axes, it is standard to parameterize the relative rotation using a set of 3-2-1 Euler angles [11,24], where the first angle ψ is known as flexion-extension rotation, the second angle θ is known as varus-valgus rotation and the third angle ϕ is known as internal-external rotation. The constraints on the relative motion can be written in several equivalent forms. The simplest form is to assume that the condyles remain in contact with the tibial plateau. Omitting details as they are similar to those presented above, Lagrange's prescription would yield a pair of reaction forces $\mathbf{N}_1$ and $\mathbf{N}_2$ acting on the femur and an equal and opposite pair acting on the tibia (see Figure 8). The pair of normal forces is equipollent to a single constraint force $\mathbf{F}_{c_F} = \mathbf{N}_1 + \mathbf{N}_2$ acting on the femur and a constraint moment $\mathbf{M}_{c_F}$ in the $\mathbf{g}_2 = \mathbf{g}^2$ direction. An equivalent method to motivate the constraint

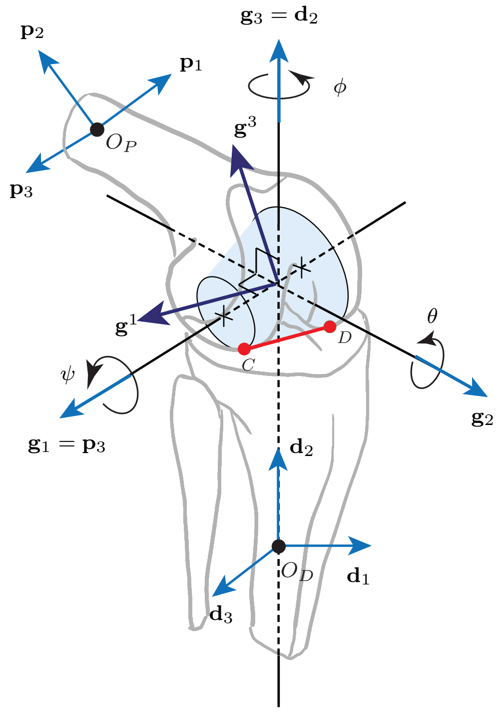

Fig. 7: *Schematic of the right knee joint showing the proximal* $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ *and distal* $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ *bases which corotate with the femur and tibia, respectively. The Euler and dual Euler basis vectors associated with the rotation of this joint and the condyles* C *and* D *are also shown. This figure is adapted from [24].*

moment is to note that the pair of condyles in contact with the tibial plane imposes a constraint which can be expressed in several equivalent forms:

$$\theta = \theta_0, \qquad \dot{\theta} = 0, \qquad (\boldsymbol{\omega}_T - \boldsymbol{\omega}_F) \cdot \mathbf{g}_2 = 0, \tag{47}$$

where $\boldsymbol{\omega}_T$ is the angular velocity of the tibia and $\boldsymbol{\omega}_F$ is the angular velocity of the femur. Then Lagrange's prescription would prescribe a moment $\mathbf{M}_{c_T} = \lambda \mathbf{g}_2$ acting on the tibia and an equal and opposite moment $\mathbf{M}_{c_F} = -\lambda \mathbf{g}_2$ acting on the femur. We leave it as an exercise for the reader to convince themselves that these moments are generated by the equal and opposite normal forces $\mathbf{N}_1$ and $\mathbf{N}_2$.

While Lagrange's prescription in either of its equivalent forms has tremendous analytical advantages and, following Gauss [10] in 1829, the remarkable feature that it provides the minimum generalized force needed to enforce a constraint (cf. [2,25,28,33] and references therein), it has long been realized that it is not universally applicable. To this end there are many additional prescriptions (or as they are sometimes known constitutive equations) for constraint forces and constraint moments. These include dynamic Coulomb fric-

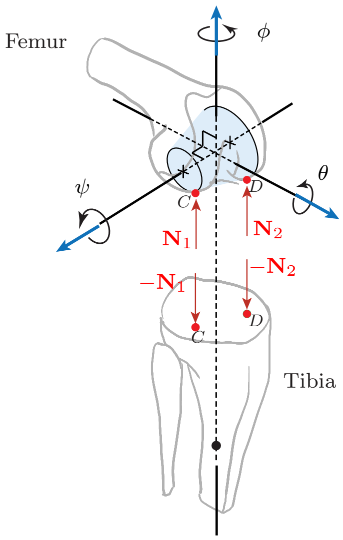

Fig. 8: *Schematic of the normal forces at the condyles. For the illustrated case* $\theta < 0$. *This figure is adapted from [24].*

tion, Coulomb-Contensou friction [16,17] and memory-type generalized constraint forces in vakonomic mechanics [2,18,25]. We refer the interested reader to these references for further discussions and examples on this rich topic.

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