

A Simplex Contained in a Sphere

Andrew Vince

Abstract. Let Δ be an n -dimensional simplex in Euclidean space \mathbb{E}^n contained in an n -dimensional closed ball B . The following question is considered. Given any point $x \in \Delta$, does there exist a reflection $r : \mathbb{E}^n \rightarrow \mathbb{E}^n$ in one of the facets of Δ such that $r(x) \in B$?

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1. Euler's inequality

Among the vast literature on Euclidean geometry can be found numerous elegant inequalities concerning the centroid, circumcenter, incenter, and orthocenter of a triangle. Some of these have natural extensions to a simplex in n -dimensional Euclidean space, a classic example being Euler's inequality

$$R \geq nr,$$

where R and r are the circumradius and the inradius of the simplex, respectively. Each *facet*, i.e., maximal proper face, of an n -simplex Δ is an $(n-1)$ -dimensional simplex. The *inradius* of an n -simplex Δ is the maximum of the radii of balls contained within Δ . The center of this unique ball is called the *incenter* of Δ . The boundary of the maximum ball is a sphere that meets each facet in a single point. The *circumradius* of Δ is the minimum of the radii of balls containing Δ . The center of this minimum ball is called the *circumcenter* of Δ . The boundary of this unique minimum ball is not necessarily the sphere through the vertices of Δ . In fact, it is only if the center of the minimum ball is inside Δ .

Extensions of and variations on Euler's inequality have since appeared, for example [3, 6]. For completeness, we provide a simple proof of Euler's theorem which, as was pointed out by the referee, is essentially the proof for the triangle that appears

in [4]. Fejes–Tóth credits the proof to I. Ádám. Klamkin and Tsintsifas [2] provide another short proof.

The main contribution of this note is a related statement. Although a short proof of this statement is given for a triangle in the plane in Section 4, it is vexing (vexing for me and a challenge to the reader) that the natural generalization to an n -dimensional simplex remains a conjecture. The statement appears in four alternate forms in Section 2. In one form the statement resembles Euler’s inequality. Two other forms, the ones that motivated the title of this note, concern an n -simplex Δ contained in a closed ball B . Let F denote the set of facets of Δ and $r_f : \mathbb{E}^n \rightarrow \mathbb{E}^n$ the reflection in the hyperplane containing facet $f \in F$. The statements are as follows.

- For any point $x \in \Delta$ there is an $f \in F$ such that $r_f(x) \in B$.
- There exists a point $x \in \Delta$ such that $r_f(x) \in B$ for all $f \in F$.

All four statements appear in Section 2. The equivalence of the four statements is the content of Section 3. In the 2-dimensional case, the point x in the second statement above is the orthocenter, the point of concurrency of the altitudes of the triangle. The proof for the 2-dimensional case does not generalize due to the interesting fact that the altitudes of an n -simplex, in general, are not concurrent for $n > 2$.

In the following proof of Euler’s inequality, $F = \{f_0, f_1, \dots, f_n\}$ is the set of facets of an n -simplex Δ , and $R(\Delta)$ is the circumradius of Δ . If x_0, x_1, \dots, x_n are affinely independent points in \mathbb{E}^n then $\Delta(x_0, x_1, \dots, x_n)$ denotes the n -simplex with these points as vertices. The Euclidean distance between points x and y is denoted $d(x, y)$.

Proof of Euler’s inequality. Let Δ be an n -simplex with vertices v_0, v_1, \dots, v_n . Denote the centroid of facet f_i by a_i . Simplices Δ and $\Delta(a_0, a_1, \dots, a_n)$ are similar with ratio n . This is easy to check since $a_k = \frac{1}{n}(v_0 + v_1 + \dots + v_{k-1} + v_{k+1} + \dots + v_n)$ which implies that $d(a_i, a_j) = \frac{1}{n}d(v_i, v_j)$ for all $i \neq j$. This similarity implies $R(\Delta) = nR(\Delta(a_0, a_1, \dots, a_n))$. A ball of radius less than that of the inscribed ball cannot meet every facet of Δ . Therefore $R(\Delta(a_0, a_1, \dots, a_n)) \geq r$ and

$$R = nR(\Delta(a_0, a_1, \dots, a_n)) \geq nr. \quad \square$$

2. Four conjectures

If O is the circumcenter of n -simplex Δ , let $\overline{\Delta}$ denote the n -simplex whose vertices are the feet of the perpendiculars from O to the facets of Δ .

Conjecture 1. $R(\Delta) \geq 2R(\overline{\Delta})$.

It is interesting to compare the inequality in Conjecture 1 with Euler’s inequality. Let b_0, b_1, \dots, b_n be the feet of the perpendiculars from the circumcenter to the

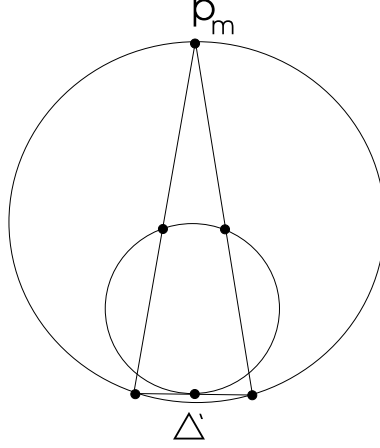


FIGURE 1. Conjecture 1 is best possible.

hyperplanes containing f_0, f_1, \dots, f_n , respectively, and let a_0, a_1, \dots, a_n be the feet of the perpendiculars from the incenter to the facets f_0, f_1, \dots, f_n , respectively. Euler's inequality is

$$\frac{R(\Delta(a_0, a_1, \dots, a_n))}{R(\Delta)} \leq \frac{1}{n}$$

while Conjecture 1 claims only

$$\frac{R(\Delta(b_0, b_1, \dots, b_n))}{R(\Delta)} \leq \frac{1}{2}.$$

On the other hand, if Conjecture 1 is true, it is best possible in the sense that $R(\overline{\Delta})/R(\Delta)$ can be made arbitrarily close to $1/2$. Let Δ' be a regular $(n-1)$ -simplex with edge length 1 embedded in the hyperplane $x_n = 0$ in \mathbb{R}^n so that its centroid is at the origin. Let Δ_m be the n -simplex that is the convex hull in \mathbb{R}^n of Δ' and the point $p_m = (0, 0, \dots, 0, m)$. See Figure 1. Then clearly

$$\lim_{m \rightarrow \infty} \frac{R(\overline{\Delta_m})}{R(\Delta_m)} = \frac{1}{2}.$$

Conjecture 2. *If n -simplex Δ is contained in a closed n -ball B , then for any point $x \in \Delta$ there is an $f \in F$ such that $r_f(x) \in B$.*

Conjecture 3. *If n -simplex Δ is contained in a closed n -ball B , then there exists a point $x \in \Delta$ such that $r_f(x) \in B$ for all $f \in F$.*

The fourth conjecture, the one that motivated this paper, arose in the analysis of an algorithm for discrete line generation in computer graphics [5]. Concerning notation, let V be the set of vertices of the n -simplex Δ . If $v \in V$ and $x \in \Delta$, let Δ_v^x denote the simplex with vertex set $V \cup \{x\} \setminus \{v\}$. The sphere *circumscribed* about Δ

will be defined to be the sphere containing the vertices of Δ . Let $R'(\Delta)$ denote the radius of the circumscribed sphere. As previously noted, the circumscribed sphere is not necessarily the boundary of the smallest ball containing Δ , hence $R(\Delta)$ and $R'(\Delta)$ are not necessarily equal.

Conjecture 4. *If $x \in \Delta$, then $R'(\Delta) \leq \max_{v \in V} R'(\Delta_v^x)$.*

3. Equivalence

This section contains the proof of the equivalence of the four conjectures stated in Section 2.

Lemma 5. *Let V be the set of vertices of an n -simplex Δ and C_v , $v \in V$, a closed set containing the facet of Δ opposite v . If $\bigcup_{v \in V} C_v = \Delta$ then $\bigcap_{v \in V} C_v \neq \emptyset$.*

Proof. For convenience, let $V = \{0, 1, \dots, n\}$. Fix a positive number ϵ and triangulate Δ such that $\text{diam}(\delta) \leq \epsilon$ for each simplex δ in the triangulation. The vertex set V_δ of the triangulation is required to contain V . For a subset $J \subseteq V$, let $J+1 = \{j+1 \pmod{n+1} \mid j \in J\}$, and let f_J denote the face of Δ that is the convex hull of J . Because C_k contains the facet opposite vertex k , we have

$$f_J \subseteq C_k \quad \text{for all } k \notin J. \quad (1)$$

Let $g : V_\delta \rightarrow V$ be a labeling of the vertices of the triangulation defined as follows. If x is an interior point of Δ , let $g(x) = k$ for any $k \in V$ for which $x \in C_k$. This is possible because $\bigcup_{v \in V} C_v = \Delta$. If x lies on the boundary of Δ , then there is a unique J such that $x \in f_J$ but $x \notin f_K$ for any $K \subsetneq J$. In this case let $g(x)$ equal any element in the set $(J+1) \setminus J$. This labeling has the following properties, the first following from (1) and the definition of g , the second from the definition of g on boundary points.

- (a) $x \in C_{g(x)}$
- (b) If $x \in f_J$, then $g(x) \in g(J)$.

Property (b) is the hypothesis of Sperner's lemma, the conclusion being that there is a simplex δ in the triangulation whose set of vertex labels is exactly V . Using property (a), the fact that the C_v are closed sets, and the usual limiting argument as ϵ tends to 0, we conclude that there exists a point in Δ that lies in C_v for all $v \in V$. \square

Let $B(\Delta)$ denote the closed ball with minimum radius containing simplex Δ and $B'(\Delta)$ the closed ball determined by the sphere circumscribed about Δ .

Lemma 6. *If Conjecture 2 is true for either $B(\Delta)$ or $B'(\Delta)$, then it is true for any closed ball containing Δ .*

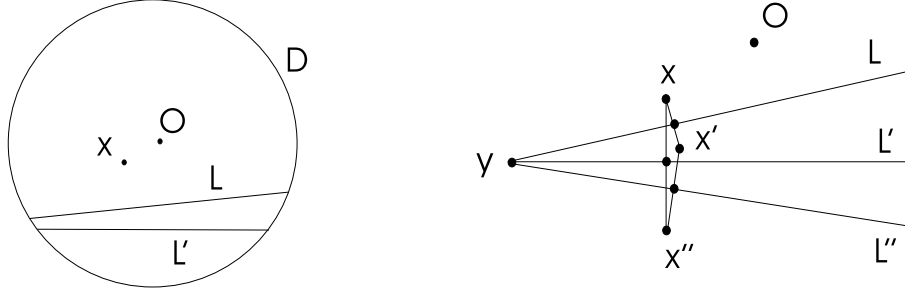


FIGURE 2. Figure accompanying the proof of Lemma 6.

Proof. Assume that Conjecture 2 holds for either $B(\Delta)$ or $B'(\Delta)$. Let B be any ball containing Δ and denote its center by Q . Let x be an arbitrary point in Δ . If $Q \notin \Delta$, then clearly $r_f(x) \in B$ for some facet f . So assume that this is not the case. Define V' to be the set of points obtained by radially projecting the set V of vertices of Δ from Q to the boundary of B . Let Δ' be the n -simplex with vertex set V' . Then $\Delta \subseteq \Delta'$ and, since $Q \in \Delta'$, we have $B = B(\Delta') = B'(\Delta')$. By assumption there is a facet f' of Δ' such that $r_{f'}(x) \in B$. It suffices to show that $r_f(x) \in B$ for the corresponding facet f of Δ .

Let S and S' be the intersections of B with the hyperplanes H and H' containing f and f' , respectively. Take point Q as the origin of \mathbb{R}^d . If $d > 2$, let u and u' be the vectors (from Q) orthogonal to the hyperplanes H and H' , respectively. Let $W = \text{span}\{u, u'\}$ and $W_x = x + W$, a 2-dimensional affine subspace of \mathbb{R}^d containing x . (If $u = u'$ then let W be the span of u and any other vector.) If proj_{W_x} denotes orthogonal projection onto W_x , then $L = \text{proj}_{W_x}(H)$ and $L' = \text{proj}_{W_x}(H')$ are lines and $O = \text{proj}_{W_x}(Q)$ is the center of the disk $D = B \cap W_x$. Note that both O and x lie in the same halfspace of W_x determined by L (or L'). See the first diagram in Figure 2. If r_L and $r_{L'}$ denote the reflections in L and L' respectively then the proof is now reduced to the 2-dimensional case, to showing that if $r_{L'}(x) \in D$ then $r_L(x) \in D$.

In Figure 2 we denote $x' = r_L(x)$ and $x'' = r_{L'}(x)$. Line L'' is the perpendicular bisector of the line segment $\overline{x'x''}$. The three perpendicular bisectors of $\triangle xx'x''$ are concurrent at point y . Any point above line L'' is closer to x' than to x'' . Since the center O of D is above L and L' , hence above L'' , we have $d(O, x') < d(O, x'')$. In other words, if $r_{L'}(x) \in D$ then $r_L(x) \in D$. \square

The following is a technical lemma required only for the proof of Theorem 8.

Lemma 7. *Consider a 3-ball B with center O and boundary sphere S , a plane σ , a line L through O and perpendicular to σ , a point $u \in S \cap \sigma$, and a point $x \in B$. Further let O_x be the unique point on L such that $d(O_x, x) = d(O_x, u)$. If r denotes reflection in σ , then $d(O_x, u) \geq d(O, u)$ if and only if $r(x) \in B$.*

Proof. Let L' be the line joining x and $r(x)$. Then L' intersects σ in a point, say b and intersects S in two points a, a' , where we take a to be in the same halfspace determined by σ as x . If line L intersects S in points A, A' , then let A be the one in the same halfspace as a and x . Further let $c = r(a')$ and $O' = r(O)$.

Let x vary from $x = a$ to $x = b$. If $x = a$, then $O_x = O$ because $a, u \in S$. As x moves along L' from a to b , a tedious analytic geometry/calculus calculation shows that O_x moves monotonically from O toward A' . When (if) $x = c$, we have $O_x = O'$ because $d(O', c) = d(r(O'), r(c)) = d(O, a') = d(O, u) = d(r(O), r(u)) = d(O', u)$. Now $r(x) \in B$ if and only if x lies on the segment \overline{bc} if and only if O_x does not lie on the segment $\overline{OO'}$ if and only if $d(O_x, u) \geq d(O, u)$. \square

Theorem 8. *The four conjectures are equivalent in the sense that, for a given dimension $n \geq 2$, they are either all true or all false.*

Proof. Let V be the vertex set of Δ . For this proof it will be convenient to denote by $r_v : \mathbb{E}^n \rightarrow \mathbb{E}^n$ the reflection in the hyperplane containing the facet of Δ opposite vertex $v \in V$.

(3 \Rightarrow 2): Assume that Conjecture 3 is true and that x is the point whose existence is guaranteed by Conjecture 3. Since the ball B is convex, $r_v(\Delta_v^x) \subset B$. Let y be an arbitrary point in Δ . Because $\Delta = \bigcup_{v \in V} \Delta_v^x$, there is a $v \in V$ such that $y \in \Delta_v^x$. For this v we have $r_v(y) \in B$.

(2 \Rightarrow 3): Assume that Conjecture 2 is true. If $C_v = \{y \in \Delta \mid r_v(y) \in B\}$, then $\bigcup_{v \in V} C_v = \Delta$. By Lemma 5 there exists a point $x \in \bigcap_{v \in V} C_v$. Then $r_v(x) \in B$ for all $v \in V$.

We next show the equivalence of statements (3) and (1).

(3 \Rightarrow 1): Assume that Conjecture 3 is true, and let B be the unique ball of smallest radius containing Δ , with radius R and center O . Let x be the point whose existence is guaranteed by Conjecture 3. Let b_v denote the foot of the perpendicular projection of O on the hyperplane containing the facet f_v opposite vertex v , and define c_v on the line $\overline{Ob_v}$ such that $d(O, c_v) = 2d(O, b_v)$ and such that b_v lies between O and c_v . Hence $c_v = r_v(O)$. Now

$$d(x, c_v) = d(x, r_v(O)) = d(r_v(x), O) \leq R,$$

the inequality following from the statement of Conjecture 3. Let $\overline{\Delta}$ be the convex hull of the b_v , $v \in V$, and $\widehat{\Delta}$ the convex hull of the c_v , $v \in V$. By the inequality above, the simplex $\widehat{\Delta}$ is contained in a ball of radius R centered at x , i.e., $R(\widehat{\Delta}) \leq R$. Since $\widehat{\Delta}$ and $\overline{\Delta}$ are similar with ratio $1/2$, we have

$$R(\overline{\Delta}) = \frac{1}{2} R(\widehat{\Delta}) \leq \frac{1}{2} R.$$

(1 \Rightarrow 3): Assume that Conjecture 1 is true, and let B be the unique ball of smallest radius containing Δ , with radius, say R . With the notation O , $\widehat{\Delta}$ and $\overline{\Delta}$ the same

as in (3 \Rightarrow 1), let x be the circumcenter of $\widehat{\Delta}$. Then for all $v \in V$ we have

$$d(O, r_v(x)) = d(r_v(O), x) \leq R(\widehat{\Delta}) = 2R(\overline{\Delta}) \leq R.$$

So $r_v(x) \in B$ for all $v \in V$. Lemma 6 implies Conjecture 2, hence also Conjecture 3, for any ball containing Δ .

Finally the equivalence of Conjectures 2 and 4 is proved.

(2 \Rightarrow 4): Assume that Conjecture 2 is true, and let x be any point in Δ . Let B be the ball containing all vertices of Δ and O the center of Δ . (Note that O is not necessarily the circumcenter of Δ .) By the statement of Conjecture 2 there is a $v \in V$ such that $r_v(x) \in B$. Let L denote the line that is the intersection of the perpendicular bisecting hyperplanes of all line segments $\overline{uu'}$, $u, u' \in V$, $u, u' \neq v$. Then L passes through O and through the center O_x of the sphere circumscribed about Δ_v^x . Let L' be the line through x and $r_v(x)$. Since both L and L' are perpendicular to the facet of Δ opposite v , the lines L and L' are parallel. Let u be any vertex in $V \setminus \{v\}$, and, to simplify matters, consider only the (at most) 3-dimensional affine subspace X of \mathbb{E}^n containing the lines L and L' and point u . Then $B' = B \cup X$ is a 3-ball with center O . It now suffices to show that $d(O_x, u) = R'(\Delta_v^x) \geq R'(\Delta) = d(O, u)$. Note the following:

- (a) O_x is the point on L equidistant from x and u .
- (b) If σ is the plane that is the perpendicular bisector of segment $\overline{xr_v(x)}$ and S is the boundary sphere of B' , then $u \in S \cap \sigma$.
- (c) $r_v(x) \in B'$.

The required result $d(O_x, u) \geq d(O, u)$ now follows from Lemma 7.

(4 \Rightarrow 2): Assume that Conjecture 4 is true. We will prove that Conjecture 2 is true when B is the circumscribed ball, i.e., the ball whose boundary contains all vertices of Δ . Conjecture 2 will then follow in general from Lemma 6. Let x be an arbitrary point in Δ and let $v \in V$ be such that $R'(\Delta_v^x) \geq R'(\Delta)$. Let L, L', σ, B', Q be defined exactly as in the proof (2 \Rightarrow 4). Conjecture 4 gives us $d(O_x, u) = R'(\Delta_v^x) \geq R'(\Delta) = d(O_x, u)$, from which $r_v(x) \in B$ follows from Lemma 7. \square

4. Dimension 2 case

Everything in this section takes place in the Euclidean plane, and is related to classical material on orthocentric systems. In particular the existence part of Theorem 10 is implied by classical results on the subject [1, pages 265–267].

Lemma 9. *Let D_1, D_2, D_3 be three distinct closed disks with equal radius, with centers O_1, O_2, O_3 , and with boundary circles C_1, C_2, C_3 , respectively. If $C_1 \cap C_2 \cap C_3 \neq \emptyset$ and $\triangle O_1 O_2 O_3$ is an acute triangle, then $D_1 \cap D_2 \cap D_3 = C_1 \cap C_2 \cap C_3$.*

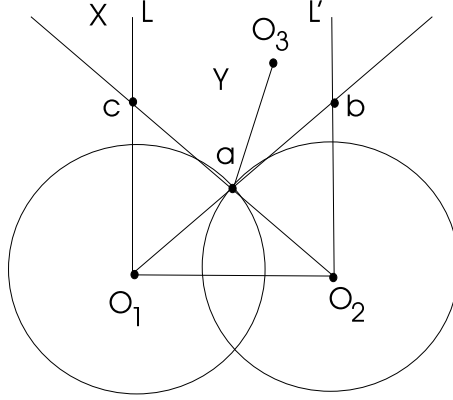


FIGURE 3. Figure accompanying the proof of Lemma 9.

Proof. Figure 3 shows circles C_1 and C_2 . Three distinct circles with equal radius ρ can have at most one point in common. Assume that $C_1 \cap C_2 \cap C_3 = \{a\}$. Let L, L' be lines perpendicular to $\overline{O_1O_2}$. Then

$$d(O_1, a) = d(O_2, a) = d(a, b) = d(a, O_3) = \rho.$$

Let X be the closed region consisting of all points that lie above both lines $\overline{O_1b}$ and $\overline{O_2c}$, and Y the closed region consisting of all points that lie between the lines L and L' . Then $D_1 \cap D_2 \cap D_3 = \{a\}$ if and only if $O_3 \in X \cap Y$ if and only if $\Delta O_1O_2O_3$ is acute. \square

The following theorem is the 2-dimensional version of Conjecture 3. Let $r_i : \mathbb{E}^2 \rightarrow \mathbb{E}^2$, $i = 1, 2, 3$, denote the reflections in each of the three lines containing a side of triangle Δ .

Theorem 10. *If Δ is a triangle in the plane and D is the circumscribed closed disk, then there exists a point O such that $r_i(O) \in D$ for $i = 1, 2, 3$. Moreover, the point O is unique and is the orthocenter of Δ if Δ is an acute triangle and is the vertex of the obtuse angle if Δ is not acute.*

Proof. The proof for the case of an obtuse triangle is easy and left to the reader. So assume that Δ is acute. Referring to Figure 4, the orthocenter is denoted O , the segments $\overline{v_1a}$ and $\overline{v_2c}$ are altitudes, and point b is the intersection of the circumcircle with the extension of altitude $\overline{v_1a}$. The measure of the arc v_1v_2 is denoted θ . Also $\alpha, \beta, \gamma, \phi$ are the designated angles. Then

$$\alpha = \frac{\pi}{2} - \beta = \frac{\pi}{2} - \left(\frac{\pi}{2} - \gamma\right) = \gamma = \frac{1}{2}\theta = \phi.$$

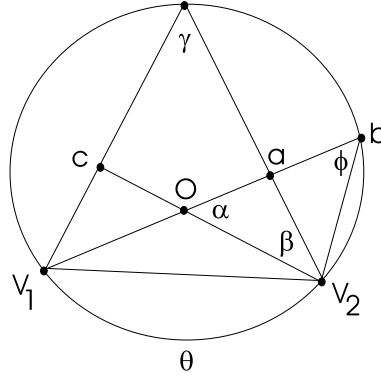


FIGURE 4. Figure accompanying the proof of Theorem 10.

This implies $d(a, O) = d(a, b)$ and hence $r_1(O) = b$ is contained in D , on the boundary of D to be precise.

It only remains to show uniqueness. Let $D_i = r_i(D)$, $i = 1, 2, 3$. Assume that x is any point in D such that $r_i(x) \in D$ for all i . Then $x \in D_1 \cap D_2 \cap D_3$. It is routine to show that $\Delta \cong \Delta'$, where Δ' is the triangle formed by joining the centers of D_1, D_2, D_3 . Hence Δ is an acute triangle if and only if Δ' is an acute triangle. By Lemma 9 we have $x \in D_1 \cap D_2 \cap D_3 = C_1 \cap C_2 \cap C_3 = \{O\}$. Hence $x = O$. \square

Note that Lemma 6 implies that the first statement in Theorem 10 holds for any closed disk D containing Δ .

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Andrew Vince
Department of Mathematics
Little Hall, PO Box 118105
University of Florida
Gainesville, FL. 32611-8105
USA
e-mail: vince@math.ufl.edu

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