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REPRINT

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Reprint Series
no. 24

CentER REPRINT

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A SIMPLICIAL ALGORITHM FOR STATIONARY POINT PROBLEMS ON POLYTOPES*†

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A simplicial variable dimension restart algorithm for the stationary point problem or variational inequality problem on a polytope is proposed. Given a polytope C in $\mathbf{R}^n$ and a continuous function $f: C \to \mathbf{R}^n$, find a point $\hat{x}$ in C such that $f(\hat{x}) \cdot \hat{x} \geq f(\hat{x}) \cdot x$ for any point x in C. Starting from an arbitrary point v in C, the algorithm generates a piecewise linear path of points in C. This path is followed by alternating linear programming pivot steps to follow a linear piece of the path and replacement steps in a simplicial subdivision of C. Within a finite number of function evaluations and linear programming pivot steps the algorithm finds an approximate stationary point. The algorithm leaves the starting point v along a ray pointing to one of the vertices w of C. The vertex w is obtained from the optimum solution of the linear programming problem maximize $f(v) \cdot x$ subject to $x \in C$.

1. Introduction. In order to compute zero points of continuous functions on the Euclidean space $\mathbf{R}^n$, many so-called simplicial variable dimension algorithms have been introduced. Such an algorithm subdivides $\mathbf{R}^n$ into n-dimensional simplices and searches for a simplex that contains an approximate zero point or solution. Starting in an arbitrarily chosen grid point of the triangulation the algorithm generates, through alternating linear programming pivot steps in a system of typically $n + 1$ linear equations and replacement steps in the triangulation, a sequence of adjacent simplices of varying dimension. Given some coercivity condition the algorithm then generates within a finite number of steps an approximate solution. When the accuracy is not satisfactory, the algorithm can be restarted at the approximate solution with a finer triangulation in the hope that within a small number of iterations a better approximate solution is found.

Simplicial variable dimension restart algorithms differ from each other in the number of rays along which the algorithm may leave the starting point. Such an algorithm with $n + 1$ rays, the $(n + 1)$-ray algorithm, was proposed in van der Laan and Talman [12]. The $2n$-ray algorithm was also introduced in [12], the 2^n-ray algorithm in [18], and the $(3^n - 1)$-ray algorithm in [10]. A unifying approach for these algorithms was given in van der Laan and Talman [13], see also Yamamoto [19]. In [13] the piecewise linear (abbreviated by pl) path traced by the algorithm when generating the sequence of adjacent simplices of varying dimension is interpreted as a curve of stationary points to the underlying problem with respect to an expanding set containing the starting point in its interior.

For the nonlinear complementarity problem on the n-dimensional unit simplex and the cross product of several unit simplices, S, simplicial restart algorithms have also

been developed. Contrary to the unbounded region $\mathbb{R}^n$, the set S is bounded for this problem. In Doup and Talman [3] a simplicial variable dimension restart algorithm on S is introduced which follows from the starting point v in S a curve of stationary points with respect to the expending set $(1 - t)\{v\} + tS$, $0 \leqslant t \leqslant 1$. This algorithm has as many rays as there are vertices of S.

Based on these ideas we propose a simplicial (variable dimension) restart algorithm in order to solve the nonlinear stationary point problem on a given polytope C in $\mathbb{R}^n$ and for a given function $f : C \to \mathbb{R}^n$, i.e., find a point $\hat{x}$ in C such that

(1.1) $f(\hat{x}) \cdot \hat{x} \geqslant f(\hat{x}) \cdot x$ for any x in C.

We assume that the set C, being a convex, compact set, is represented by the intersection of m half-spaces, $a^i \cdot x \leqslant b_i$, $i = 1, \dots, m$, where none of the constraints is redundant. Starting in an arbitrary point v of C, the algorithm generates a sequence of adjacent simplices of a specific simplicial subdivision of C. This sequence contains a piecewise liner path of points leading from v to an approximate stationary point. Again, alternating linear programming pivot steps in a linear system and replacement steps in the triangulation are made. The triangulation is closely related to the V-triangulation of S proposed in [3] and is completely determined by projections of v on the faces of C. These projections are subsequently determined during the performance of the algorithm by making pivot steps in the dual system of linear equations. The algorithm leaves the starting point v in the direction of the vertex of C that solves the linear programming problem

$$\max f(v) \cdot x   \text{ subject to } x \in C,$$

so that the number of rays is equal to the number of vertices of C. Each point x along the piecewise linear path traced by the algorithm solves the stationary point problem with respect to the piecewise linear approximation $\bar{f}$ to f on the set $(1 - t)\{v\} + tC$, for some t, $0 \leqslant t \leqslant 1$. The point v is the unique stationary point if $t = 0$ and when t reaches 1, as we trace the path, a stationary point for $\bar{f}$ on C is obtained, and hence an approximate stationary point for f.

The stationary point problem or variational inequality problem arises e.g. from economic equilibrium problems, noncooperative games, traffic assignment problems, and nonlinear optimization problems (see e.g. [8] and [16]). In a typical traffic assignment problem, the variable vector x represents the traffic load pattern on links and $f(x)$ the link cost. Assuming that the link cost function f is uniformly monotone Dafermos [1] proposed an algorithm for this problem, which repeatedly solves the stationary point problem with f replaced by an appropriate affine function and generates a convergent sequence of points to a stationary point (see also [2]). It is worth mentioning that the algorithm we will propose in this paper needs no assumptions on f except continuity.

If the function f is affine, no simplicial subdivision of the underlying set is needed. For such a linear stationary point problem, Eaves ([6, 7]) introduced a pivoting algorithm which traces a piecewise linear path of stationary points with respect to a specific expanding set.

The organization of this paper is as follows. In §2 we review the unifying framework for restart fixed point algorithms based on the primal-dual pair of subdivided manifolds proposed in Kojima and Yamamoto [5]. In §3 we specify the primal-dual pair of subdivided manifolds which induce our algorithm and present the basic system. We also prove the convergence of the algorithm and derive the accuracy of an approximate solution. In §4 we give a formal description of the algorithm under the assumption that

the polytope C is simple, full-dimensional, and that the linear inequalities defining C are nonredundant. §5 is devoted to the description of a specific triangulation of C that could underlie the algorithm. In §6 we discuss the cases when the polytope C is not full-dimensional or not simple.

2. Preliminaries. In this section we give a brief description of a subdivided manifold, a primal-dual pair of subdivided manifolds, and the basic theorem for simplicial algorithms.

We call a nonempty convex polyhedral set a cell. A cell of dimension m is called an m-cell. If a cell B is a face of a cell C, we write $B < C$.

Let $\mathcal{M}$ be a finite or countable collection of m-cells. We denote $\{B|B$ is a face of some m-cell of $\mathcal{M}\}$ by $\overline{\mathcal{M}}$ and $\bigcup(C|C \in \mathcal{M})$ by $|\mathcal{M}|$. We call $\mathcal{M}$ a subdivided m-manifold if and only if

(2.1) for any $B, C \in \mathcal{M}$, $B \cap C = \varnothing$ or $B \cap C < B$ and C;

(2.2) for each $(m - 1)$-cell B of $\overline{\mathcal{M}}$ at most two m-cells of $\mathcal{M}$ have B as a facet;

(2.3) $\mathcal{M}$ is locally finite: each point x of $|\mathcal{M}|$ has a neighborhood which intersects only a finite number of m-cells of $\mathcal{M}$.

We call the collection of $(m - 1)$-cells of $\overline{\mathcal{M}}$ that lie in exactly one m-cell of $\mathcal{M}$ the boundary of $\mathcal{M}$ and denote it by $\partial \mathcal{M}$.

A continuous function H from $|\mathcal{M}|$ into some Euclidean space is said to be a pl function on $\mathcal{M}$ if the restriction of H to each cell of $\mathcal{M}$ is an affine function. For a subdivided $(n + 1)$-manifold $\mathcal{M}$ and a pl function H on $\mathcal{M}$ into $\mathbf{R}^n$ we say that $c \in \mathbf{R}^n$ is a regular value of $H: |\mathcal{M}| \to \mathbf{R}^n$ if $B \in \overline{\mathcal{M}}$ and $H^{-1}(c) \cap B \neq \varnothing$ always imply $\dim H(B) = n$.

The following theorem is a basic theorem for simplicial algorithms (see Eaves [5]).

THEOREM 2.1. *Let $\mathcal{M}$ be a subdivided $(n + 1)$-manifold and H a pl function on $\mathcal{M}$ into $\mathbf{R}^n$. Suppose that $c \in \mathbf{R}^n$ is a regular value of H. Then $H^{-1}(c)$ is a disjoint union of paths and loops, where a path is a subdivided 1-manifold homeomorphic to one of the intervals $(0, 1)$, $(0, 1]$ and $[0, 1]$, and where a loop is a subdivided 1-manifold homeomorphic to the 1-dimensional sphere. Furthermore, $H^{-1}(c)$ satisfies the following conditions*

(2.4) $H^{-1}(c) \cap B$ *is either empty or a 1-cell for each $B \in \mathcal{M}$,*

(2.5) *a loop of $H^{-1}(c)$ does not intersect $|\partial \mathcal{M}|$.*

(2.6) *if a path S of $H^{-1}(c)$ is compact, the boundary* bd S *of S consists of two distinct points in $|\partial \mathcal{M}|$.*

Let $\mathcal{P}$ and $\mathcal{D}$ be two subdivided manifolds. If $\mathcal{P}$ and $\mathcal{D}$ satisfy the following conditions with some positive integer m and an operator $d: \overline{\mathcal{P}} \cup \overline{\mathcal{D}} \to \overline{\mathcal{P}} \cup \overline{\mathcal{D}} \cup \{\varnothing\}$, we say that $(\mathcal{P}, \mathcal{D}; d)$ is a primal-dual pair of subdivided manifolds (abbreviated by PDM) with degree m.

(2.7) for every $X \in \overline{\mathcal{P}}$, $X^d = \varnothing$ or $X^d \in \overline{\mathcal{D}}$.

(2.7)' for every $Y \in \overline{\mathcal{D}}$, $Y^d = \varnothing$ or $Y^d \in \overline{\mathcal{P}}$.

(2.8) if $Z \in \overline{\mathcal{P}} \cup \overline{\mathcal{D}}$ and $Z^d \neq \varnothing$, then $(Z^d)^d = Z$ and $\dim Z + \dim Z^d = m$.

(2.9) if $X_1, X_2 \in \overline{\mathcal{P}}$, $X_1 < X_2$, $X_1^d \neq \varnothing$ and $X_2^d \neq \varnothing$, then $X_2^d < X_1^d$.

(2.9)' if $Y_1, Y_2 \in \overline{\mathcal{D}}$, $Y_1 < Y_2$, $Y_1^d \neq \varnothing$ and $Y_2^d \neq \varnothing$, then $Y_2^d < Y_1^d$.

We call the operator d the dual operator.

For a PDM $(\mathcal{P}, \mathcal{D}; d)$ with degree m let

$$\langle \mathcal{P}, \mathcal{D}; d \rangle = \{ X \times X^d | X \in \overline{\mathcal{P}}, X^d \neq \varnothing \}.$$

or equivalently

$$\langle \mathcal{P}, \mathcal{D}; d \rangle = \{ Y^d \times Y | Y \in \overline{\mathcal{D}}, Y^d \neq \varnothing \}.$$

Then we have the following theorems. See Kojima and Yamamoto [9] for the proofs and for more details.

THEOREM 2.2 (Theorems 3.2 and 3.3 in [9]).   *Let* $(\mathcal{P}, \mathcal{D}; d)$ *be a PDM with degree* m. *Then* $\mathcal{L} = \langle \mathcal{P}, \mathcal{D}; d \rangle$ *is a subdivided* m-*manifold and*

$$\partial \mathcal{L} = \{ X \times Y | X \times Y \text{ is an } (m - 1)\text{-cell of } \bar{\mathcal{L}}, \ X \in \bar{\mathcal{P}}, \ Y \in \bar{\mathcal{D}}$$

$$\text{and either } X^d \text{ or } Y^d \text{ is empty} \}.$$

Let $\mathcal{2}$ be a refinement of $\mathcal{P}$, i.e. $\mathcal{2}$ is a subdivided manifold of the same dimension as $\mathcal{P}$, each cell of $\mathcal{2}$ is contained in some cell of $\mathcal{P}$, and $|\mathcal{2}| = |\mathcal{P}|$. For each cell X of $\bar{\mathcal{P}}$ let

$$\mathcal{2}|X = \{ \sigma | \sigma \in \bar{\mathcal{2}}, \sigma \subset X, \dim \sigma = \dim X \}.$$

THEOREM 2.3 (Theorem 4.1 in [9]).   *Let* $(\mathcal{P}, \mathcal{D}; d)$ *be a PDM with degree* m *and* $\mathcal{2}$ *be a refinement of* $\mathcal{P}$. *Then*

$$\mathcal{M} = \{ \sigma \times Y | Y \in \bar{\mathcal{D}}, Y^d \neq \varnothing, \sigma \in \mathcal{2}|Y^d \}$$

is a subdivided m-*manifold and a refinement of* $\mathcal{L} = \langle \mathcal{P}, \mathcal{D}; d \rangle$.

Note that $\partial \mathcal{M}$ is also a refinement of $\partial \mathcal{L}$ and that $|\partial \mathcal{M}| = |\partial \mathcal{L}|$.

Now consider a PDM $(\mathcal{P}, \mathcal{D}; d)$ with degree $n + 1$, a refinement $\mathcal{2}$ of $\mathcal{P}$, and a pl function F on $\mathcal{2}$ into $\mathbf{R}^n$. Let

$$(2.10) \qquad\qquad H(x, y) = y - F(x) \quad \text{for each } (x, y) \in |\mathcal{M}|,$$

where $\mathcal{M}$ is the refinement of $\mathcal{L} = \langle \mathcal{P}, \mathcal{D}; d \rangle$ as in Theorem 2.3. Then H is a pl function on $\mathcal{M}$. If we assume that $0 \in \mathbf{R}^n$ is a regular value of H, then we can apply Theorem 2.1 to the system of pl equations

$$(2.11) \qquad\qquad H(x, y) = 0, \qquad (x, y) \in |\mathcal{M}|.$$

This system is a basic model of the class of variable dimension algorithms and also gives the foundation of the algorithm to be presented.

3.   The path of the algorithm.   Before giving the PDM for our algorithm we rewrite the stationary point problem (1.1). Let $\mathcal{F}$ be the collection of all faces of the polytope C. For each face $F \in \mathcal{F}$ let F^* be the set of all n-dimensional coefficient vectors y such that any point of F is an optimum solution of the linear programming problem

$$\text{maximize } y \cdot x \quad \text{subject to} \quad x \in C = \{ x \in \mathbf{R}^n | a^i \cdot x \leqslant b_i, i = 1, \ldots, m \}.$$

Then the stationary point problem on C is the problem of finding a point $\hat{x}$ in C such that $f(\hat{x}) \in F^*$ for some face F containing $\hat{x}$. By the inclusion reversing property of F and F^* this is equivalent to $f(\hat{x}) \in F^*$ for a minimum face F of C containing the point $\hat{x}$. Note that if for a given F we define $I = \{ i | a^i \cdot x = b_i \text{ for all } x \in F \}$, then duality theory allows us to write $F^* = \{ y | y = \Sigma_{i \in I} \mu_i a^i, \mu_i \geqslant 0 \text{ for } i \in I \}$, and also $C^* = \{ 0 \}$.

Now let v be the starting point in C of the algorithm. Take an initial guess of a stationary point as v. Since an initial guess usually lies on the boundary of C, we allow the starting point v to lie on the boundary of C. For a face F of C which does not

contain the starting point v let vF be the convex hull of v and F, i.e.,

$$vF = \{ x | x = \alpha v + (1 - \alpha)z \text{ for some } z \in F \text{ and some } \alpha \in [0,1] \}.$$

Note that $\dim vF = \dim F + 1$. To make a PDM we define

$$(3.1) \qquad \mathscr{P} = \{ vF | v \notin F \in \mathscr{F}, \dim F = n - 1 \}.$$

Then $\mathscr{P}$ is a subdivided n-manifold and

$$(3.2) \qquad \overline{\mathscr{P}} = \{ vF | v \notin F \in \mathscr{F} \} \cup \{ F | v \notin F \in \mathscr{F} \} \cup \{ \{ v \} \}.$$

It should be noted that

$$(3.3) \qquad |\mathscr{P}| = C.$$

Figure 3.1 shows two examples of $\mathscr{P}$ for two distinct starting points, where the convex polytope C is a pentagon defined by the five linear inequalities $a^i \cdot x \leq b_i$, $i = 1, \ldots, 5$, and $F(I) = \{ x \in C | a^i \cdot x = b_i \text{ for } i \in I \}$. Let

$$(3.4) \qquad \mathscr{D} = \{ F^* | F \in \mathscr{F}, \dim F = 0 \}$$

$$= \{ \{ u \}^* | u \text{ is a vertex of } C \}.$$

Then $\mathscr{D}$ is also a subdivided n-manifold and

$$(3.5) \qquad \overline{\mathscr{D}} = \{ F^* | F \in \mathscr{F} \}.$$

$$(3.6) \qquad |\mathscr{D}| = \mathbf{R}^n.$$

Now let the dual operator d be defined by

$$(vF)^d = F^* \quad \text{if } v \notin F \in \mathscr{F},$$

$$F^d = \varnothing \quad \text{if } v \notin F \in \mathscr{F},$$

$$\{ v \}^d = \varnothing,$$

$$(F^*)^d = vF \quad \text{if } v \notin F \in \mathscr{F},$$

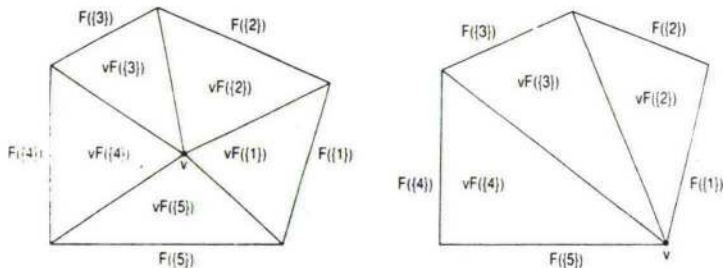$$= \varnothing \quad \text{if } v \in F \in \mathscr{F}.$$

FIGURE 3.1. A subdivided manifold $\mathscr{P}$.

It is readily seen that $(\mathscr{P}, \mathscr{D}; d)$ is a PDM with degree $n + 1$. By Theorem 2.2 we have the following lemma.

LEMMA 3.1. Let $\mathscr{L} = \langle \mathscr{P}, \mathscr{D}; d \rangle$. Then $\mathscr{L}$ is a subdivided $(n + 1)$-manifold and

$$(3.7) \qquad \partial \mathscr{L} = \{\{v\} \times \{u\}^* | u \text{ is a vertex of } C, u \neq v\}$$

$$\cup \{F \times F^* | v \notin F \in \mathscr{F}\}$$

$$\cup \{vE \times F^* | v \in F \in \mathscr{F}, \dim F > 0, E \text{ is a facet of } F,$$

$$\text{and } v \notin E\}.$$

PROOF. We only prove (3.7). Suppose that an n-cell $X \times Y$ lies in the boundary $\partial \mathscr{L}$ of $\mathscr{L}$. Then by Theorem 2.2 $\dim X + \dim Y = n$ and either $X^d = \emptyset$ or $Y^d = \emptyset$. Suppose first $X^d = \emptyset$. Then the unique $(n + 1)$-cell of $\mathscr{L}$ containing $X \times Y$ is $Y^d \times Y$. When $X = \{v\}$, Y^d is a 1-cell of $\mathscr{P}$ having $\{v\}$ as a facet, i.e., $Y^d = v\{u\}$ for some vertex u of C with $u \neq v$, and hence $Y = (v\{u\})^d = \{u\}^*$. When $X = F$ for some face $F \in \mathscr{F}$ with $v \notin F$, $Y^d = vF$. Therefore $Y = F^*$. Next suppose $Y^d = \emptyset$, i.e., $Y = F^*$ for some face F of C containing v. Then $X \times Y$ lies in $X \times X^d$ and $X^d = E^*$ for some face E of C such that $v \notin E$ and E^* has F^* as a facet. Therefore $X = (E^*)^d = vE$. By the inclusion reversing property of F and F^* we see that E is a facet of F and $\dim F > 0$.

Since it is readily seen that these cells above are members of $\partial \mathscr{L}$, we have proved (3.7). ∎

COROLLARY 3.2.

$$(3.8) \qquad |\partial \mathscr{L}| = (\cup(\{v\} \times \{u\}^* | u \text{ is a vertex of } C, u \neq v))$$

$$\cup (\cup(F \times F^* | v \notin F \in \mathscr{F}))$$

$$\cup (\cup(F \times F^* | v \in F \in \mathscr{F}, \dim F > 0)).$$

If the starting point v is not a vertex of C, then

$$(3.9) \qquad |\partial \mathscr{L}| = (\{v\} \times \mathbf{R}^n) \cup (\cup(F \times F^* | F \in \mathscr{F})).$$

Notice that F is equal to $\cup(vE | E$ is a facet of F and $v \notin E)$ if F contains the starting point v, so that the third class of cells in (3.7) gives the third subset of $|\partial \mathscr{L}|$ in (3.8). (3.9) is readily obtained from (3.6) and (3.8).

Now let T be a triangulation of C such that the restriction $T | X$ also triangulates X for each cell X of $\bar{\mathscr{P}}$. A specific triangulation is described in §5. Then T is a refinement of $\mathscr{P}$. We denote by $\mathscr{M}$ the subdivided $(n + 1)$-manifold defined by the refinement T of $\mathscr{P}$, $\mathscr{L}$, and the dual operator d (see Theorem 2.3). Let $\tilde{f}$ be the pl approximation of the function f with respect to the triangulation T and let

$$(3.10) \qquad H(x, y) = y - \tilde{f}(x) \quad \text{for each } (x, y) \in |\mathscr{M}|.$$

Then the function $H: |\mathscr{M}| \to \mathbf{R}^n$ is a pl function on $\mathscr{M}$. We consider the system of pl equations

$$(3.11) \qquad H(x, y) = 0, \quad (x, y) \in |\mathscr{M}|$$

as the basic model of our algorithm. By applying Theorem 2.1 to (3.11) we have the following main theorem.

THEOREM 3.3. *Suppose that the starting point v in C is not a stationary point. Then $(v, f(v))$ lies in $H^{-1}(0) \cap |\partial \mathcal{M}|$. Suppose further that $0 \in \mathbf{R}^n$ is a regular value of the function $H: |\mathcal{M}| \to \mathbf{R}^n$. Then the connected component S of $H^{-1}(0)$ containing $(v, f(v))$ is a path and it leads to a point $(\bar{x}, \bar{f}(\bar{x}))$ in $|\partial \mathcal{M}|$ such that $\bar{x}$ is a stationary point of the pl approximation $\bar{f}$ of f.*

Moreover, for any point (x, y) in S, let $t, 0 \leqslant t \leqslant 1$, be such that $x = (1 - t)v + tz$ for the unique point z in the face F for which x lies in vF, then x is a stationary point of $\bar{f}$ on the set $(1 - t)\{v\} + tC$.

PROOF. Since the starting point v is not a stationary point, $f(v)$ does not lie in F^* for any face F of C containing the point v. Therefore whether v is a vertex of C or not, we obtain from (3.8) that $(v, f(v))$ is in $\{v\} \times \bigcup(\{u\}^*|u$ is a vertex of C, $u \neq v) \subset |\partial \mathcal{M}|$. Since the point v is a vertex of the triangulation T, $\bar{f}(v) = f(v)$ and consequently $(v, f(v)) \in H^{-1}(0)$.

By (2.5) in Theorem 2.1 the connected component S of $H^{-1}(0)$ containing $(v, f(v))$ is a path. Since $\bar{f}$ is continuous and C is compact, $\bar{f}(C)$ is also compact. Hence $H^{-1}(0)$ is bounded and so is S. It is easily seen that the intersection of S and each cell of $\mathcal{M}$ is a bounded 1-cell if it is not empty. By the local finiteness property (2.3) of $\mathcal{M}$, S intersects finitely many cells of $\mathcal{M}$. Therefore S is compact and, by (2.6) of Theorem 2.1, bd S consists of two points in $|\partial \mathcal{M}|$, one of which is $(v, f(v))$. Let $(\bar{x}, \bar{y})$ be the other point of bd S and suppose $(\bar{x}, \bar{y}) \in \{v\} \times \bigcup(\{u\}^*|u$ is a vertex of C, $u \neq v)$. Since $(\bar{x}, \bar{y}) \in S \subset H^{-1}(0)$, $\bar{y} = \bar{f}(\bar{x}) = f(v)$. This contradicts that $(\bar{x}, \bar{y}) \neq (v, f(v))$. Then by Corollary 3.2 we have $(\bar{x}, \bar{y}) = (\bar{x}, \bar{f}(\bar{x}))$ lies in $\bigcup(F \times F^*|F \in \mathcal{F}, F \neq \{v\})$. This implies that $\bar{x}$ is a stationary point for $\bar{f}$.

The second part of the theorem follows from the fact that if $(x, y) \in S$ and $x \neq v$, then $y = \bar{f}(x)$ and $(x, y) \in vF \times F^*$ for some $F \in \mathcal{F}$. Notice that $F^* = F(t)^*$, where $F(t) = (1 - t)\{v\} + tF, 0 \leqslant t \leqslant 1$.

Theorem 3.3 shows that the path S when projected on C is the pl path of stationary points of $\bar{f}$ on $(1 - t)\{v\} + tC, 0 \leqslant t \leqslant 1$, that originates for $t = 0$ at the point v and terminates with an approximate stationary point $\bar{x}$. If $f(\bar{x})$ happens to lie in F^* for some face F of C containing the point $\bar{x}$, then $\bar{x}$ is a stationary point for f. Otherwise it is only an approximate stationary point. If the distance between $f(\bar{x})$ and F^* is not satisfactorily small, we take $\bar{x}$ as a new starting point, take a finer triangulation of C, and restart the algorithm. In the following lemma we discuss the accuracy of an approximate solution. As norm we use the Euclidean norm in $\mathbf{R}^n$.

LEMMA 3.4. *Let $\gamma = \sup\{\text{diam } f(\sigma)|\sigma \in T\}$, where for a set B, $\text{diam } B = \sup\{\|z^1 - z^2\| |z^1, z^2 \in B\}$. Let x be an approximate stationary point obtained by the algorithm, so that $x \in F$ and $\bar{f}(x) \in F^*$ for some face F of C. Then $f(x)$ lies in the γ-neighborhood of F^*, i.e., there is a $y \in F^*$ such that $\|y - f(x)\| \leqslant \gamma$.*

PROOF. Let $w^1, \ldots, w^{t+1}$ be the vertices of a t-simplex of T containing x, then $\bar{f}(x) = \sum_{j=1}^{t+1} \lambda_j f(w^j)$, where $\lambda_1, \ldots, \lambda_{t+1}$ are the convex combination coefficients such that $x = \sum_{j=1}^{t+1} \lambda_j w^j$ and $\sum_{j=1}^{t+1} \lambda_j = 1$. Therefore

$$\|\bar{f}(x) - f(x)\| = \left\| \sum_{j=1}^{t+1} \lambda_j f(w^j) - f(x) \right\| = \left\| \sum_{j=1}^{t+1} \lambda_j (f(w^j) - f(x)) \right\|$$

$$\leqslant \sum_{j=1}^{t+1} \lambda_j \|f(w^j) - f(x)\| \leqslant \gamma. \qquad \blacksquare$$

Since a polytope is compact and f is continuous on C, the error γ goes to zero as the mesh size $\delta = \sup\{\text{diam } \sigma | \sigma \in T\}$ of the triangulation T goes to zero. Let x^h be an approximate stationary point and γ^h be the error in Lemma 3.4 for a triangulation with mesh size δ^h. Suppose δ^h converges to zero as h goes to infinity. Then the sequence $\{x^h | h = 1, 2, \ldots\}$ has a cluster point $\bar{x}$ in C. For simplicity of notation we assume that this sequence itself converges to $\bar{x}$. Since the number of faces of C is finite, there is a face F of C and a subsequence $\{z^h | h = 1, 2, \ldots\}$ such that $z^h \in F$ and $f(z^h)$ is in the γ^h-neighborhood of F^* for all h. Therefore by the closedness of F and F^* we obtain that $\bar{x} \in F$ and $f(\bar{x}) \in F^*$. We thus have the following corollary.

COROLLARY 3.5. *Let x^h be the approximate stationary point found by the algorithm for a triangulation with mesh size δ^h, for $h = 1, 2, \ldots$. Suppose δ^h converges to zero as h goes to infinity. Then the sequence $\{x^h | h = 1, 2, \ldots\}$ has a cluster point and any cluster point is a stationary point of f on C.*

4. **Description of the algorithm.** In this section we will give a formal description of the steps of the algorithm for following the path S, under the assumption that the polytope C is full-dimensional and simple and that the m linear inequalities $a^i \cdot x \leqslant b_i$, $i = 1, \ldots, m$ are not redundant.

The system (3.11) is equivalent to the system

$$(4.1) \qquad\qquad y - \tilde{f}(x) = 0, \qquad (x, y) \in \sigma \times F^*,$$

where σ is a simplex of $T|_{v}F$ and F is a face of C not containing the starting point v. Let $I = \{i | 1 \leqslant i \leqslant m, a^i \cdot x = b_i \text{ for any point } x \text{ of } F\}$ and let $w^1, \ldots, w^{t+1}$ be the vertices of the simplex σ. Then (4.1) has a solution (x, y) if and only if the following system of linear equations has a solution $(\mu, \lambda) \in \mathbb{R}^{m+t+1}$,

$$(4.2) \qquad\qquad \sum_{i=1}^{m} \mu_i a^i - \sum_{j=1}^{t+1} \lambda_j f(w^j) = 0,$$

$$\sum_{j=1}^{t+1} \lambda_j = 1,$$

$$\mu_i \geqslant 0 \quad \text{for } i = 1, \ldots, m, \qquad \mu_i = 0 \quad \text{for } i \notin I,$$

$$\lambda_j \geqslant 0 \quad \text{for } j = 1, \ldots, t + 1.$$

A line segment of solutions (μ, λ) to (4.2) corresponds to a linear piece of the path S and can be followed by making a linear programming pivot step in (4.2). At the start of the algorithm we have to find the simplex σ and the cone F^* such that $(v, f(v)) \in \sigma \times F^*$. The cone F^* can be found by solving the linear programming problem

$$(4.3) \quad \text{minimize } b \cdot \mu \quad \text{subject to } \sum_{i=1}^{m} \mu_i a^i - \lambda_1 f(v) = 0,$$

$$\mu_i \geqslant 0 \quad \text{for } i = 1, \ldots, m,$$

$$\lambda_1 = 1,$$

which is the dual problem of

$$\text{maximize } f(v) \cdot z \quad \text{subject to } z \in C.$$

The optimal solution of (4.3) gives us the cell of $\mathscr{L}$ in which the end point $(v, f(v))$ of the path S lies. Namely, let I be the set of indices such that $\mu_i > 0$ at the optimal solution and let F be the face of C defined by the system of equations $a^i \cdot x = b_i$ for $i \in I$. Then $(v, f(v)) = (v, \Sigma_{i \in I} \mu_i a^i)$ lies in $\{v\} \times F^* \subset vF \times F \in \mathscr{L}$. Barring degeneracy of the linear programming problem (4.3), the set I has exactly n elements, so that F is a vertex, say $\{u\}$, of C and $\dim vF = 1$. Define the simplex σ to be the 1-dimensional simplex of $T|vF$ having the starting point v as a facet, i.e., $\sigma = \{v, w\}$ with w a vertex in $T|vF$. Thus we leave the starting point v along the line segment $vF = [v, u]$ in the direction of the vertex u of C by making a pivot step in (4.2) with the variable λ_2, whose column entry is $(-f^T(w), 1)^T$.

We now show that in general the set of solutions of (4.2) is bounded. Suppose, to the contrary, that the set of solutions has an unbounded ray $\{(\mu^0, \lambda^0) + \alpha(\Delta\mu, \Delta\lambda) | \alpha \geq 0\}$. Since $\Sigma_{j=1}^{t+1}(\lambda_j^0 + \alpha \Delta\lambda_j) = 1$ and $\lambda_j^0 + \alpha \Delta\lambda_j \geq 0$ for any $\alpha \geq 0$, we have $\Delta\lambda = 0$. Therefore $\Sigma_{i \in I} \Delta\mu_i a^i = 0$ and $\Delta\mu_i \geq 0$ for $i \in I$. Since a point of F satisfies $a^i \cdot x = b_i$ for any $i \in I$, this implies $\dim C < n$, a contradiction. Hence the set of solutions to (4.2) is bounded and consequently has two distinct basic solutions. When some λ_j vanishes at a basic solution, the point $(x, y) = (\Sigma_{j=1}^{t+1} \lambda_j w^j, \Sigma_{i \in I} \mu_i a^i)$ lies in a facet $\tau \times F^*$ of $\sigma \times F^*$, where τ is a facet of σ. Then either τ is a facet of just one other simplex $\bar{\sigma}$ of the triangulation of vF or τ lies in the boundary of vF. On the other hand, when some μ_i vanishes, we in general cannot conclude that (x, y) lies on a facet of $\sigma \times F^*$. This is due to the fact that the cone F^* could contain more vectors a^i than its dimension. When the polytope C is a simple polytope and the system of linear inequalities defining C is nonredundant, the number of inequalities such that $a^i \cdot x = b_i$ for any point x of F is equal to $n - \dim F = \dim F^*$ so that F^* has exactly $\dim F^*$ coefficient vectors a^i. In this case we might conclude that (x, y) is on a facet $\sigma \times E^*$ of $\sigma \times F^*$.

For a subset I of the index set $\{1, \dots, m\}$, let

$$F(I) = \{x \in C | a^i \cdot x = b_i \text{ for all } i \in I\}.$$

Then $F(I)$ is a face of C unless it is empty. Let $\mathscr{I}$ be the class of index sets $I \subset \{1, \dots, m\}$ such that $F(I)$ is a nonempty face of C. Under the above assumption that $\dim C = n$, C is a simple polytope, and the linear inequalities defining C are nonredundant, we have the following properties

(i) for each face F of C the set $I \in \mathscr{I}$ such that $F(I) = F$ is unique and identical with the set $\{i | a^i \cdot x = b_i \text{ for all } x \in F\}$;

(ii) $\dim F(I) = n - |I|$;

(iii) G is a facet of $F(I)$ if and only if $G = F(I \cup \{j\})$ for some $j \notin I$ with $I \cup \{j\} \in \mathscr{I}$;

(iv) G has $F(I)$ as a facet if and only if $G = F(I \setminus \{k\})$ for some $k \in I$.

Note that $I \setminus \{k\} \in \mathscr{I}$ for all $I \in \mathscr{I}$ and any $k \in I$. Now starting at v the algorithm generates the path S by making alternating lp pivot steps in (4.2) and replacement steps in the triangulation of $vF(I)$, for varying $I \in \mathscr{I}$, as described in the flow chart given in Figure 4.1.

The algorithm terminates as soon as one of the following cases occurs:

(1) τ lies in $F(I)$.

(2) μ_i becomes 0 for some $i \in I$ and $F(I \setminus \{i\})$ contains the starting point v (including the case where $I \setminus \{i\}$ is empty).

In both cases let x be equal to $\Sigma_{j=1}^{t+1} \lambda_j w^j$. In case (1), $(x, \hat{f}(x)) = (x, y)$ lies in $F(I) \times F(I)^*$. In case (2) we have $vF(I) \subset F(I \setminus \{i\})$ because $F(I)$ is a facet of $F(I \setminus \{i\})$ and $F(I \setminus \{i\})$ contains the starting point v. Since x lies in a simplex σ
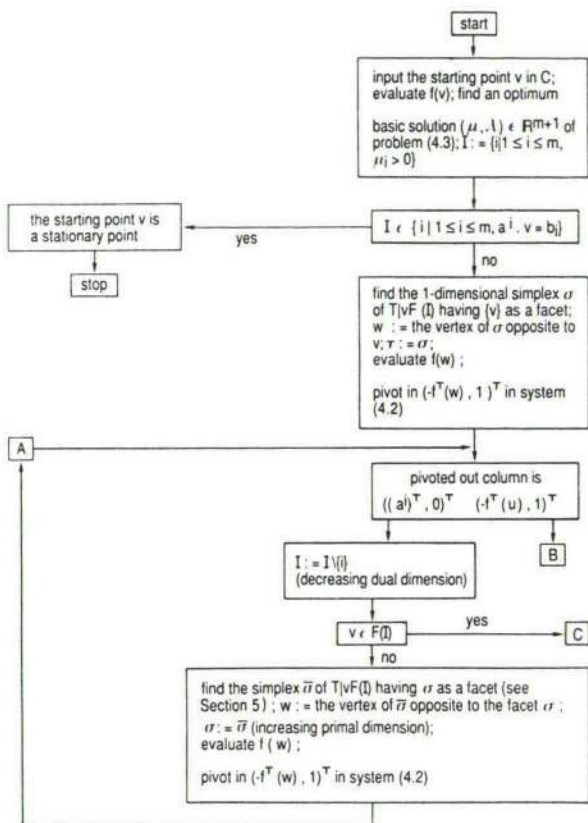
FIGURE 4.1. Flow chart of the steps of the algorithms.

in $vF(I)$, we have $(x, \tilde{f}(x)) = (x, y)$ lies in $F(I \setminus \{i\}) \times F(I \setminus \{i\})^*$. Thus in either case we have that x is a stationary point for the pl approximation $\tilde{f}$ of f.

We show two examples of the trajectory of the algorithm in Figure 4.2. In the first example $f(v)$ lies in $F(\{1, 2\})^*$. Then the algorithm leaves the starting point v along the line segment $vF(\{1, 2\})$. When the column $(-f^T(w^2), 1)^T$ has been pivoted in, the column $((a^1)^T, 0)^T$ is pivoted out, i.e., I becomes $\{2\}$ and the dual dimension decreases. To increase the primal dimension the vertex w^3 is found and $(-f^T(w^3), 1)^T$ is pivoted in. After several pivot operations and replacements in $vF(\{2\})$ we obtain the 1-simplex $\rho = \{w^7, w^8\}$ in $vF(\{2, 3\})$, and the primal dimension decreases. To increase the dual dimension we pivot in the column $((a^3)^T, 0)^T$. Then the column $((a^2)^T, 0)^T$ is pivoted out and the dual dimension does not change. To move in $vF(\{3\})$ we find the vertex w^9. After several iterations we have the simplex $\tau = \{w^{12}, w^{13}\}$ which lies in $F(\{3\})$. Case (1) occurs and the algorithm terminates with an approximate stationary point in $F(\{3\})$.

In the second example $f(v)$ lies in $F(\{3, 4\})^*$, and we go along $vF(\{3, 4\})$. After several iterations we obtain $\tau = \{u^7, u^8\}$ in $vF(\{4, 5\})$, and the primal dimension decreases. To increase the dual dimension $((a^5)^T, 0)^T$ is pivoted in, $(-f^T(u^7), 1)^T$ is pivoted out, and we continue in $vF(\{4, 5\})$. When $(-f^T(u^9), 1)^T$ is pivoted in,
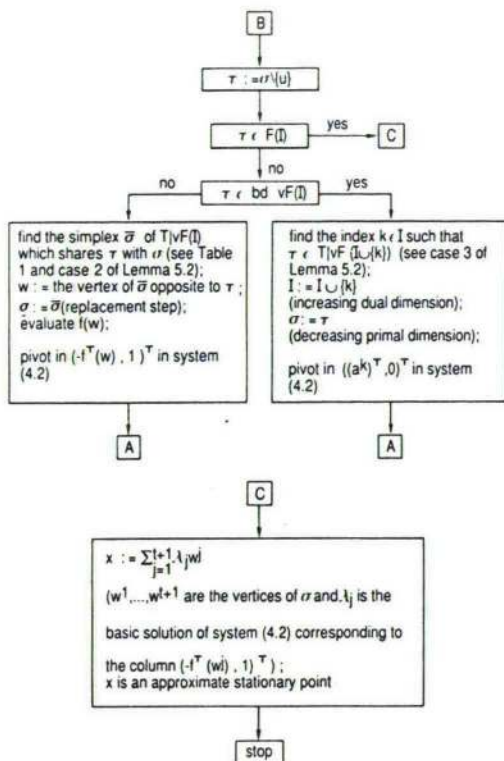
FIGURE 4.1. *Continued*

$((a^4)^T, 0)^T$ is pivoted out, and I becomes $\{5\}$. The face $F(\{5\})$ of C contains the starting point v. Case (2) occurs so that the algorithm terminates with an approximate stationary point in the simplex $\{u^8, u^9\}$ in $F(\{5\})$.

5. **Triangulation of the polytope.** The simplicial subdivision of the set C which underlies the algorithm presented in this paper must be such that it subdivides each set $vF(I)$, $I \in \mathscr{I}$, for which $v \notin F(I)$, into t-dimensional simplices with $t = n - |I| + 1$. In this section we briefly describe such a triangulation. The triangulation is closely related to the V-triangulation of the product space of unit simplices introduced in [3]. We again assume that the set C is full-dimensional in R^n, that no constraint $a^i \cdot x = b_i$, $i = 1, \ldots, m$, is redundant, and that C is simple so that at each vertex of C, $a^i \cdot x = b_i$ holds for exactly n indices i, $1 \leqslant i \leqslant m$. The triangulation of each set $vF(I)$, $v \notin F(I)$, is completely determined by projections of v on the faces of $F(I)$ and on $F(I)$ itself. The projection of the starting point v on a face $F(I)$ of C can be any relative interior point of $F(I)$ and is denoted by $v(I)$. These points are automatically generated during the algorithm, but once a projection $v(I)$ has been chosen it is fixed during the rest of the algorithm. Clearly, if I consists of n elements, then $F(I) = \{v(I)\}$ is a vertex of C. For general I, for which $v \notin F(I)$, let $v(I(n))$ be a vertex of $F(I)$ i.e., $|I(n)| = n$ and $I \subset I(n) \in \mathscr{I}$, and let $(\gamma_1, \ldots, \gamma_{t-1}) = \gamma(I(n) \setminus I)$ be a permutation of the $t - 1 = n - |I|$ elements of the set $I(n) \setminus I$. Then the subset
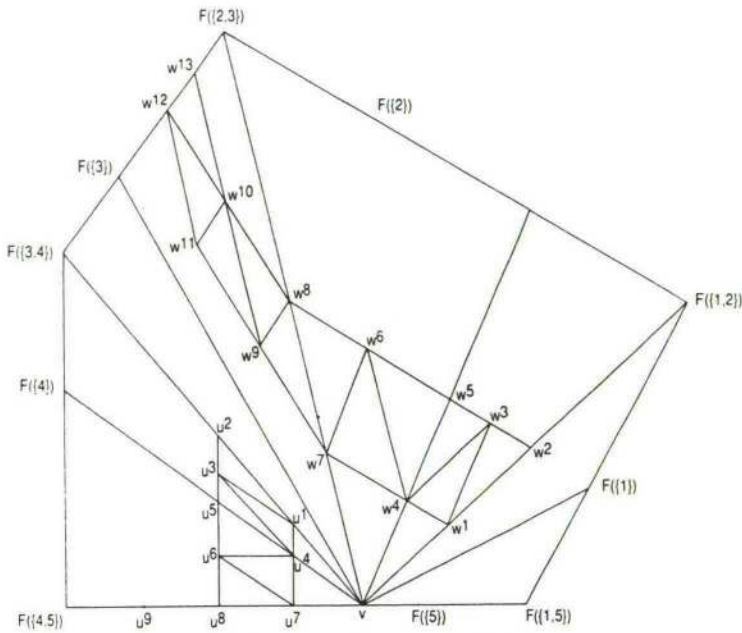
FIGURE 4.2. Two possible trajectories of the algorithm.

$vF(I, \gamma(I(n) \setminus I))$ of $vF(I)$ is defined to be the convex hull of v and the projections $v(I(n))$, $v(I(n) \setminus \{\gamma_1\}), \ldots, v(I(n) \setminus \{\gamma_1, \ldots, \gamma_{t-2}\})$, and $v(I)$, i.e.,

$$vF(I, \gamma(I(n) \setminus I)) = \left\{ x \in C \,|\, x = v + \sum_{j=0}^{t-1} \alpha_j q(j), \right.$$

$$\left. 0 \leqslant \alpha_{t-1} \leqslant \cdots \leqslant \alpha_1 \leqslant \alpha_0 \leqslant 1 \right\}$$

where $q(0) = v(I(n)) - v$, and for $j = 1, \ldots, t - 1$,

$$q(j) = v(I \cup \{\gamma_{j+1}, \ldots, \gamma_{t-1}\}) - v(I \cup \{\gamma_j, \ldots, \gamma_{t-1}\}).$$

LEMMA 5.1. *The collection of the sets $vF(I, \gamma(I(n) \setminus I))$ over all permutations $\gamma(I(n) \setminus I)$ and over all vertices $v(I(n))$ of $F(I)$ is a triangulation of the set $vF(I)$. Furthermore, the collection of the sets $vF(\{i\}, \gamma(I(n) \setminus \{i\}))$ over all permutations $\gamma(I(n) \setminus \{i\})$ and vertices $v(I(n))$ of $F(\{i\})$ and over all i, is a triangulation of C.*

For $n = 2$, the simplicial subdivision of C in the subsets $vF(I, \gamma(I(n) \setminus I))$ is illustrated in Figure 5.1. In this figure the set $vF(\{2\})$ consists of the two triangles $vF(\{2\}, (3))$ and $vF(\{2\}, (1))$, for $v(\{2, 3\})$ and $v(\{2, 1\})$ are the two vertices of $F(\{2\})$.

A simplicial subdivision with arbitrary small mesh size is obtained by triangulating each simplex $vF(I, \gamma(I(n) \setminus I))$ into t-dimensional simplices in the same way as the well-known Q-triangulation triangulates the n-dimensional unit simplex, for example see [11]. Let d be an arbitrary positive integer.
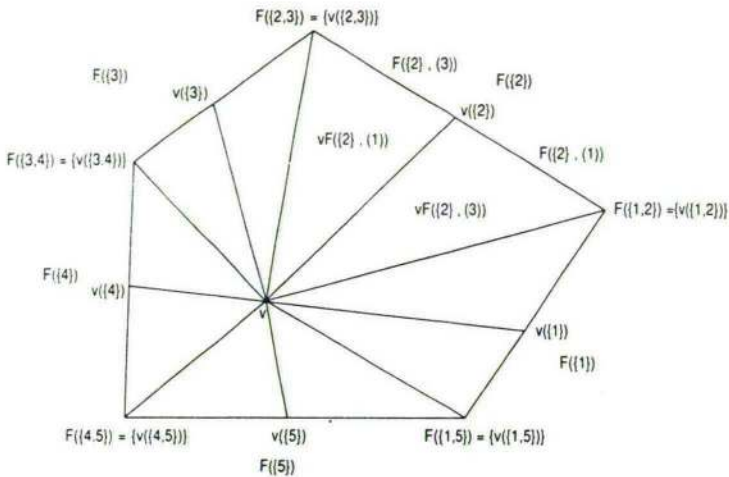
FIGURE 5.1. Triangulation of C into $vF(I, \gamma(I(n) \setminus I))$'s.

DEFINITION 5.1. The set $G^d(I, \gamma(I(n) \setminus I))$ is the collection of t-simplices $\sigma(w^1, \pi)$ in $vF(I, \gamma(I(n) \setminus I))$ with vertices $w^1, \ldots, w^{t+1}$ such that

(i) $w^1 = v + \sum_{j=0}^{t-1} a(j) d^{-1} q(j)$ where $a(0), \ldots, a(t-1)$ are integers such that $0 \leqslant a(t-1) \leqslant \cdots \leqslant a(0) \leqslant d - 1$;

(ii) $\pi = (\pi_1, \ldots, \pi_t)$ is a permutation of the t elements of the set $\{0, \ldots, t-1\}$ such that $s < s'$ if for some $j \in \{0, \ldots, t-2\}$, $\pi_s = j$, $\pi_{s'} = j + 1$, and $a(j) = a(j+1)$;

(iii) $w^{t+1} = w^t + d^{-1} q(\pi_i)$, $i = 1, \ldots, t$.

The set $G^d(I, \gamma(I(n) \setminus I))$ is a simplicial subdivision of $vF(I, \gamma(I(n) \setminus I))$ with grid size d^{-1} or refinement factor d. Taking the union over all permutations γ and over all admissible index sets $I(n)$ and I, we obtain a triangulation of C with grid size d^{-1} having mesh size less than or equal to $d^{-1} \operatorname{diam} C$.

As described in §4, the algorithm follows the pl path S defined in Theorem 3.3 leading from v to an approximate solution by making alternating linear programming pivot steps in system (4.2) and replacement steps in the underlying triangulation. In this way the algorithm generates a sequence of adjacent simplices in $vF(I)$ for varying I of varying dimension $t = n - |I| + 1$. When, with respect to some t-simplex $\sigma(w^1, \pi)$ in $vF(I, \gamma(I(n) \setminus I))$, the variable λ_s, for some s, $1 \leqslant s \leqslant t + 1$, becomes zero through a linear programming pivot step in (4.2), then the replacement step determines the unique t-simplex $\bar{\sigma}$ in $vF(I)$ sharing with σ the facet τ opposite the vertex w^s unless this facet lies in the boundary of $vF(I)$. If τ does not lie in the boundary of the set $vF(I, \gamma(I(n) \setminus I))$, then $\bar{\sigma} = \sigma(\bar{w}^1, \bar{\pi})$ as given in Table 1, where $a_{j+1} = a(j)$, $j = 0, \ldots, t-1$, $a_i = 0$, $i > t$, and $e(j-1)$ is the jth unit vector in $\mathbf{R}^n$, $j = 1, \ldots, n$.

TABLE 1

Parameters of $\bar{\sigma}$ if the vertex w^s of $\sigma(w^1, \pi)$ is replaced

	$\bar{w}^1$	$\bar{\pi}$	$\bar{a}$
$s = 1$	$w^1 + d^{-1} q(\pi_1)$	$(\pi_2, \ldots, \pi_t, \pi_1)$	$a + e(\pi_1)$
$1 < s < t + 1$	w^1	$(\pi_1, \ldots, \pi_{s-2}, \pi_s, \pi_{s-1}, \ldots, \pi_t)$	a
$s = t + 1$	$w^1 - d^{-1} q(\pi_t)$	$(\pi_t, \pi_1, \ldots, \pi_{t-1})$	$a - e(\pi_t)$

The algorithm continues with $\bar{\sigma}$ by making an lp pivot step in (4.2) with $(-f^T(\bar{w}), 1)^T$, where $\bar{w}$ is the vertex of $\bar{\sigma}$ opposite the facet τ.

LEMMA 5.2.   *The facet τ opposite to the vertex w^s of $\sigma(w^1, \pi)$ in $vF(I, \gamma(I(n) \setminus I))$ lies in the boundary of this set if and only if one of the following cases holds:*
  (1) $s = 1$, $\pi_1 = 0$, *and* $a(0) = d - 1$;
  (2) $1 < s < t + 1$, $\pi_s = h + 1$, $\pi_{s-1} = h$, *and* $a(h) = a(h + 1)$ *for some* $h \in \{0, 1, \ldots, t - 2\}$;
  (3) $s = t + 1$, $\pi_t = t - 1$, *and* $a(t - 1) = 0$.

In case (1) the facet τ lies in the facet of $vF(I, \gamma(I(n) \setminus I))$ opposite to v and the algorithm terminates with the approximate solution $\bar{x} = \sum_{i=2}^{t+1} \lambda_i w^i$ in the face $F(I)$ of C. In case (2) and if $h \geq 1$, the facet τ is a facet of the t-simplex $\bar{\sigma} = \sigma(w^1, \pi)$ in $vF(I)$ lying in the subset $vF(I, \bar{\gamma}(I(n) \setminus I))$ with

$$\bar{\gamma}(I(n) \setminus I) = (\gamma_1, \ldots, \gamma_{h-1}, \gamma_{h+1}, \gamma_h, \gamma_{h+2}, \ldots, \gamma_{t-1}).$$

If not determined before, a projection $v(I \cup \{\gamma_h, \gamma_{h+2}, \ldots, \gamma_{t-1}\})$ of v on the face $F(I \cup \{\gamma_h, \gamma_{h+2}, \ldots, \gamma_{t-1}\})$ of C can be obtained as follows. Make a pivot step in the (primal) system

$$(5.1) \qquad Ax + \sum_{i \notin I(n)} \rho_i e(h) = b, \qquad \rho_i \geq 0, \quad i \notin I(n),$$

at $x = v(I(n))$ by increasing $\rho_{\gamma_{h+1}}$ from zero, where A is the $m \times n$ matrix with $(a^i)^T$ as the ith row, $i = 1, \ldots, m$. Let ρ_k become zero for some (unique) index $k \notin I(n) \cup \{\gamma_{h+1}\}$. Then $v(I \cup \{\gamma_h, \gamma_{h+2}, \ldots, \gamma_{t-1}\})$ can be chosen as the barycenter of (or any other interior point in) the convex hull of the new vertex $v(I(n) \cup \{k\} \setminus \{\gamma_{h+1}\})$ of C obtained from the pivot step in (5.1) and the projections $v(I \cup \{\gamma_h, \ldots, \gamma_{t-1}\}), \ldots, v(I \cup \{\gamma_1, \ldots, \gamma_{t-1}\})$. If, in case (2), $h = 0$, then τ is a facet of the t-simplex $\bar{\sigma} = \sigma(w^1, \pi)$ in $vF(I)$ lying in the subset $vF(I, \gamma(\bar{I}(n) \setminus I))$ with $\bar{I}(n)$ and γ defined as follows. Make an lp pivot step in (5.1) at $x = v(I(n))$ by increasing ρ_{γ_1} from zero and let ρ_k become zero for some $k \notin I(n) \cup \{\gamma_1\}$. Then $\bar{I}(n) = I(n) \cup \{k\} \setminus \{\gamma_1\}$ and $\gamma(\bar{I}(n) \setminus I) = (k, \gamma_2, \ldots, \gamma_{t-1})$. The vertex $v(\bar{I}(n))$ of C is the new solution for x in (5.1) after the pivot step. In both subcases of case (2) the algorithm continues with making an pivot step in (4.2) with $(-f^T(\bar{w}), 1)^T$, where $\bar{w}$ is the vertex of the new t-simplex $\bar{\sigma}$ opposite the facet τ. In case (3) of Lemma 5.2 the facet τ lies in the facet $vF(I \cup \{\underline{\gamma}_{t-1}\})$ of $vF(I)$. More precisely, τ is the $(t - 1)$-simplex $\sigma(w^1, \bar{\pi})$ in $vF(\bar{I}, \gamma(I(n) \setminus \bar{I}))$, where $\bar{I} = I \cup \{\gamma_{t-1}\}$, $\gamma(I(n) \setminus \bar{I}) = (\gamma_1, \ldots, \gamma_{t-2})$, and $\bar{\pi} = (\pi_1, \ldots, \pi_{t-1})$. The algorithm now continues with making a pivot step in (4.2) with $((a^h)^T, 0)^T$ where $h = \gamma_{t-1}$.

Finally, if, through a linear programming pivot step in (4.2), μ_h becomes 0 for some $h \in I$, then the algorithm terminates with the approximate solution $\bar{x} = \sum_j \lambda_j w^j$ if $I = \{h\}$ or the starting point v lies in the face $F(I \setminus \{h\})$ of C. Otherwise, the simplex $\sigma(w^1, \pi)$ is a facet of a unique $(t + 1)$-simplex $\bar{\sigma}$ in $vF(I \setminus \{h\})$. More precisely, $\bar{\sigma}$ is the $(t + 1)$-simplex $\sigma(w^1, \bar{\pi})$ in $vF(\bar{I}, \gamma(I(n) \setminus \bar{I}))$, where $\bar{I} = I \setminus \{h\}$, $\gamma(I(n) \setminus \bar{I}) = (\gamma_1, \ldots, \gamma_{t-1}, h)$, and $\bar{\pi} = (\pi_1, \ldots, \pi_t, t)$. The algorithm continues by making a pivot step in (4.2) with $(-f^T(\bar{w}), 1)^T$, where $\bar{w}$ is the vertex of $\bar{\sigma}$ opposite the facet σ. Let k be the (unique) index for which ρ_k becomes 0 if a pivot step is made in (5.1) at $x = v(I(n))$ when ρ_h is increased from zero, then the projection $v(I \setminus \{h\})$ can be chosen as the barycenter of the convex hull of the vertex $v(I(n) \cup \{k\} \setminus \{h\})$ obtained from the pivot step in (5.1) and the projections $v(I), \ldots, v(I \cup \{\gamma_1, \ldots, \gamma_{t-1}\})$.

6. Concluding remarks.   In the previous sections we assumed that the polytope C in $\mathbf{R}^n$ is n-dimensional and simple and that none of the constraints $a^i \cdot x \leq b_i$ is redundant. If the set C is a lower-dimensional set in $\mathbf{R}^n$ we may assume under the latter two conditions that C can be expressed as

$$C = \left\{ x \in \mathbf{R}^n | a^i \cdot x \leq b_i,\ i = 1, \ldots, m, \text{ and } c^i \cdot x = d_i,\ i = 1, \ldots, m^1 \right\}$$

while $\dim C = n - m^1$, for some m^1, $0 \leq m^1 \leq n$. Then for index sets I, $I \subset \mathcal{I}$, the cone $F(I)^*$ is equal to

$$F(I)^* = \left\{ y \in \mathbf{R}^n | y = \sum_{i \in I} \mu_i a^i + \sum_{i=1}^{m^1} \nu_i c^i,\ \mu_i \geq 0 \text{ for } i \in I \right\},$$

where again

$$F(I) = \left\{ x \in C | a^i \cdot x = b_i \text{ for } i \in I \right\}.$$

The dimension of such an $F(I)$ is equal to $t - 1 = n - m^1 - |I|$. The algorithm is now the same as described in §4 except that the pivot steps are made in the system

$$\sum_{i \in I} \mu_i a^i + \sum_{i=1}^{m^1} \nu_i c^i - \sum_{j=1}^{t+1} \lambda_j f(w^j) = 0,$$

$$\sum_{j=1}^{t+1} \lambda_j = 1,$$

$$\mu_i \geq 0 \quad \text{for } i \in I, \qquad \lambda_j \geq 0 \quad \text{for } j = 1, \ldots, t+1.$$

An initial starting point v for the algorithm can be obtained by applying the simplex method for linear programming on C, bringing the columns c^i, $i = 1, \ldots, m^1$, in the basis and yielding one of the vertices of C as starting point.

In case the polytope C is not simple there might be more than $n - t + 1$ constraints $a^i \cdot x = b_i$, $i \in I$, where $t = n - |I| + 1$, which determine a $(t - 1)$-dimensional face of C. Let for a particular $(n - k)$-dimensional face F of C the set I be the set of indices such that

$$I = \left\{ i | a^i \cdot x = b_i \text{ for all } x \in F \right\}.$$

Assuming nonredundancy, for any subset I^+ of I consisting of k elements the cone $F(I^+)^*$ is k-dimensional and the cone $F(I)^*$ is then the union of $F(I^+)^*$ over all subsets I^+ of I having k elements. The algorithm now makes linear programming pivot steps in (4.2) with I replaced by some I^+. If, by a pivot step in (4.2), μ_h becomes zero for some $h \in I^+$, then it is first checked whether the vector $y = \sum_{i \in I} \mu_i a^i$ keeps moving in the same direction when some μ_p, $p \in I \setminus I^+$, is increased in (4.2) from zero. If so, then I^+ becomes $I^+ \cup \{p\} \setminus \{h\}$ and a pivot step is made with the variable μ_p. Otherwise I^+ becomes $I^+ \setminus \{h\}$ and the algorithm continues in $vF(I')$ with I' containing $I^+ \setminus \{h\}$ as before. Notice that the set I' is automatically determined when making a pivot step in (5.1).

Special cases of the set C are cubes or simplices. In case the set C is the n-dimensional cube $C = \{x \in \mathbf{R}^n | a \leq x \leq b\}$ for two vectors a and b in $\mathbf{R}^n$ with $a_i < b_i$, $i = 1, \ldots, n$, the stationary point problem reduces to finding an x^* in C such

that for all i

$$x_i^* = b_i \text{ implies } f_i(x^*) \geq 0,$$

$$a_i < x_i^* < b_i \quad \text{implies } f_i(x^*) = 0, \text{ and}$$

$$x_i^* = a_i \text{ implies } f_i(x^*) \leq 0.$$

A simplicial algorithm for this problem was introduced in [14]. However, that algorithm has only $2n$ rays to leave the arbitrarily chosen starting point, one ray to each facet of C. The algorithm deviced in this paper has 2^n rays, one to each vertex of C. The difference between both algorithms can be compared with the difference of Lemke's algorithm and the algorithm proposed in [15] for solving the linear complementarity problem with upper and lower bounds. In the latter paper it has been argued that the algorithm with 2^n rays is very natural.

In case C is an n-dimensional simplex $\tau(w^1, \ldots, w^{n+1})$ the algorithm proposed in this paper is similar to the algorithm proposed in [3]. The latter algorithm was developed for problems on the n-dimensional unit simplex in $\mathbb{R}^{n+1}$ with w^i the ith unit vector in $\mathbb{R}^{n+1}$.

References

[1] Dafermos, S. (1982). Relaxation Algorithm for the General Asymmetric Traffic Equilibrium Problem. *Transportation Sci.* **16** 231–240.

[2] _____. (1983). An Iterative Scheme for Variational Inequalities. *Math. Programming* **26** 40–47.

[3] Doup, T. M. and Talman, A. J. J. (1987). A New Variable Dimension Simplicial Algorithm to Find Equilibria on the Product Space of Unit Simplices. *Math. Programming* **37** 319–355.

[4] _____, van den Elzen, A. H. and Talman, A. J. J. (1987). Simplicial Algorithms for Solving the Nonlinear Complementarity Problem on the Simplotope. in: *Computation and Modelling of Economic Equilibria.* A. J. J. Talman and G. van der Laan (eds.), North-Holland, Amsterdam, 125–153.

[5] Eaves, B. C. (1976). A Short Course in Solving Equations with PL Homotopies. *SIAM-AMS Proc.* **9** 73–143.

[6] _____. (1978). Computing Stationary Points. *Math. Programming Study* **7** 1–14.

[7] _____. (1978). Computing Stationary Points, Again. in: *Nonlinear Programming. 3,* O. L. Mangasarian et al. (eds.), Academic Press, New York, 391–405.

[8] Garcia, C. B. and Zangwill, W. I. (1981). *Pathways to Solutions, Fixed Points and Equilibria.* Prentice-Hall, Englewood Cliffs, NJ.

[9] Kojima, M. and Yamamoto, Y. (1982). Variable Dimension Algorithms: Basic Theory, Interpretations and Extensions of Some Existing Methods. *Math. Programming* **24** 177–215.

[10] _____ and _____. (1984). A Unified Approach to the Implementation of Several Restart Fixed Point Algorithms and a New Variable Dimension Algorithm. *Math. Programming* **28** 288–328.

[11] van der Laan, G. and Talman, A. J. J. (1979). A Restart Algorithm for Computing Fixed Points without an Extra Dimension. *Math. Programming* **17** 74–84.

[12] _____ and _____. (1981). A Class of Simplicial Subdivisions for Restart Fixed Point Algorithms without an Extra Dimension. *Math. Programming* **20** 33–48.

[13] _____ and _____. (1986). Simplicial Algorithms for Finding Stationary Points, a Unifying Description. *J. Optim. Theory Appl.* **50** 262–281.

[14] _____ and _____. (1987). Simplicial Approximation of Solutions to the Nonlinear Complementarity Problem with Lower and Upper Bounds. *Math. Programming* **37** 1–17.

[15] _____ and _____. (1985). An Algorithm for the Linear Complementarity Problem with Upper and Lower Bounds. Research Memorandum FEW 200, Tilburg University, Tilburg, The Netherlands, December; *J. Optim. Theory Appl.* (to appear).

[16] Smith, M. J. (1979). The Existence, Uniqueness and Stability of Traffic Equilibria. *Transportation Res.*
      13B 295–304.
[17] Todd, M. J. (1978). Improving the Convergence of Fixed Point Algorithms. *Math. Programming
      Studies* **7** 151–169.
[18] Wright, A. H. (1981). The Octahedral Algorithm, a New Simplicial Fixed Point Algorithm. *Math.
      Programming* **21** 47–69.
[19] Yamamoto, Y. (1984). A Unifying Model based on Retraction for Fixed Point Algorithms. *Math.
      Programming* **28** 192–197.

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         SDU uitgeverij, 1989, pp. 81 - 106.

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