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A SIMPLIFIED METHOD OF ELASTIC-STABILITY ANALYSIS FOR THIN CYLINDRICAL SHELLS

By S. B. Batdorf

SUMMARY

This paper develops a new method for determining the buckling stresses of cylindrical shells under various loading conditions. For convenience of exposition, it is divided into two parts.

In part I, the equation for the equilibrium of cylindrical shells introduced by Donnell in NACA Report No. 479 to find the critical stresses of cylinders in torsion is applied to find critical stresses for cylinders with simply supported edges under other loading conditions. It is shown that by this method solutions may be obtained very easily and the results in each case may be expressed in terms of two nondimensional parameters, one dependent on the critical stress and the other essentially determined by the geometry of the cylinder. The influence of boundary conditions related to edge displacements in the shell median surface is discussed. The accuracy of the solutions found is established by comparing them with previous theoretical solutions and with test results. The solutions to a number of problems concerned with buckling of cylinders with simply supported edges on the basis of a unified viewpoint are presented in a convenient form for practical use.

In part II, a modified form of Donnell's equation for the equilibrium of thin cylindrical shells is derived which is equivalent to Donnell's equation but has certain advantages in physical interpretation and in ease of solution, particularly in the case of shells having clamped edges. The solution of this modified equation by means of trigonometric series and its application to a number of problems concerned with the shear buckling stresses of cylindrical shells are discussed. The question of implicit boundary conditions also is considered.

INTRODUCTION

The recent emphasis on aircraft designed for very high speed has resulted in a trend toward thicker skin and fewer stiffening elements. As a result of this trend, a larger fraction of the load is being carried by the skin and thus ability to predict accurately the behavior of the skin under load has become more important. Accordingly, it was considered desirable to provide the designer with more information on the buckling of curved sheet than has been available in the past. In carrying out a theoretical research program for this purpose, a method of analysis was developed which is believed to be simpler to apply than those generally appearing in the literature. The specific problems solved as a part of this research program are treated in detail in other papers. The purpose of this paper, which is discussed in two parts, is to present the method of analysis that was developed to solve these problems.

In part I, the stability of a stressed cylindrical shell is analyzed in terms of Donnell's equation, a partial differential equation for the radial displacement w, which takes into account the effects of the axial displacement u and the circumferential displacement v. Part I shows the manner in which this equation can be used to obtain relatively easy solutions to a number of problems concerning the stability of cylindrical shells with simply supported edges. The results of the solution of this equation are shown to take on a simple form by the use of the parameter k (similar to the buckling-stress coefficients for flat plates) to represent the state of stress in the shell and the parameter Z to represent the dimensions of the shell, where Z is defined by the following equations:

For a cylinder of length L

$$Z=\frac{L^2}{rt}\sqrt{1-\mu^2}$$

and for a curved panel of width b

$$Z=\frac{b^2}{rt}\sqrt{1-\mu^2}$$

where

r radius of curvature

t thickness of shell

and

μ Poisson's ratio for material

The accuracy of Donnell's equation is established by comparisons of the results found by its use with the results found by other methods and by experiment.

In the simplest method that has been found for solving Donnell's equation, the radial displacement w is represented by a trigonometric series expansion. This method can be used to great advantage for cylinders or curved panels with simply supported edges but leads to incorrect results when applied uncritically to cylinders or panels with clamped edges.

In part II, an equation is derived which is equivalent to Donnell's equation but is adapted to solution for clamped as well as simply supported edges by means of trigonometric series. This modified equation retains the advantages of Donnell's equation in ease of solution and simplicity of results. The solution of the modified equation by means of the Galerkin method is explained, and the results obtained by this approach in a number of problems concerned with the shear buckling stresses of cylindrical shells are given in graphical form and discussed briefly. Boundary conditions implied by the method of solution of the modified equation are also discussed.

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SYMBOLS

a length of curved panel (longer dimension)

b width of curved panel (shorter dimension)

d diameter of cylinder

i, j, m, n, p, q integers

p lateral pressure, positive inward

r radius of cylindrical shell

t thickness of cylindrical shell

u displacement in axial (x-) direction of point on shell median surface

v displacement in circumferential (y-) direction of point on shell median surface

w displacement in radial direction of point on shell median surface; positive outward

x axial coordinate

y circumferential coordinate

$a_{mn}, b_{mn}, c_{mn}, d_{mn}$ numerical coefficients

k_s shear-stress coefficient $\left(\frac{\tau t L^2}{D\pi^2}\right.$ for cylinder or $\frac{\tau t b^2}{D\pi^2}$ for curved panel or infinitely long curved strip$\left.\right)$

k_x axial compressive-stress coefficient $\left(\frac{\sigma_x t L^2}{D\pi^2}\right.$ for cylinder or $\frac{\sigma_x t b^2}{D\pi^2}$ for curved panel or infinitely long curved strip$\left.\right)$

k_y circumferential compressive-stress coefficient $\left(\frac{\sigma_y t L^2}{D\pi^2}\right.$ for cylinder or $\frac{\sigma_y t b^2}{D\pi^2}$ for curved panel or infinitely long curved strip$\left.\right)$

C_p hydrostatic-pressure coefficient $\left(\frac{prL^2}{D\pi^2}\right)$

w_0 amplitude of deflection function

D plate flexural stiffness per unit length $\left(\frac{Et^3}{12(1-\mu^2)}\right)$

E Young's modulus

F Airy's stress function for the median-surface stresses produced by the buckle deformation $\left(\frac{\partial^2 F}{\partial y^2}\right.$, stress in axial direction; $\frac{\partial^2 F}{\partial x^2}$, stress in circumferential direction; $-\frac{\partial^2 F}{\partial x \partial y}$, shear stress$\left.\right)$

L length of cylinder

Q, Q_1, Q_2 mathematical operators

Z curvature parameter $\left(\frac{L^2}{rt}\sqrt{1-\mu^2}\right.$ for cylinder or $\frac{b^2}{rt}\sqrt{1-\mu^2}$ for curved panel or infinitely long curved strip$\left.\right)$

β L/λ for cylinder or b/λ for infinitely long curved strip

λ half wave length of buckles; measured circumferentially in cylinders and axially in infinitely long curved strips

ξ dimensionless axial coordinate (x/b)

η dimensionless circumferential coordinate (y/b)

μ Poisson's ratio

τ applied shear stress

τ_{cr} critical shear stress

σ_x applied axial stress, positive for compression

σ_y applied circumferential stress, positive for compression

R_s shear-stress ratio; ratio of shear stress present to critical shear stress when no other stress is acting

R_x axial-compressive-stress ratio; ratio of direct axial stress present to critical compressive stress when no other stress is acting

∇^4 operator $\left(\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2 \partial y^2}+\frac{\partial^4}{\partial y^4}\right)$

$\nabla_G{}^4$ operator $\left(\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)^2\right)$

∇^8 operator $\left(\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^4\right)$

$\nabla_G{}^8$ operator $\left(\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)^4\right)$

∇^{-4} inverse operator defined by equation $(\nabla^{-4}(\nabla^4 f)=\nabla^4(\nabla^{-4}f)=f)$

1. DONNELL'S EQUATION

THEORETICAL BACKGROUND

In most theoretical treatments of the buckling of cylindrical shells (see references 1 to 3) three simultaneous partial differential equations have been used to express the relationship between the components of shell median-surface displacement u, v, and w in the axial, circumferential, and radial directions, respectively. No general agreement has been reached, however, on just what these equations should be. In 1934 Donnell (reference 4) pointed out that the differences in the various sets of equations arose from the inclusion or omission of a number of relatively unimportant terms (referred to in the present paper as higher-order terms), and proposed the use of simpler equations in which only the most essential terms (first-order terms) were retained. The omitted terms were shown to be small, and thus the simplified equations to be applicable, if the cylinders have thin walls and if the square of the number of circumferential waves is large compared with unity. Donnell further showed that the three simplified equations can be transformed into a single eighth-order partial differential equation in w (see appendix A of the present paper) in which the effects of the displacements u and v are properly taken into account; this equation will hereinafter be referred to as Donnell's equation.

When higher-order terms are included in the three partial differential equations previously mentioned, the resulting theoretical buckling stresses are usually very complicated functions of the cylinder dimensions and the elastic properties of the material. A family of curves is ordinarily drawn giving the critical stress as a function of the length-diameter ratio for specified values of the radius-thickness ratio and for given elastic properties (references 2, 3, and 5). When the higher-order terms are omitted from the equations and the requirements of an integral number of circumferential waves is removed, new parameters can be introduced which combine the cylinder dimensions and material properties in such a way that the results can be given in terms of a single curve. These parameters have been used, with slight variations in detail, by Donnell, Kromm, Leggett, and Redshaw (references 4 and 6 to 9). The omission of the higher-order terms also greatly simplifies the calculations, and the calculations are simplest if Donnell's equation, rather than the set of three simultaneous equations, is employed. Donnell's equation, or an equivalent equation, may therefore be presumed to be the most promising for use in solving hitherto unsolved problems in the stability of cylindrical shells.

In spite of the fact that it was introduced some time ago, Donnell's equation has not achieved the wide acceptance for use in the stability analysis of cylindrical shells which it appears to merit. Some investigators have continued to use simultaneous differential equations in which higher-order terms appear, presumably on the assumption that the errors arising from neglect of these terms might be undesirably large. Others have dropped second-order terms but have continued to employ simultaneous equations, probably in order to specify directly edge-restraint conditions having to do with displacements in the axial and circumferential directions, which cannot be done with Donnell's equation.

The purposes of part I are to establish the accuracy of the equation by comparing the results found by the use of Donnell's equation with the results found by other methods and with experimental results and to investigate the question of boundary conditions on u and v. The additional purpose is achieved of presenting the solutions of a number of problems concerned with buckling of cylinders with simply supported edges on the basis of a unified viewpoint and in a convenient form for practical use.

BUCKLING STRESSES OF CYLINDERS WITH SIMPLY SUPPORTED EDGES

Lateral pressure.—The theory for the lateral pressure (uniform external pressure applied to walls only) at which a cylinder will buckle is given in appendix B in which it is assumed that the lateral pressure causes the buckling by producing a circumferential stress σ_y and that it affects the buckling in no other way. The results are shown in a logarithmic plot in figure 1. The ordinate in this figure is the stress coefficient k_y which appears in the flat-plate buckling equation (see, for example, reference 3, p. 339)

$$\sigma_y = k_y \frac{\pi^2 D}{L^2 t}$$

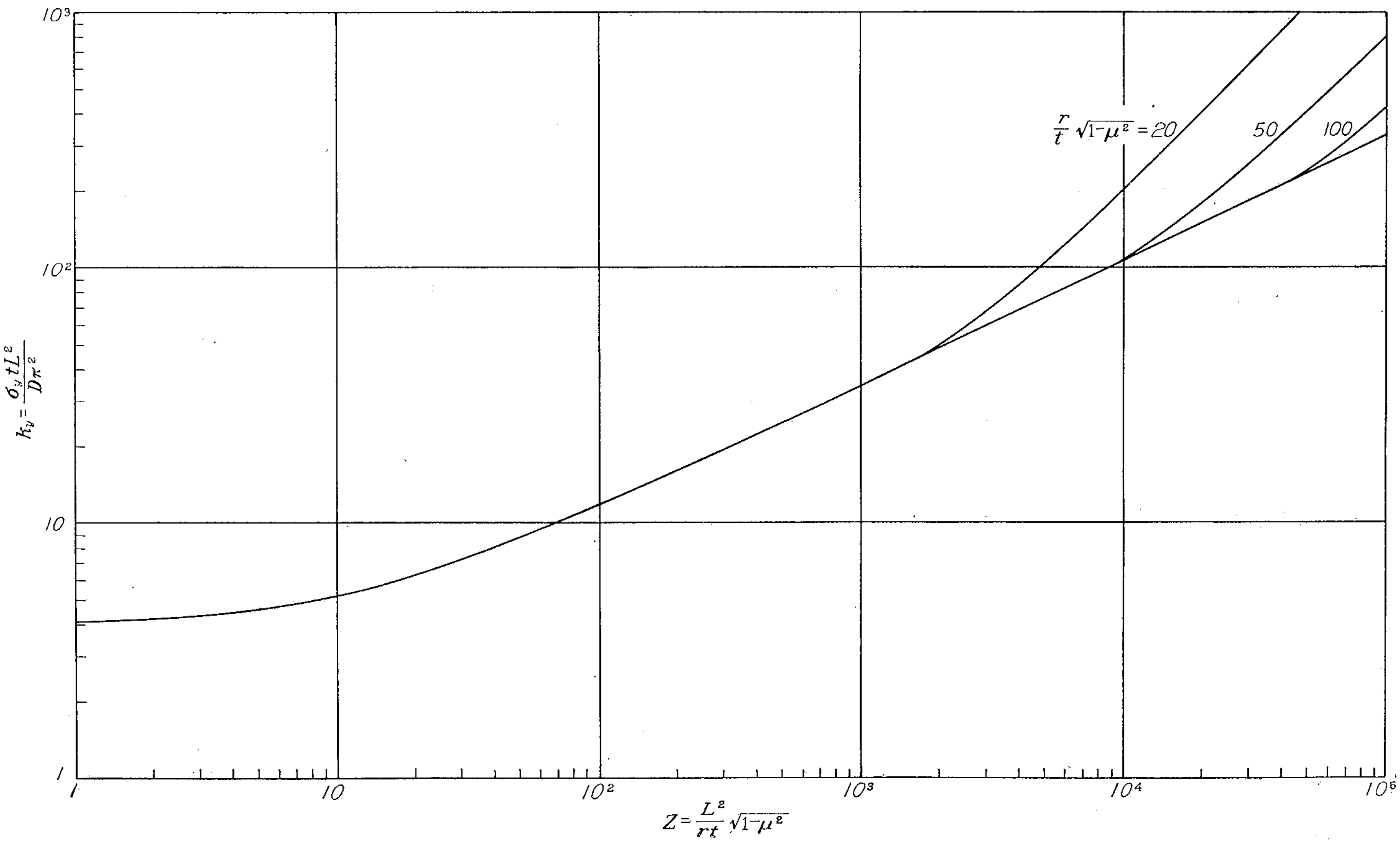


FIGURE 1.—Critical circumferential-stress coefficients for cylinders with simply supported edges.

(The discussion given in the section of the present paper entitled "Parameters Appearing in Buckling Curves" shows the relationship between a cylinder of length L and an infinitely long flat plate of width $b=L$.) The abscissa

$$Z=\frac{L^2}{rt}\sqrt{1-\mu^2}=\left(\frac{L}{r}\right)^2\frac{r}{t}\sqrt{1-\mu^2}$$

may be regarded either as a measure of the curvature, or, for any given ratio of radius to thickness, as a measure of the length-radius ratio of the cylinder. Figure 1 shows that for small curvature k_y approaches the value 4, which applies in the case of simply supported long flat plates in longitudinal compression (reference 3, p. 327). As the curvature parameter Z increases, the stress coefficient k_y also increases. For large values of Z, the curve approaches a straight line of slope 1/2. This straight line is expressed by the formula

$$k_y \doteq 1.04Z^{1/2}$$

As the length-radius ratio increases, for a given value of r/t, the number of circumferential waves n diminishes. Although n must be an integer, the curves of figure 1 were obtained on the assumption that n is free to vary continuously. Only small conservative errors are involved in this assumption. Because $n=1$ corresponds merely to a lateral displacement of the entire circular cross section, the minimum value of n is 2, which corresponds to deformation of the section into an ellipse. This limitation on n results in splitting the curve of figure 1 into a number of curves for different values of r/t when Z becomes large. A cylinder having a value of $\frac{r}{t}=20$ buckles into an ellipse when L/r is about 10, and the value of L/r at which such buckling occurs increases with increasing r/t.

In figure 2 the curve of figure 1 is compared with results based on more complicated calculations given in reference 3 and in reference 5. At fairly large values of Z the results given in reference 3 and in reference 5 are in good agreement with the results of the present paper. At small values of Z the curve based on reference 3 (Timoshenko) is definitely too low, because k_y should approach the flat-plate value of 4 as Z approaches zero. An interesting feature of the comparison is that one calculation gives results below, and the other calculation results above, those given herein. The test data, taken from reference 5, are in reasonable agreement with and show more scatter than the theoretical curves.

In the case of cylinders so long that $n=2$, the requirement for the validity of Donnell's equation that $n^2 \gg 1$ is no longer satisfied and appreciable error is to be expected. Indeed it may be shown that for very long cylinders when $n=2$ Donnell's equation gives $4D/r^3$ as the critical value of the applied lateral pressure, whereas the accepted theoretical

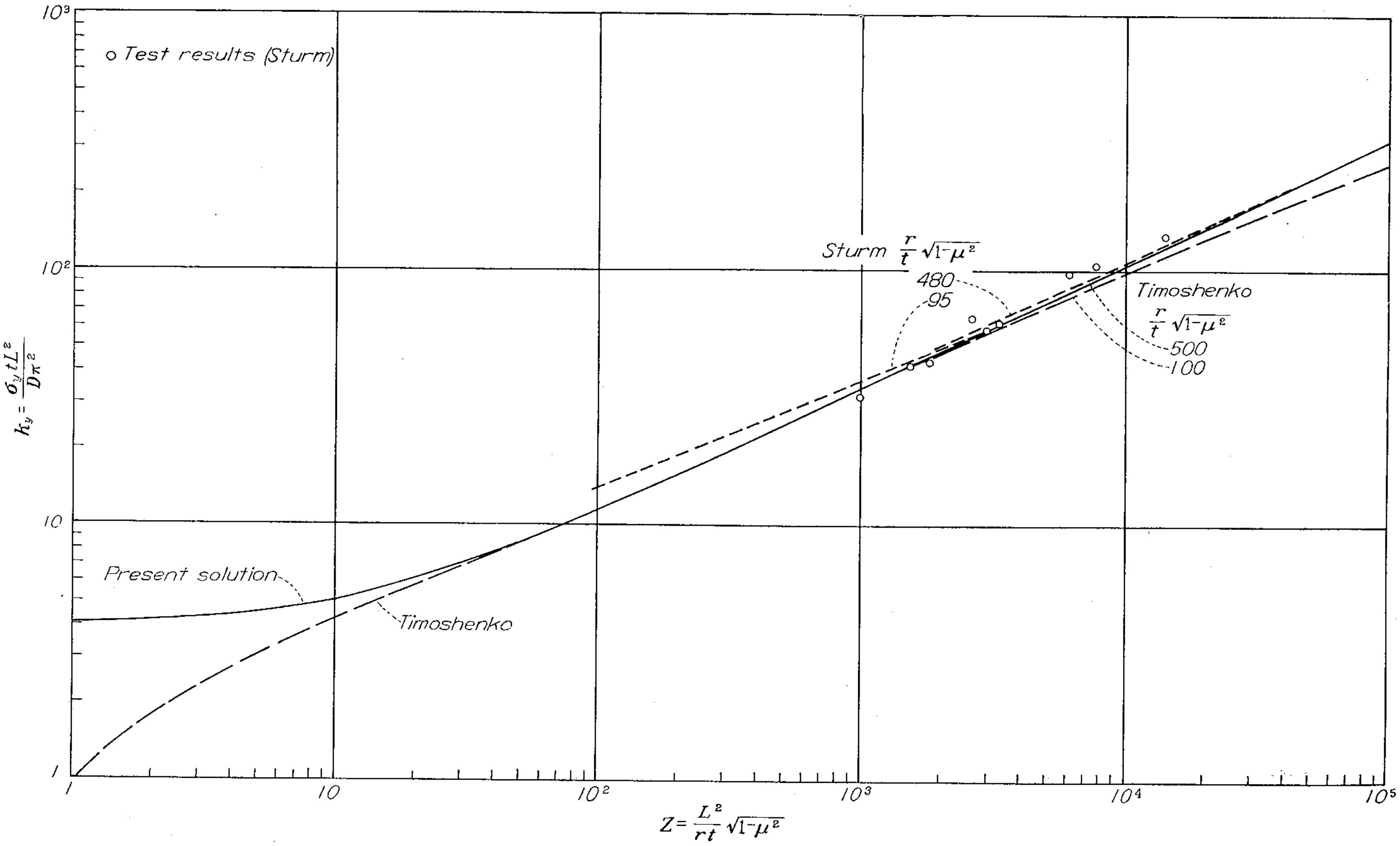


FIGURE 2.—Comparison of present solution for critical circumferential-stress coefficients for simply supported cylinders with other theoretical solutions and with test results. (Timoshenko's solution is from reference 3 and Sturm's data and solution are from reference 5.)

result is $3D/r^3$ (by use of the formula given on p. 450 of reference 3). The curves for $n=2$ will probably not often be needed, however, since they apply only when $\left(\frac{L}{r}\right)^2 > \left(\frac{5r}{t}\right)$, which in the case of thin cylinders corresponds to a very large length-radius ratio, and if needed, the curves for $n=2$ can be applied in conjunction with a correction factor 0.75.

Axial compression.—The theory for the axial stress at which a cylinder will buckle is given in appendix B, and the results are shown in figure 3. The ordinate is analogous to, and the abscissa identical with, the corresponding coordinates used in figure 1. Figure 3 shows that for small values of Z, k_x approaches the value 1, which applies in the case of long flat plates in transverse compression with long edges simply supported (reference 3). For large values of Z, the curve becomes a straight line of slope 1. This straight line is expressed by the formula

$$k_x = \frac{4\sqrt{3}}{\pi^2} Z = 0.702Z$$

For any fixed value of r/t some value of Z always exists above which L/r is so large that the cylinder fails as an Euler strut rather than by buckling of the cylinder walls. Pin-ended Euler buckling of cylinders is indicated in figure 3 by means of dashed curves.

The result just given for the critical-stress coefficient for a cylinder in axial compression leads to the following expression for the critical stress:

$$\sigma_x = \frac{1}{\sqrt{3(1-\mu^2)}} \frac{Et}{r} \tag{1}$$

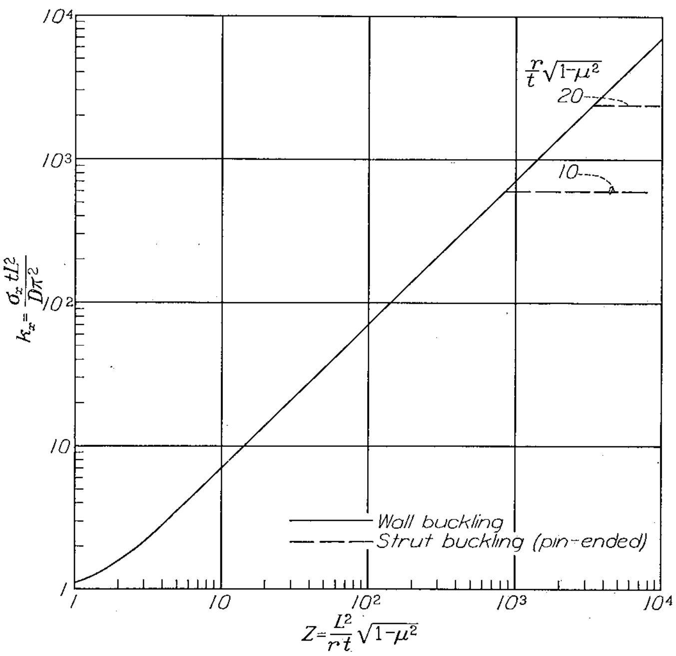


FIGURE 3.—Critical axial-stress coefficients for cylinders with simply supported edges.

The value given in equation (1) for the critical stress of a moderately long cylinder in axial compression by use of Donnell's equation is identical with the value found by a number of investigators using other equations as starting points (references 1 to 3). In the case of cylinders under axial compression the errors involved in dropping the second-order terms are therefore concluded to be small.

The buckling stresses given by equation (1) are nevertheless in serious disagreement with the buckling stresses obtained by experiment (reference 10). For a discussion of the degree of correlation that can be found between theory and experiment for cylinders under axial compression, see reference 11.

Hydrostatic pressure on closed cylinders.—When closed cylinders are subjected to external pressure, both axial and circumferential stress are present. The theory for buckling under these combined loads is given in appendix B. The results are shown in figure 4. The ordinate C_p used in this figure is a nondimensional measure of the pressure p defined as follows:

$$C_p = \frac{prL^2}{\pi^2 D}$$

The coefficient C_p can be directly related to the corresponding stress coefficients k_x and k_y. By definition

$$k_y = \frac{\sigma_y t L^2}{\pi^2 D}$$

and, according to the hoop-stress formula,

$$\sigma_y = \frac{pr}{t}$$

It follows from the three preceding equations that C_p is numerically equal to k_y. Similarly C_p can be shown to be numerically equal to $2k_x$.

At low values of Z, C_p approaches the value 2, which implies that $k_x=1$ and $k_y=2$. That these values of k represent a critical combination of stresses for an infinitely long flat plate was shown in reference 12. At large values of Z, the curve approaches the curve given in figure 1 for buckling under lateral pressure alone and, like that curve, has branches representing buckling into two circumferential waves.

In figure 5 the computed values of the pressure coefficient C_p at which the cylinder would buckle if only the axial pressure were acting and if only lateral pressure were acting are compared with the results when both are acting because of hydrostatic pressure. At large values of Z the circumferential stress at which buckling occurs under hydrostatic pressure is substantially the same as it would be if no axial stress were present, as in the case of lateral pressure. The reason that the circumferential stress appears as the main factor in buckling at high values of Z presumably is that at these values of Z the axial stress required to produce buckling is many times the circumferential stress required, whereas under hydrostatic pressure the axial stress actually present is only one-half the circumferential stress.

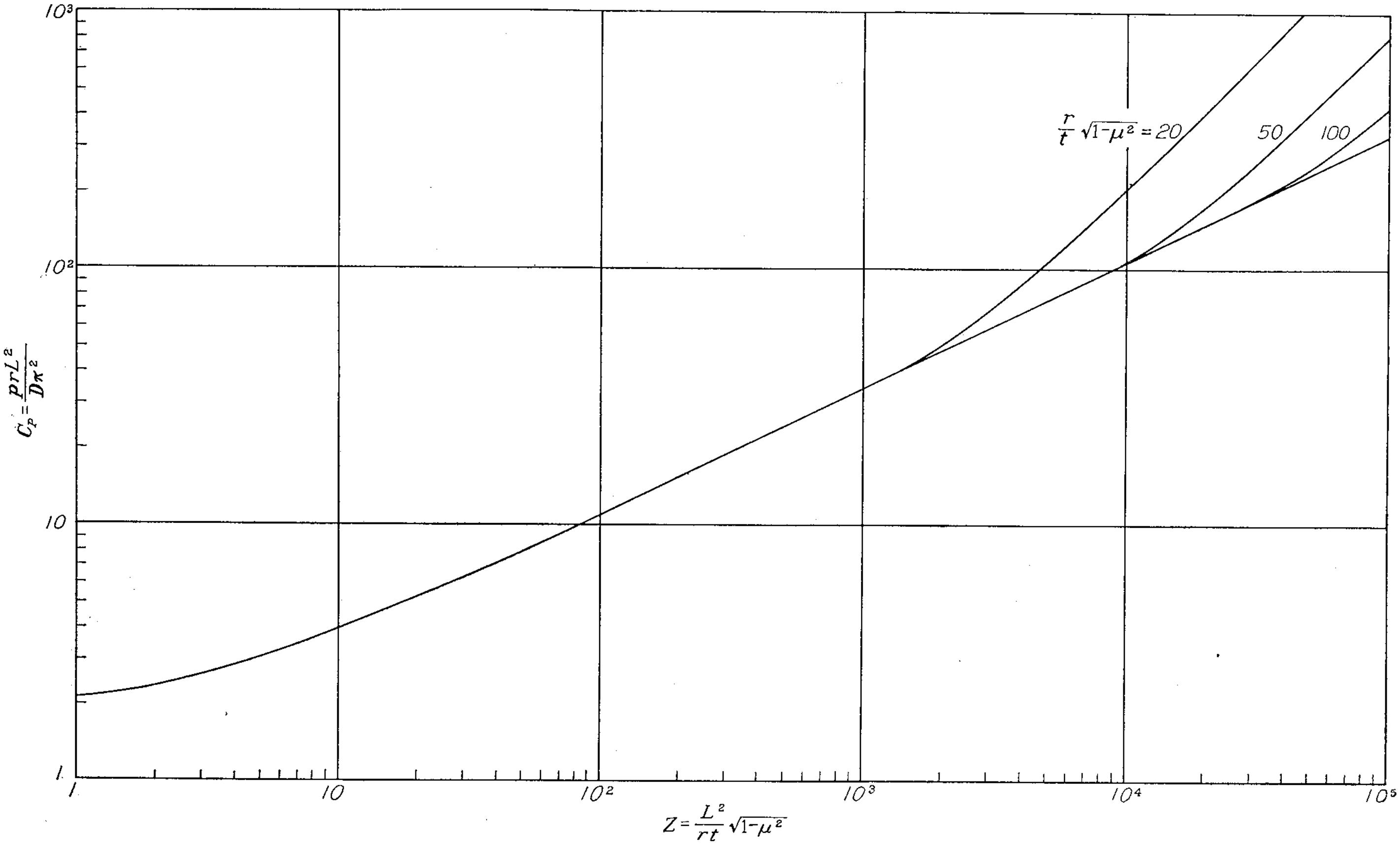


FIGURE 4.—Theoretical solution for hydrostatic pressure under which simply supported cylinders buckle.

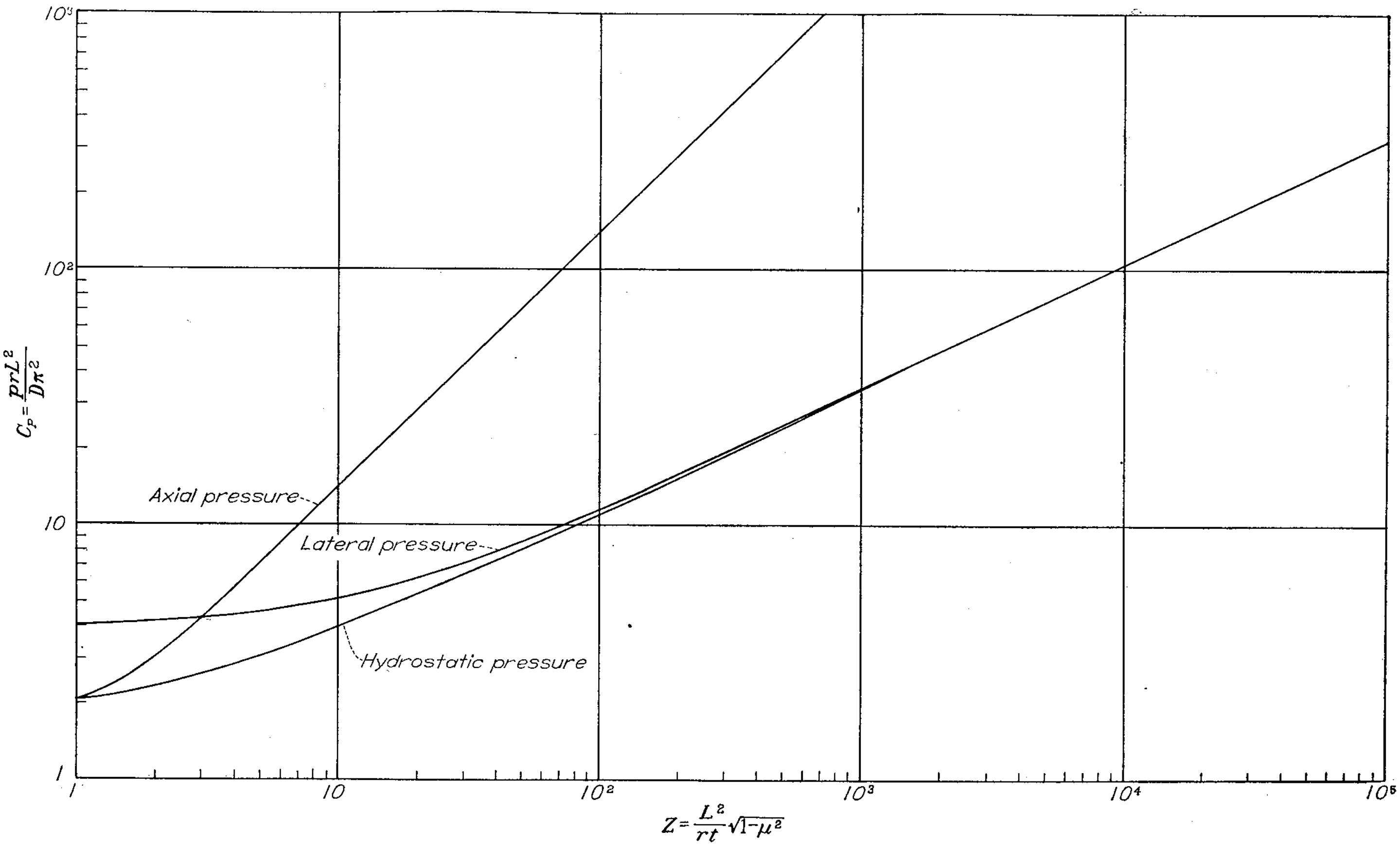


FIGURE 5.—Comparison of solution for buckling of simply supported cylinders under hydrostatic pressure with solutions for buckling under axial pressure alone and lateral pressure alone.

In figure 6 the curve of figure 4 is compared with curves representing Sturm's theoretical results (reference 5) and with a curve based on the following formula developed at the United States Experimental Model Basin (reference 13, equation (9)):

$$p=\frac{2.42E}{(1-\mu^2)^{3/4}}\frac{\left(\frac{t}{d}\right)^{5/2}}{\left[\frac{L}{d}-0.45\left(\frac{t}{d}\right)^{1/2}\right]}$$

This formula is an approximation based on theoretical results obtained by Von Mises (reference 3, p. 479) which are identical with the results in the present paper for buckling under hydrostatic pressure. Figure 6 shows that Sturm's theoretical results (reference 5) are in reasonable agreement with those of the present paper and that the formula from the United States Experimental Model Basin practically coincides with the present results except at very low values of Z.

Test results from references 5 and 13 are included in figure 6. The test data are in good agreement with the theoretical results except at low values of the curvature parameter Z at which the theoretical results are appreciably above those obtained experimentally. A possible explanation of the discrepancy between the theoretical and experimental results at low curvature is suggested by the relative importance of axial and circumferential stress in causing buckling. The axial stress becomes important only at low values of the curvature parameter Z. It is known experimentally that buckling under axial stresses may occur far below the theoretical value of the critical stress. At low values of Z cylinders under hydrostatic pressure may therefore be expected to buckle well below the theoretical critical load just as cylinders do under axial compression.

Torsion.—The problem of the determination of the buckling stresses of cylinders in torsion was solved by Donnell (reference 4) who gave an approximate solution of the equation of equilibrium. A somewhat more accurate solution of this equation is given in reference 14. The essential results are shown in figure 7 taken from reference 14. At low values of Z the buckling-stress coefficient k_s approaches the value 5.34 appropriate to infinitely long flat plates loaded in shear (reference 15). At higher values of Z the curve approaches a straight line given by

$$k_s=0.85Z^{3/4}$$

At very high values of the curvature parameter the curve splits up into a number of other curves, depending on the value of r/t. The curves for various r/t values at high values of Z represent buckling into two circumferential waves. As mentioned before, Donnell's equation is not reliable for the case $n=2$ $\left(\text{a case which occurs for cylinders in torsion when } \left(\frac{L}{r}\right)^2>10\,\frac{r}{t}\right)$. A solution for this case given by Schwerin and discussed in reference 4 results in critical stresses about 20 percent below those of the present paper. Because Schwerin's solution does not satisfy the condition $w=0$ at the end of the cylinder, however, it is probable that the error in the present solution for $n=2$ is less than 20 percent.

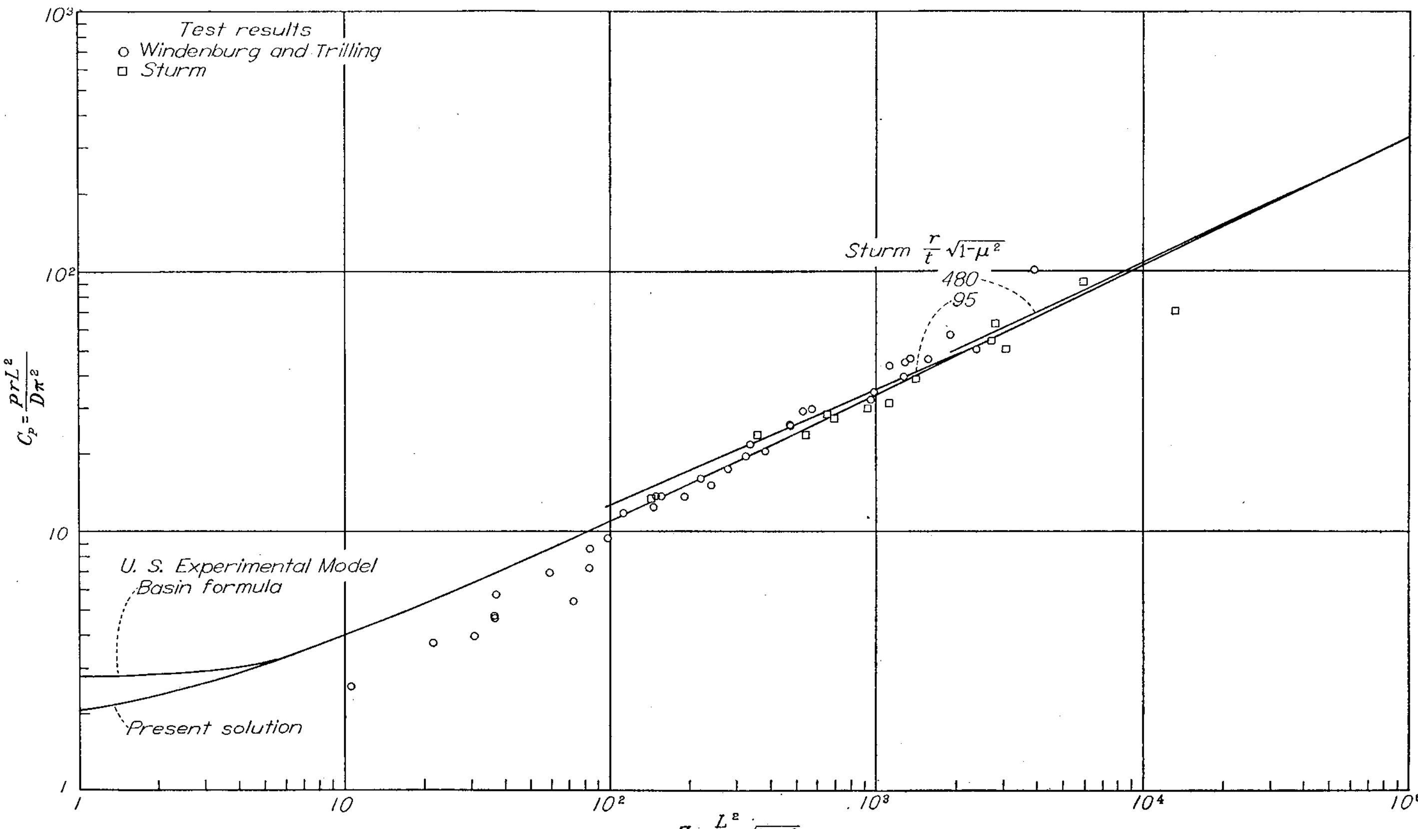


FIGURE 6.—Comparison of present solution for buckling of simply supported cylinders under hydrostatic pressure with other theoretical solutions and test results. (Sturm's results are from reference 5 and Windenburg and Trilling's results are from reference 13.)

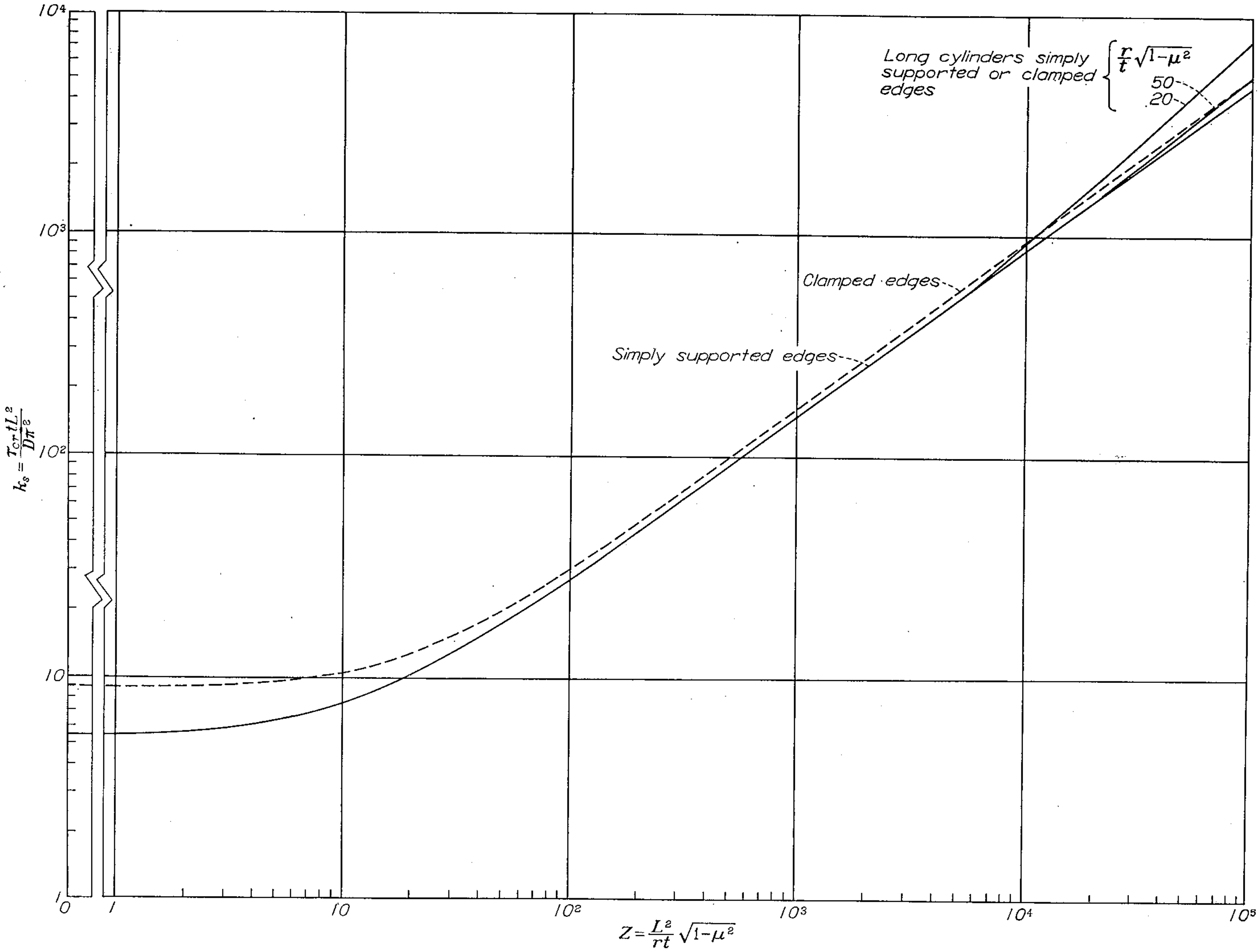


FIGURE 7.—Critical-shear-stress coefficients for cylinders in torsion. (Fig. 1 of reference 14.)

In experimental investigations of cylinders in torsion the maximum rather than the critical loads have usually been reported. Because these maximum loads usually exceed the critical loads by only a small margin, it is common practice to check theoretical buckling stresses by comparison with the average stresses at maximum load. Such a comparison is provided in figure 8 which incorporates test data from references 4, 10, 16, and 17 For this figure the test results average about 15 percent below those given by theory.

DISCUSSION

Parameters appearing in buckling curves.—The fact that the buckling of a cylinder under axial compression, lateral pressure, hydrostatic pressure, or torsion involves substantially the same parameters is not a mere coincidence but is a direct consequence of the differential equation. The differential equation implies that when the requirement of an integral number of circumferential waves is removed the six variables L, r, t, E, μ, and the load may be combined into two nondimensional parameters, one (k_x, k_y, k_s, or C_p) describing the stress condition, and the other (Z) essentially determined by the geometry. (See appendix C.) It is also shown in appendix C that the buckling of a curved rectangular plate of any given length-width ratio may be represented in terms of these parameters. The critical stress of a cylinder or a curved plate of given length-width ratio may therefore be given by a single curve relating the two parameters provided that the number of circumferential waves may be regarded as continuously variable. This restriction becomes important at very large values of Z, for which the curves may split into a number of curves for cylinders of different values of r/t buckling into two circumferential waves.

Except for hydrostatic pressure, each type of loading considered results in a single uniform stress in the cylinder, and the nondimensional parameter k describing this stress is defined as follows in analogy to the parameter used in describing the buckling of a flat plate:

$$k=\frac{\sigma(\text{or }\tau)}{\frac{\pi^2 D}{L^2 t}}$$

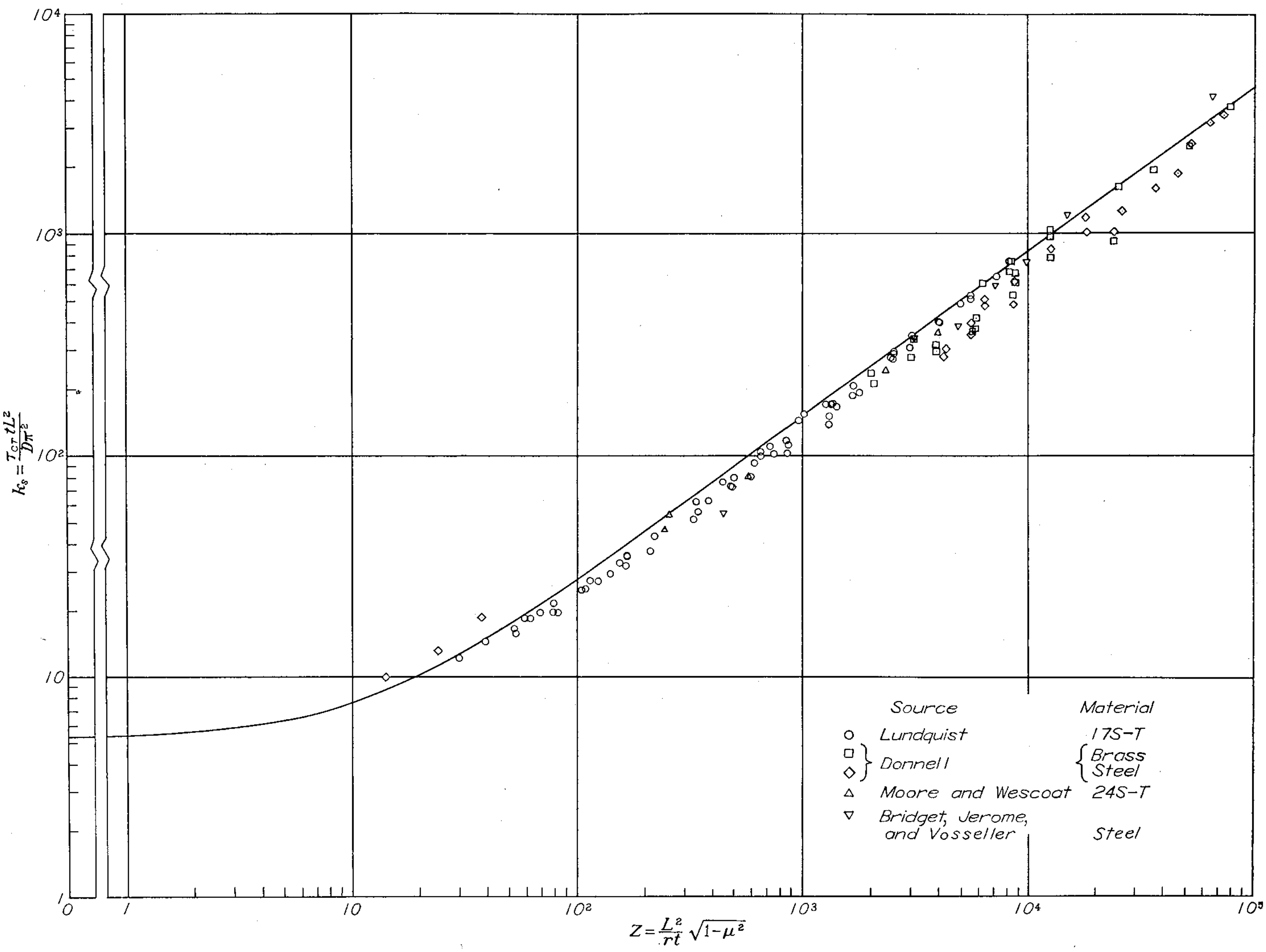


FIGURE 8.—Comparison of theoretical solution for critical shear stress of simply supported cylinders in torsion with experimental ultimate stresses. (Lundquist's data are from reference 10, Donnell's data are from reference 4, Moore and Wescoat's data are from reference 16, and Bridget, Jerome, and Vosseller's data are from reference 17.)

As the radius of the cylinder increases toward infinity (the other dimensions remaining constant), the cylinder approaches an infinitely long flat plate of the same thickness as the cylinder, having a width b equal to the length L of the cylinder. Accordingly, as the radius approaches infinity, the critical-stress coefficient k for the cylinder approaches the value of the corresponding stress coefficient for an infinitely long flat plate under the appropriate loading condition.

The other nondimensional parameter Z is defined by the equation

$$Z=\frac{L^2}{rt}\sqrt{1-\mu^2}=\left(\frac{L}{r}\right)^2\frac{r}{t}\sqrt{1-\mu^2}$$

If the small correction due to Poisson's ratio is neglected, a direct physical significance can be assigned to Z when its magnitude is small. The maximum distance from a slightly curved arc of length L and radius r to its chord can be shown to be given by the expression $L^2/8r$, which is called the "bulge" by some writers (see references 8 and 9). Accordingly, in the case of a curved strip of length L in the circumferential direction, $L^2/8rt$ is the bulge divided by the thickness and is thus a nondimensional measure of the deviation from flatness of the strip. As applied to a short cylinder, $L^2/8rt$ is the deviation from flatness of a square panel of the cylinder, each side of which is equal to the length of the cylinder. For cylinders having a length greater than a few tenths of the diameter, the parameter Z loses this simple physical significance and is perhaps best regarded as a nondimensional measure of the length of the cylinder. Some indication of the variety of cylinder shapes corresponding to a fixed value of Z is given in figure 9.

Boundary conditions.—When problems in the stability of cylindrical shells are solved by the use of Donnell's equation, boundary conditions on u and v cannot be imposed directly because only w appears in the equation. The method of solution, however, may in some cases imply boundary conditions on u or v. In appendix D it is shown that for simply supported cylinders the method used in the present paper (a solution using one or more terms of a Fourier series satisfying the boundary conditions on w term by term) implies that at

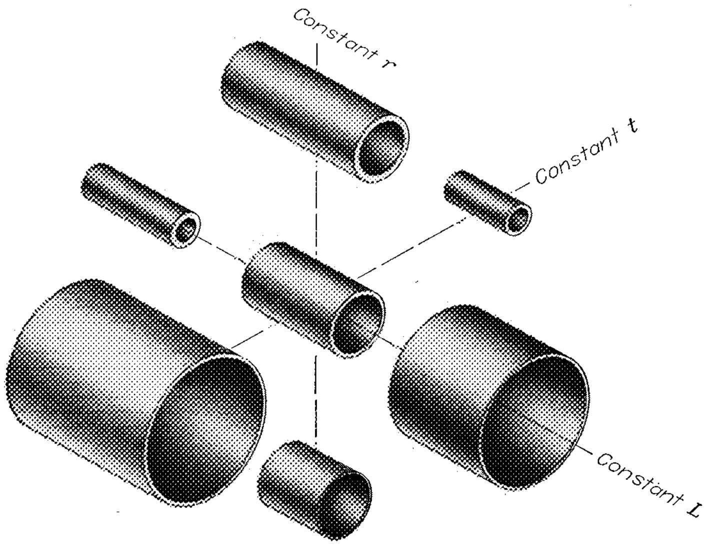


FIGURE 9.—Representative cylinders corresponding to the same value of Z (Z about 150).

both ends of the cylinder the circumferential displacement v is zero, but that the cylinder edges are free to warp in the axial direction ($u \neq 0$). For a simply supported rectangular curved panel, the present method implies (with regard to displacements within the panel median surface) zero displacement along the four edges of the panel and free warping normal to the edges. These edge conditions on u and v are appropriate to cylinders or panels bounded by light bulkheads or deep stiffeners which are stiff in their own planes but may be readily warped out of their planes.

Relatively few calculations of the stability of a cylinder take into account the boundary conditions on u and v. A calculation for the case of torsion, however, was recently made by Leggett (reference 18). The results of this calculation, computed for $u=v=0$ at the edges of the cylinder, are given only for $Z<50$. Throughout the range for which they are given, however, they agree very closely with the results found by the method employed in the present paper, which implies that at the edge of the cylinder $v=0$ and $u \neq 0$. Restraining the ends of the cylinder from warping in the axial direction may therefore be assumed to have a negligible effect upon the buckling stress. This assumption receives added support from the form of the equation of equilibrium (appendix A) for the case of constant pressure

$$D\nabla^4 w + t\left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial x^2}\frac{1}{r}\right) = 0$$

In this equation, σ_x, σ_y, and τ are the stresses present just before buckling and $\frac{\partial^2 F}{\partial x^2}$ is the circumferential stress produced by the buckling itself. The equation indicates that the only difference between the buckling behavior of a cylindrical sheet and that of a flat plate (found by omitting the last term in the foregoing equation) is due to the effect of the circumferential stresses caused by the buckling deformations.

Because the restraint against warping in the axial direction requires the application of axial rather than circumferential stresses, this restraint might be expected to have only small effects on buckling stresses. Circumferential stresses would have to be applied to the straight sides of a curved strip to prevent warping normal to these edges during buckling. Because the circumferential stress due to buckling appears explicitly in the equation of equilibrium, the imposition of the restraint $v=0$ to the straight sides of a panel should have an appreciable effect on the buckling stress (except when the straight sides of the panel are short compared with the curved sides).

Theoretical results on the buckling of curved strips infinitely long in the axial direction are available to test the foregoing conclusion. In figure 10 the critical axial compressive stress for an infinitely long curved strip with u and v both zero along the edges (reference 8) is compared with the critical axial compressive stress when u is zero along the edges, and the edges are free to warp in the circumferential direction. (See appendix B for solution.) The critical axial stress is appreciably increased by the constraint $v=0$ in a certain range of small curvature. In figure 11 the critical shear stresses are compared under the same sets of edge conditions (references 6 and 7). The critical shear stress is conspicuously increased by the constraint $v=0$ except near the limiting case of flat plates.

It appears from the foregoing discussion that the effect on the buckling stresses of preventing free warping normal to the curved edges of a cylinder or panel is very small but that the effect on the buckling stresses of a similar restraint on the straight edges of a panel may be quite important.

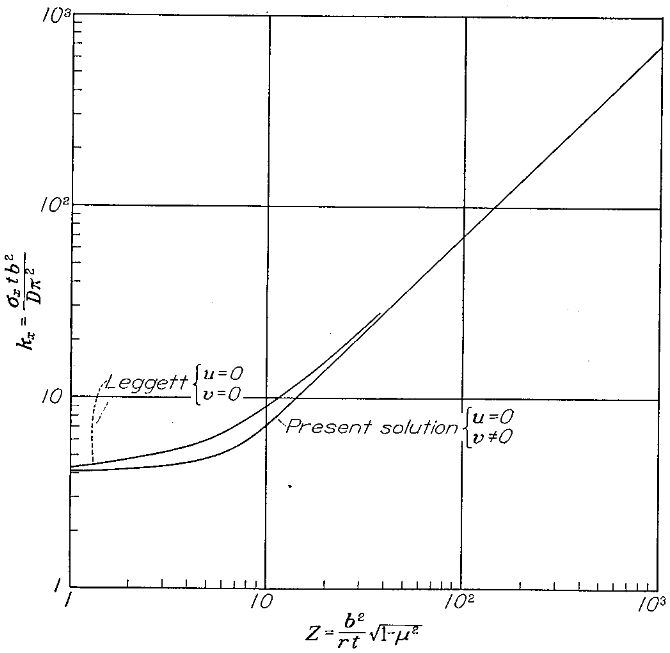


FIGURE 10.—Comparison of the present solution for the buckling under axial compression of a curved strip infinitively long in the axial direction, with solution found by Leggett (reference 8).

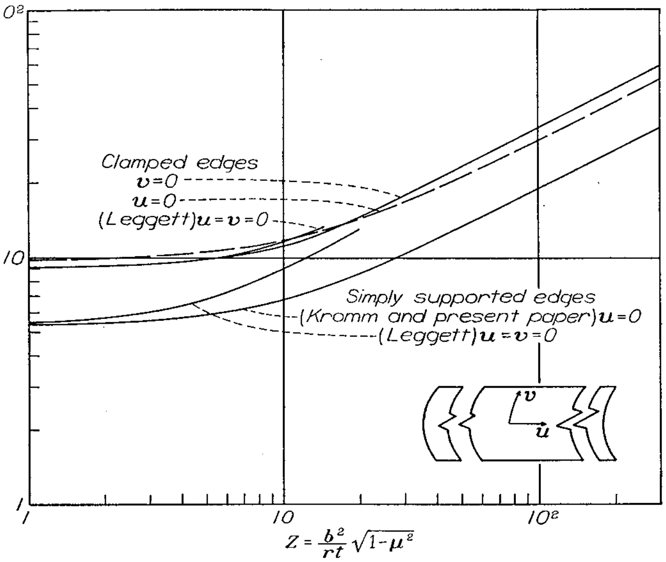


FIGURE 11.—Comparison of Leggett's solutions with present solutions for critical-shear-stress coefficients of a long curved strip. (Fig. 2 of reference 20.)

Simplicity of results.—The theoretical results based on Donnell's equation for the critical stresses of cylinders under a given loading condition appear particularly simple when presented as a logarithmic plot of buckling coefficient k against the curvature parameter Z. As r approaches infinity, and therefore as Z approaches zero, k approaches the value appropriate to a flat plate. At large values of Z the curve approached a straight line in each of the cases investigated. These straight lines had slopes 0.5, 0.75, and 1 and are given approximately by the following equations which have already been given in the present paper and are reassembled here and provided with upper and lower limits for easy reference:

$$k_y=1.04Z^{1/2} \qquad \left(100<Z<5\left(\frac{r}{t}\right)^2(1-\mu^2)\right)$$

$$k_s=0.85Z^{3/4} \qquad \left(50<Z<10\left(\frac{r}{t}\right)^2(1-\mu^2)\right)$$

$$k_x=0.702Z \qquad \left(3<Z<6\left(\frac{r}{t}\right)^2(1-\mu^2)\right)$$

These equations can also be written (when μ is taken to be 0.316)

$$\sigma_y=0.926\frac{Et}{r}\left(\frac{tr}{L^2}\right)^{1/2}=0.926E\left(\frac{t}{r}\right)^{3/2}\left(\frac{r}{L}\right) \qquad \left(100\frac{t}{r}<\left(\frac{L}{r}\right)^2<5\frac{r}{t}\right)$$

$$\tau=0.747\frac{Et}{r}\left(\frac{tr}{L^2}\right)^{1/4}=0.747E\left(\frac{t}{r}\right)^{5/4}\left(\frac{r}{L}\right)^{1/2} \qquad \left(50\frac{t}{r}<\left(\frac{L}{r}\right)^2<10\frac{r}{t}\right)$$

$$\sigma_x=0.608\frac{Et}{r} \qquad \left(3\frac{t}{r}<\left(\frac{L}{r}\right)^2<6\frac{r}{t}\right)$$

II. MODIFIED EQUATION

THEORY

DERIVATION OF MODIFIED EQUATION

The equation of equilibrium for a flat plate may be written

$$D\nabla^4w+t\left(\sigma_x\frac{\partial^2w}{\partial x^2}+2\tau\frac{\partial^2w}{\partial x\partial y}+\sigma_y\frac{\partial^2w}{\partial y^2}\right)+p=0 \tag{2}$$

where p is lateral pressure. (This equation is equivalent to equation (197) of reference 3.)

For a cylindrically curved plate having a radius of curvature r, the following pair of simultaneous equations of equilibrium may be written (as a generalization of equations (11) and (10) of reference 7):

$$D\nabla^4w+t\left(\sigma_x\frac{\partial^2w}{\partial x^2}+2\tau\frac{\partial^2w}{\partial x\partial y}+\sigma_y\frac{\partial^2w}{\partial y^2}\right)+p-\frac{t}{r}\left(\frac{\partial^2F}{\partial x^2}+\sigma_y\right)=0 \tag{3}$$

$$\nabla^4F+\frac{E}{r}\frac{\partial^2w}{\partial x^2}=0 \tag{4}$$

where F is Airy's stress function for the median-surface stresses produced by the buckle deformation (reference 19). Equation (3) differs from equation (2) only in the addition of the term $-\frac{t}{r}\left(\frac{\partial^2F}{\partial x^2}+\sigma_y\right)$, which expresses the effect of the curvature. Equation (4) shows that, unlike flat plates, cylindrical shells experience stretching of the median surface when originally straight lines in the surface are bent slightly. Elimination of F between equations (3) and (4) by suitable differentiations and additions gives the following single equation in w for the equilibrium of cylindrical shells:

$$D\nabla^8w+\frac{Et}{r^2}\frac{\partial^4w}{\partial x^4}+t\nabla^4\left(\sigma_x\frac{\partial^2w}{\partial x^2}+2\tau\frac{\partial^2w}{\partial x\partial y}+\sigma_y\frac{\partial^2w}{\partial y^2}-\frac{\sigma_y}{r}\right)+\nabla^4p=0 \tag{5}$$

Equation (5), which was first derived by Donnell (reference 4), was treated in part I.

An alternative method for obtaining a single equation in w for the equilibrium of a cylindrical shell is to solve equation (4) for F and substitute the result into equation (3). This procedure can readily be carried out in the following manner. Differentiation of equation (4) twice with respect to x gives

$$\nabla^4\frac{\partial^2F}{\partial x^2}+\frac{E}{r}\frac{\partial^4w}{\partial x^4}=0 \tag{6}$$

The symbolic solution of equation (6) for $\frac{\partial^2F}{\partial x^2}$ is

$$\frac{\partial^2F}{\partial x^2}=-\frac{E}{r}\nabla^{-4}\frac{\partial^4w}{\partial x^4}$$

Substitution of this result into equation (3) gives

$$D\nabla^4 w+\frac{Et}{r^2}\nabla^{-4}\frac{\partial^4 w}{\partial x^4}+t\left(\sigma_x\frac{\partial^2 w}{\partial x^2}+2\tau\frac{\partial^2 w}{\partial x\partial y}+\sigma_y\frac{\partial^2 w}{\partial y^2}-\frac{\sigma_y}{r}\right)+p=0 \quad (7)$$

Equation (7) is simply equation (5) modified by multiplication by the operator ∇^{-4}. In the present paper, equations (5) and (7) are referred to as Donnell's equation and the modified equation, respectively.

ADVANTAGES OF MODIFIED EQUATION

One of the quickest and most convenient methods for obtaining solutions of flat-plate buckling problems to any desired degree of approximation uses a Fourier series type of expansion for the deflection surface w. Both Donnell's equation and the modified equation can be solved by this method in the case of buckling problems involving curved plates having simply supported edges.

As mentioned in the "Introduction," however, Donnell's equation is not well adapted to solution by Fourier series of problems involving the stability of shells with clamped edges. The cause of the trouble appears to be that the calculation of some of the high-order derivatives found in Donnell's equation sometimes leads to divergent trigonometric series when the edges are clamped. The modified equation, however, is applicable to clamped-edge problems as well as to problems involving simply supported edges because lower-order derivatives are involved.

Besides its advantages in the solution of problems involving shells with clamped edges, equation (7) has the additional advantage that each term has a definite physical significance: The first term gives the restoring force per unit area of the deflected surface due to bending and twisting stiffnesses; the second term gives the restoring force per unit area due to stretching stiffness; and the remaining terms give the deflecting or restoring forces per unit area due to applied loads. Because of these advantages, the modified equation was adopted for general use in references 11, 14, and 20 to 23.

Both Donnell's equation and the modified equation result in the same critical stresses for simply supported cylindrical shells, and the two methods require essentially equivalent mathematical processes. (See appendix E.) The characteristics of solutions by means of Donnell's equation in the case of simply supported shells—namely, the theoretical cylinder parameters, the simplicity of calculations and results, and the implied boundary conditions on u and v—are characteristics, also, of solutions by means of the modified equation. The same characteristics, except for a change in the implied boundary conditions on u and v, also apply to solutions of clamped-edge shell problems by means of the modified equation. This change is discussed in the section entitled "Boundary Conditions."

SOLUTION OF MODIFIED EQUATION BY GALERKIN METHOD

An approximate method of solving vibration and buckling problems closely paralleling that of Ritz was introduced in 1915 by Galerkin. (See, for example, references 24 and 25.) The main distinction between the Ritz and Galerkin methods is that the Ritz method begins with an energy expression, whereas the Galerkin method begins with an equation of equilibrium. The Galerkin method is readily adaptable to the solution of equation (7) and is now described briefly.

Let the equation of equilibrium be written

$$Q(w)=0 \quad (8)$$

where Q is some operator in x and y which for the purposes of this paper is taken to be linear. According to the Galerkin method, the equation may be solved by expanding the unknown function w in terms of a suitable set of functions $f_i(x)g_j(y)$, each of which satisfies the boundary conditions but not in general the equation of equilibrium:

$$w=\sum_i\sum_j a_{ij}f_i(x)g_j(y) \quad (9)$$

Substitution of this expression for w into equation (8) gives the following equation:

$$\sum_i\sum_j a_{ij}Q[f_i(x)g_j(y)]=0 \quad (10)$$

Because the functions $f_i(x)g_j(y)$ were chosen to satisfy the boundary conditions rather than the equation of equilibrium, equation (10) cannot, in general, be satisfied identically by any choice of the coefficients a_{ij}. These coefficients can be chosen, however, to assure the vanishing of certain weighted averages of the left-hand side of equation (10). The weighting functions used in the Galerkin method are the original expansion functions, so that the following simultaneous equations for determining the coefficients a_{ij} are obtained:

$$\sum_i\sum_j B_{mnij}a_{ij}=0 \quad (m=1,2,3,\ldots;\ n=1,2,3,\ldots) \quad (11)$$

where

$$B_{mnij}=\int\int f_m(x)g_n(y)\ Q[f_i(x)g_j(y)]\ dx\ dy \quad (12)$$

The simultaneous set of linear algebraic equations in the unknown coefficients a_{ij} (equation(11)), obtained by using the original expansion functions as weighting functions, is ordinarily the same set which would be found by the Ritz method, if the same series expansion for w were used. A solution of any desired degree of accuracy may therefore be obtained by the Galerkin method.

In applying the Galerkin method to equation (7) by use of Fourier series expansion for w, expressions of the type

$$\nabla^{-4}\sum_i\sum_j a_{ij}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

must be evaluated. The operator ∇^{-4}, the inverse of ∇^4, simply introduces into the denominator of each term of the series the expression that comes into the numerator if ∇^4 is applied. Thus,

$$\nabla^{-4}\sum_i\sum_j a_{ij}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}=\sum_i\sum_j\frac{a_{ij}}{\left[\left(\frac{i\pi}{a}\right)^2+\left(\frac{j\pi}{b}\right)^2\right]^2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \quad (13)$$

This result may readily be verified by applying the operator ∇^4 to each side of equation (13).

In writing equation (13) the quantity $\nabla^{-4}f$, as defined by the equation

$$\nabla^4\nabla^{-4}f=f$$

was tacitly assumed to be unique. The quantity actually is not unique; any number of terms which vanish when operated upon by ∇^4 could be added to the right-hand side of equation (13). The omission of such terms makes the present analysis parallel to the analysis using Donnell's equation (see part I) and implies certain boundary conditions on u and v, which are discussed in a subsequent section entitled "Boundary Conditions."

DEFLECTION FUNCTIONS

Simply supported edges.—For simply supported cylindrical shells, the following series expansions for w may be used to represent the buckle deformation to any desired degree of accuracy (in these functions, x is the coordinate in the axial direction and y, the coordinate in the circumferential direction):

(1) Rectangular curved plate (axial dimension a and circumferential dimension b)

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{14}$$

(2) Curved strip long in the axial direction (circumferential width b and axial wave length 2λ)

(a) Direct stresses only

$$w=\sin\frac{\pi x}{\lambda}\sum_{m=1}^{\infty} a_m \sin\frac{m\pi y}{b} \tag{15}$$

(b) Shear stress with or without addition of direct stress

$$w=\sin\frac{\pi x}{\lambda}\sum_{m=1}^{\infty} a_m \sin\frac{m\pi y}{b}+\cos\frac{\pi x}{\lambda}\sum_{m=1}^{\infty} b_m \sin\frac{m\pi y}{b} \tag{16}$$

(3) Complete cylinder (length L and circumferential wave length 2λ)

(a) Direct stresses only

$$w=\sin\frac{\pi y}{\lambda}\sum_{m=1}^{\infty} a_m \sin\frac{m\pi x}{L} \tag{17}$$

(b) Shear stress with or without addition of direct stress

$$w=\sin\frac{\pi y}{\lambda}\sum_{m=1}^{\infty} a_m \sin\frac{m\pi x}{L}+\cos\frac{\pi y}{\lambda}\sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{L} \tag{18}$$

Clamped edges.—Probably the simplest method of treating cylindrical shells with clamped edges is to employ the expansions in equations (14) to (18) modified by substituting functions of the type

$$\varphi_m(x)=\sin\frac{m\pi x}{a}\sin\frac{\pi x}{a}=\frac{1}{2}\left[\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\right] \tag{19}$$

wherever functions of the type $\sin\frac{m\pi x}{a}$ appear, with a similar substitution for functions of y (all terms involving summation subscripts m and n are thus changed; terms involving λ, such as $\sin\frac{\pi x}{\lambda}$ remain unchanged). The functions $\varphi_m(x)$ form a complete set so that finite expansions for w of the type suggested for shells with clamped edges as well as those for shells with simply supported edges may be used to represent the buckle deformation to any desired degree of accuracy.

BOUNDARY CONDITIONS

Simply supported edges.—Appendix D shows that, if the buckling stress of a simply supported shell is found by means of the expansions for w given in the preceding section entitled "Deflection Function," the boundary or edge conditions implied for the median-surface displacements u and v are zero displacement along each of the edges of a cylinder or curved panel and free displacement normal to each edge. (Although the proof given used equation (5), the proof could equally well have been based on equation (7).)

The boundary conditions for simple support may thus be written, at a curved edge (x=Constant),

$$w=\frac{\partial^2 w}{\partial x^2}=v=\frac{\partial^2 F}{\partial y^2}=0 \tag{20}$$

and, at a straight edge (y=Constant),

$$w=\frac{\partial^2 w}{\partial y^2}=u=\frac{\partial^2 F}{\partial x^2}=0 \tag{21}$$

Clamped edges.—By a method similar to that in appendix D solutions using the functions suggested in the preceding section for the treatment of clamped edges can be shown to correspond to the boundary conditions zero displacement normal to an edge and free displacement along an edge.

The boundary conditions for clamped edges therefore become, at a curved edge (x=Constant),

$$w=\frac{\partial w}{\partial x}=u=\frac{\partial^2 F}{\partial x^2}=0 \tag{22}$$

and, at a straight edge (y=Constant),

$$w=\frac{\partial w}{\partial y}=v=\frac{\partial^2 F}{\partial y^2}=0 \tag{23}$$

Discussion.—As mentioned in part I, the boundary conditions implied for u and v in the case of simply supported edges are appropriate for cylinders or panels bounded by light bulkheads or deep stiffeners, which are stiff in their own planes but may be readily warped out of their planes.

The boundary conditions on u and v appropriate for a clamped edge would seem to be zero displacement normal to the edge and zero, rather than free, displacement along the edge. Comparison of critical stresses for shells with clamped edges found by the method in the present paper with critical stresses found by the method in references 7 and 8, giving boundary conditions $u=v=0$, however, indicates that the imposition of the added requirement of zero displacement along the edge ordinarily has very little effect on the critical stresses.

A less satisfactory method for solving problems concerning shells with clamped edges involves the use of functions of the type

$$\frac{1}{m}\sin\frac{m\pi x}{a}-\frac{1}{m+2}\sin\frac{(m+2)\pi x}{a}$$

instead of those described by equation (19). In this method, the functions used are those for simple support taken in such combinations that the edge slope is zero. Use of such functions leads to the same boundary conditions on u and v as were described for simply supported edges; at the edge y=Constant, for example, the boundary conditions become

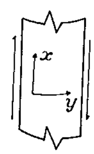
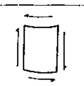
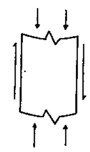
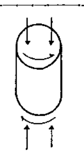
$$w=\frac{\partial w}{\partial y}=u=\frac{\partial^2 F}{\partial x^2}=0 \tag{24}$$

The use of these functions to represent shells with clamped edges is not recommended, however, for the following reasons: The associated boundary conditions seem to be artificial and unlikely to be reproduced even approximately in actual construction; the method leads in some cases to solutions that differ considerably from the solution for ideal clamped-edge conditions in which $u=v=0$; and the solutions obtained generally converge rather poorly.

APPLICATIONS AND DISCUSSION

Among the more difficult shell-stability problems to treat theoretically are those which involve shear stresses. In fact, until 1934, when Donnell's paper on critical shear stress of a cylinder in torsion was published (reference 4), such problems were generally regarded as impracticable to solve. In order to illustrate the type of solution to be found by the method of analysis just outlined and the effect of boundary conditions on critical stresses, the results obtained for a number of shell-stability problems involving shear stresses are reproduced and discussed briefly here. The problems treated are summarized in table I.

TABLE I.—INDEX OF PROBLEMS TREATED

Problem	Figure	Reference	Edge condition
	12	20	Simply supported ($u=0$, $v\neq 0$). Clamped ($u\neq 0$, $v=0$). Clamped ($u=0$, $v\neq 0$).
	11	20	Simply supported ($u=0$, $v\neq 0$). Clamped ($u\neq 0$, $v=0$). Clamped ($u=0$, $v\neq 0$).
		7 (Leggett)	Simply supported ($u=v=0$). Clamped ($u=v=0$).
	7	14	Simply supported. Clamped.
	13	21	Simply supported.
	14	21	Simply supported.
	Not shown	22	Simply supported. Clamped.
	15 (a)	23	Simply supported.
	15 (b)	23	Clamped.

Critical shear stress of long curved strip.—The critical shear stress for a long plate with transverse curvature is given by the equation

$$\tau_{cr}=k_s\frac{\pi^2 D}{b^2 t}$$

where k_s is a dimensionless coefficient, the value of which depends upon the dimensions of the strip, Poisson's ratio for the material, and the type of edge support. In figure 12 (fig. 1 of reference 20) the shear-stress coefficient k_s is given for plates with simply supported edges and with clamped edges. This solution for simply supported edges coincides with that given by Kromm (reference 6).

As indicated in the previous section entitled "Boundary Conditions," the solution corresponding to the boundary conditions of equation (24) (dashed curve of fig. 12) is poorly convergent and deviates appreciably from the results for completely fixed edges. Figure 12 shows this poor convergence in the limiting case of a flat plate, for which the critical stress is independent of boundary conditions on u and v. Even a tenth-order determinant led to a result that is 7 percent above the true solution; whereas the result using a fourth-order determinant obtained with the deflection functions recommended for clamped edges is only 1 percent above.

In figure 11 (fig. 2 of reference 20) the solutions given in figure 12 are compared with the results given by Leggett (reference 7) for simply supported and clamped edges with $u=v=0$ at each edge. Throughout the range for which they are given, Leggett's results for clamped edges differ only slightly from those of the present paper. On the other hand, the previously mentioned discrepancy between the results for completely fixed edges ($u=v=0$) and those for the boundary conditions of equation (24) (dashed curve) may be inferred from this figure to be considerable for large values of Z. A minimum measure of this discrepancy is the distance between the clamped-edge curves for $v=0$ and for $u=0$ in figure 11, since Leggett's curve must always lie above the curve for $v=0$.

The reason for the marked increase in buckling stress of simply supported curved strips when the edges are restrained against circumferential displacement during buckling is discussed in part I.

Critical shear stress of cylinder in torsion.—The critical shear stress of a cylinder subjected to torsion is given by the equation

$$\tau_{cr}=k_s\frac{\pi^2 D}{L^2 t}$$

In figure 7 (fig. 1 of reference 14) the values of k_s are given for cylinders with simply supported edges (boundary conditions of equation (20)) and cylinders with clamped edges (boundary conditions of equation (22)). At high values of Z, the values of k_s for thick cylinders are given by special curves for various values of $\frac{r}{t}\sqrt{1-\mu^2}$, as discussed in part I. At values of Z greater than about 100 only a small increase in buckling stress is caused by clamping the edges.

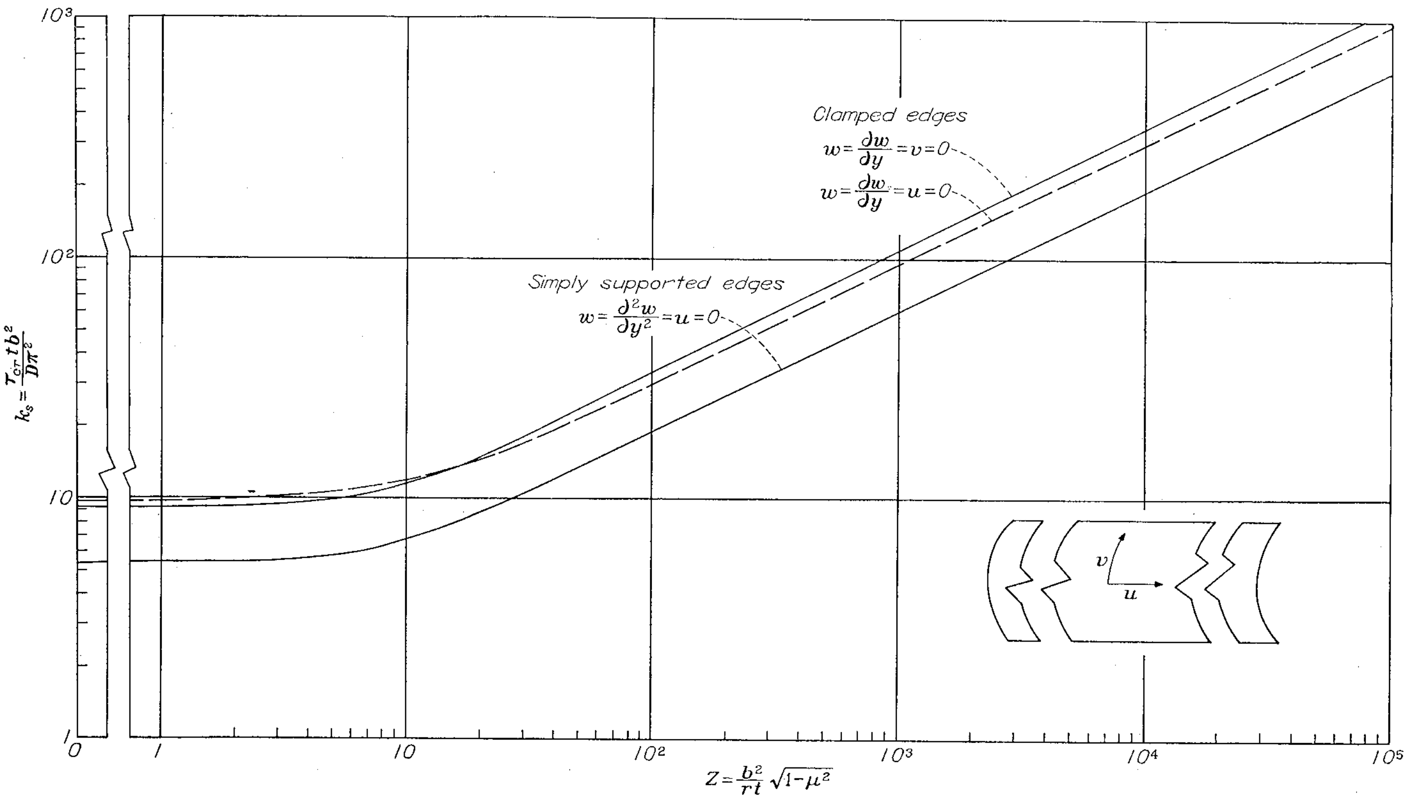


FIGURE 12.—Critical-shear-stress coefficients for a long curved strip. (Fig. 1 of reference 20.)

The results indicated in figure 7 are in very close agreement with Donnell's results for the same problem, except in the range 5 $<Z<$ 500 where the somewhat lower curves of the present paper represent a more accurate solution.

Part I shows that boundary conditions imposed upon u and v at the curved edges of a panel or cylinder have an almost insignificant effect on the buckling stresses, whereas conditions imposed on v at the straight edges may be quite important. Comparison of figure 12, in which boundary conditions on straight edges are considered, with figure 7, in which conditions on curved edges are considered, indicates that a similar situation exists with respect to restraint against edge rotation.

Critical shear stress of curved panel.—The values of k_s giving the critical shear stresses of simply supported curved rectangular panels are given in figures 13 and 14 (figs. 1 and 2, respectively, of reference 21). The corresponding boundary conditions on u and v are zero displacement parallel to

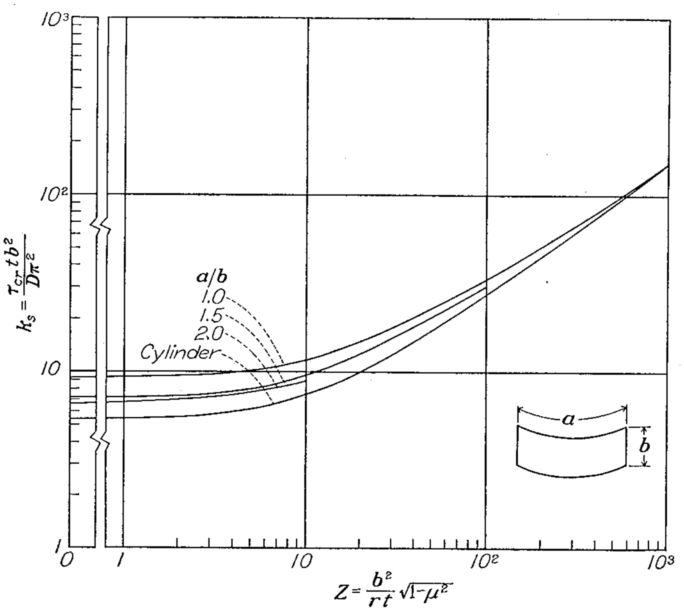


FIGURE 13.—Critical-shear-stress coefficients for simply supported curved panels having circumferential dimension greater than axial dimension. (Fig. 1 of reference 21.)

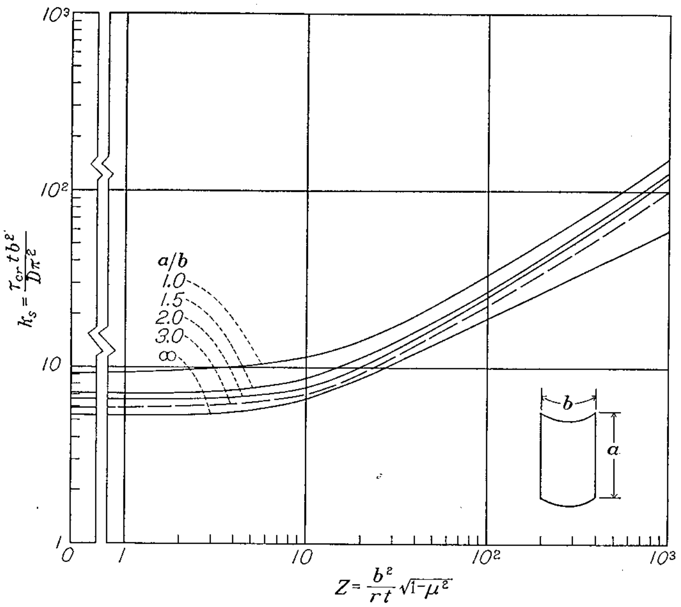


FIGURE 14.—Critical-shear-stress coefficients of simply supported curved panels having axial dimensions greater than circumferential dimension. (Dashed curve estimated.) (Fig. 2 of reference 21.)

the edges and free warping normal to the edges. Figure 13 indicates that, as the curvature parameter Z increases, the critical shear stresses of panels having a circumferential dimension greater than the axial dimension approach those for a cylinder. Figure 14 indicates that, as the curvature parameter Z increases, the critical shear stresses for panels having an axial dimension greater than the circumferential dimension deviate more and more from the critical shear stress for an infinitely long curved plate. Reference 21 shows that the reason for this deviation in figure 14 is that at high curvatures the buckling stresses of these panels, as well as those of figure 13, approach those of the cylinder obtained by extending the circumferential dimensions of the panels.

The effects of boundary conditions in the limiting cases of infinitely long curved strips (fig. 12) and of complete cylinders (fig. 7) suggest that the curves of figure 13 are substantially independent of edge restraint at large values of Z but that the curves of figure 14 would be considerably affected by a change in edge restraint.

Long curved strips under combined shear and direct axial stress.—Reference 22 shows that the theoretical interaction curve for a long curved strip under combined shear stress and direct axial stress is approximately parabolic when the edges are either simply supported or clamped, regardless of the value of Z. This parabola is given by the formula

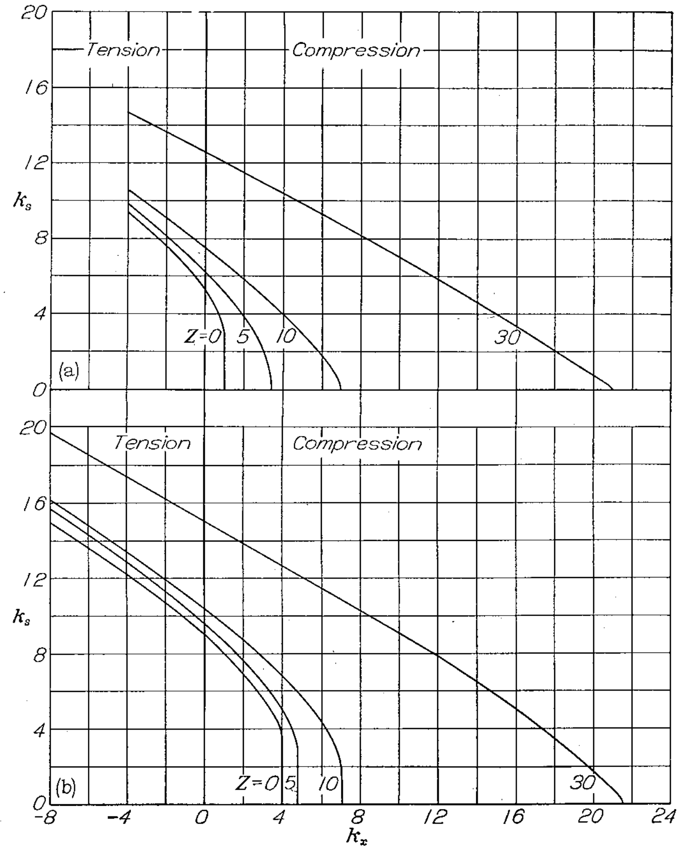
$$R_s^2 + R_x = 1$$

where R_s and R_x are the shear-stress and compressive-stress ratios, respectively.

At high values of Z curved strips, like cylinders, buckle at compressive stresses considerably below the theoretical critical stresses. In order to take this condition into account, certain modifications in the theoretical results are proposed in reference 22 for use in design.

Cylinders under combined shear and direct axial stress.—The theoretically determined combinations of shear stress and direct axial stress which cause a cylinder with simply supported and clamped edges to buckle are shown in figure 15 (fig. 1 of reference 23). Considerable variation in the shape of the interaction curves occurs for low values of Z. For high values of Z the interaction curves for either simply supported or clamped edges are similar to the curve for $Z=30$.

Because cylinders actually buckle at a small fraction of their theoretical critical compressive stress, the theoretical interaction curves of figure 15 cannot be expected to be in satisfactory agreement with experiment when a very appreciable amount of compression is present. For semiempirical curves and a check of available test data, see reference 23.



(a) Simply supported edges.
(b) Clamped edges.

FIGURE 15.—Critical combinations of shear-stress and direct-axial-stress coefficients for cylinders. (Fig. 1 of reference 23.)

CONCLUDING REMARKS

The use of Donnell's equation to find the buckling stresses of simply supported cylindrical shells leads to simpler results and involves less labor than the use of equations in which second-order terms are retained. The buckling stresses found by use of Donnell's equation are in reasonable agreement with results based on other theoretical calculations. Except for the case of axial loading, they are also in reasonable agreement with test results. Boundary conditions having to do with axial and circumferential displacements cannot be handled directly by use of Donnell's equation. This disadvantage is not considered serious, however, because the boundary conditions on axial and circumferential displacement, which are implied by the simple solutions given, correspond approximately to those that are most likely to occur in practical construction and because in many cases the buckling stress is not very sensitive to these boundary conditions. The restriction to simply supported edges in Donnell's equation can be removed by the introduction of a new equation which is equivalent to Donnell's equation but is better adapted to solution by Fourier series. This modified equation can be solved for the buckling stresses of curved sheet having either simply supported or clamped edges by established methods essentially equivalent to those in use for flat sheet. This approach permits a simple and straightforward solution to be given for a number of problems previously considered rather formidable.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *March 20, 1947.*

APPENDIX A

SIMPLIFIED EQUATIONS OF EQUILIBRIUM FOR CYLINDRICAL SHELLS

The principal sets of simplified equations currently in use for the equilibrium of cylindrical shells are listed for convenient reference. The various sets of equations are equivalent. The reference papers in which the equations are derived are also listed. The equations given are generally not identical with those in the reference papers but are modified in certain respects to include all the loading conditions studied in the present paper or to put them in the notation of the present paper.

The three following simultaneous equations in displacements u, v, and w (reference 3) are derived from the conditions of static equilibrium:

$$\frac{\partial^2 u}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\mu}{2}\frac{\partial^2 v}{\partial x \partial y}+\frac{\mu}{r}\frac{\partial w}{\partial x}=0 \tag{A1}$$

$$\frac{\partial^2 v}{\partial y^2}+\frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\mu}{2}\frac{\partial^2 u}{\partial x \partial y}+\frac{1}{r}\frac{\partial w}{\partial y}=0 \tag{A2}$$

$$D\nabla^4 w+\frac{Et}{r(1-\mu^2)}\left(\frac{\partial v}{\partial y}+\mu\frac{\partial u}{\partial x}+\frac{w}{r}\right)+ t\left(\sigma_x\frac{\partial^2 w}{\partial x^2}+2\tau\frac{\partial^2 w}{\partial x \partial y}+\sigma_y\frac{\partial^2 w}{\partial y^2}-\frac{\sigma_y}{r}\right)+p=0 \tag{A3}$$

Two simultaneous equations in deflection w and stress function F (reference 6) are as follows:

$$\nabla^4 F+\frac{E}{r}\frac{\partial^2 w}{\partial x^2}=0 \tag{A4}$$

$$D\nabla^4 w+t\left(\sigma_x\frac{\partial^2 w}{\partial x^2}+2\tau\frac{\partial^2 w}{\partial x \partial y}+\sigma_y\frac{\partial^2 w}{\partial y^2}-\frac{1}{r}\frac{\partial^2 F}{\partial x^2}-\frac{\sigma_y}{r}\right)+ p=0 \tag{A5}$$

A single equation in deflection w (Donnell's equation, reference 4) is

$$D\nabla^8 w+\frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4}+t\nabla^4\left(\sigma_x\frac{\partial^2 w}{\partial x^2}+2\tau\frac{\partial^2 w}{\partial x \partial y}+ \sigma_y\frac{\partial^2 w}{\partial y^2}-\frac{\sigma_y}{r}\right)+\nabla^4 p=0 \tag{A6}$$

The relationships between u and w and between v and w are (reference 4)

$$r\nabla^4 u=-\mu\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x \partial y^2} \tag{A7}$$

$$r\nabla^4 v=-(2+\mu)\frac{\partial^3 w}{\partial x^2 \partial y}-\frac{\partial^3 w}{\partial y^3} \tag{A8}$$

APPENDIX B

THEORETICAL SOLUTIONS

Donnell's equation for the equilibrium of cylindrical shells is used to investigate the stability of simply supported cylinders subject to lateral pressure, axial compression, and hydrostatic pressure, and of simply supported curved strips long in the axial direction subject to axial compression.

CYLINDER UNDER LATERAL PRESSURE

If bending of the cylinder wall is neglected, constant lateral pressure on a cylinder causes only circumferential stresses. Donnell's equation (equation (A6)) then reduces to

$$D\nabla^8 w+\frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4}+\sigma_y t\nabla^4\frac{\partial^2 w}{\partial y^2}=0 \tag{B1}$$

where

$$\sigma_y=\frac{pr}{t}$$

and p is the pressure applied. (By virtue of the preceding equation the terms involving p and $\frac{\sigma_y}{r}$ appearing in equation (A6) cancel in the present case.) Division of equation (B1) by D results, with proper substitutions, in the following equation:

$$\nabla^8 w+\frac{12Z^2}{L^4}\frac{\partial^4 w}{\partial x^4}+k_y\frac{\pi^2}{L^2}\nabla^4\frac{\partial^2 w}{\partial y^2}=0 \tag{B2}$$

The boundary conditions corresponding to simply supported edges (no deflection and no moment along the edges) are

$$w(0,y)=w(L,y)=0$$

$$\frac{\partial^2 w}{\partial x^2}(0,y)=\frac{\partial^2 w}{\partial x^2}(L,y)=0$$

A solution of equation (B2) satisfying the boundary conditions for simple support is

$$w=w_0\sin\frac{\pi y}{\lambda}\sin\frac{m\pi x}{L} \tag{B3}$$

where λ is the half wave length in the circumferential direction. Combining equation (B3) and equation (B2) yields the following equation:

$$(m^2+\beta^2)^4+\frac{12Z^2m^4}{\pi^4}-k_y\beta^2(m^2+\beta^2)^2=0 \tag{B4}$$

The solution of equation (B4) for k_y is

$$k_y=\frac{(m^2+\beta^2)^2}{\beta^2}+\frac{12Z^2m^4}{\pi^4\beta^2(m^2+\beta^2)^2} \tag{B5}$$

where

$$\beta=\frac{L}{\lambda}$$

The critical value for k_y is found by minimizing the right-hand side of equation (B5) with respect to m and β. If the numerator and denominator of the last term in equation (B5) are divided by m^4, it becomes evident that under the restriction of integral values of m, k_y will be a minimum when $m=1$. Equation (B5) therefore becomes

$$k_y=\frac{(1+\beta^2)^2}{\beta^2}+\frac{12Z^2}{\pi^4\beta^2(1+\beta^2)^2} \tag{B6}$$

The results found by minimizing this expression for k_y with respect to β (considered continuously variable) are shown in figure 1 by the curve independent of r/t.

At low values of Z, buckling is characterized by a large number of circumferential waves. As Z increases, the number of circumferential waves decreases until it finally becomes two $\left(\lambda=\frac{\pi r}{2}\right)$, corresponding to buckling into an elliptical cross section. The curves for buckling into two circumferential waves are shown in figure 1 as the curves for various values of $\frac{r}{t}\sqrt{1-\mu^2}$. The equations for these curves are found by substituting in equation (B5) the last of the following expressions for β:

$$\beta=\frac{L}{\lambda}=\frac{2L}{\pi r}=\frac{2}{\pi}\sqrt{\frac{Z}{\frac{r}{t}\sqrt{1-\mu^2}}}$$

CYLINDER IN AXIAL COMPRESSION

When only axial stress is present, equation (A6) becomes

$$D\nabla^8 w+\frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4}+\sigma_x t\nabla^4\frac{\partial^2 w}{\partial x^2}=0 \tag{B7}$$

Division by D results, with proper substitutions, in the following equation:

$$\nabla^8 w+\frac{12Z^2}{L^4}\frac{\partial^4 w}{\partial x^4}+k_x\frac{\pi^2}{L^2}\nabla^4\frac{\partial^2 w}{\partial x^2}=0 \tag{B8}$$

Combination of the deflection equation (B3) with equation (B8) yields the following equation:

$$(m^2+\beta^2)^4+\frac{12Z^2m^4}{\pi^4}-k_x m^2(m^2+\beta^2)^2=0 \tag{B9}$$

The solution of equation (B9) for k_x is

$$k_x=\frac{(m^2+\beta^2)^2}{m^2}+\frac{12Z^2m^2}{\pi^4(m^2+\beta^2)^2}$$

The critical value of k_x for a given value of Z may be found by minimizing k_x with respect to the parameter

$$\frac{(m^2+\beta^2)^2}{m^2}$$

If no restrictions are placed on the value that this parameter can take, the minimum value of k_x is found to be

$$k_x=\frac{4\sqrt{3}}{\pi^2}Z=0.702Z \tag{B10}$$

which coincides with the results generally given for the buckling of long cylinders.

For values of Z below 2.85, however, the straight-line formula (equation (B10)) cannot be used, since it implies either imaginary values of the circumferential wave length λ or the number of axial half waves m below unity. The critical stress coefficient k_x for $Z<2.85$ is found by substituting the limiting values $\beta=0$ and $m=1$ in equation (B9). The results are shown in figure 3.

CYLINDER UNDER HYDROSTATIC PRESSURE

Hydrostatic pressure applied to a closed cylinder produces the following axial and circumferential stresses:

$$\sigma_x=\frac{pr}{2t}$$

$$\sigma_y=\frac{pr}{t}$$

The equation of equilibrium (equation (A6)) when both circumferential and axial stress are present is (since $\nabla^4 p=0$)

$$D\nabla^8 w+\frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4}+\sigma_x t\nabla^4\frac{\partial^2 w}{\partial x^2}+\sigma_y t\nabla^4\frac{\partial^2 w}{\partial y^2}=0 \tag{B11}$$

By use of the definition

$$C_p=\frac{prL^2}{D\pi^2}$$

equation (B11) can be written

$$\nabla^8 w+\frac{12Z^2}{L^4}\frac{\partial^4 w}{\partial x^4}+C_p\frac{\pi^2}{L^2}\nabla^4\left(\frac{1}{2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0 \tag{B12}$$

If the deflection equation (equation (B3)) is combined with equation (B12), the following expression results for C_p:

$$C_p=\frac{(m^2+\beta^2)^2}{\frac{m^2}{2}+\beta^2}+\frac{12Z^2m^4}{\pi^4(m^2+\beta^2)^2\left(\frac{m^2}{2}+\beta^2\right)} \tag{B13}$$

The critical value of C_p is found by minimizing the right-hand side of equation (B13) with respect to m and β, with due regard to the values which m and β may assume. It can be shown that the minimum value of C_p is found by taking m equal to 1, so that equation (B13) becomes

$$C_p=\frac{(1+\beta^2)^2}{\frac{1}{2}+\beta^2}+\frac{12Z^2}{\pi^4(1+\beta^2)^2\left(\frac{1}{2}+\beta^2\right)} \tag{B14}$$

Equation (B14) is equivalent to an equation derived by Von Mises (reference 3, p. 479). The results of minimizing C_p with respect to β are shown in figure 4. (The curves given for various values of $\frac{r}{t}\sqrt{1-\mu^2}$ have the same significance as in the case of a cylinder buckling under lateral pressure alone.)

LONG CURVED STRIP IN AXIAL COMPRESSION

Because it merely describes equilibrium at a point, equation (B7) applies to the buckling of a long curved strip as well as to cylinder buckling. In modifying this equation to obtain nondimensional coefficients as in equation (B8), however, it is convenient to define k_x and Z in terms of the width of the strip b rather than in terms of the axial length L, which applied in the case of the cylinder. Accordingly, equations (B7) and (B8) for a cylinder in axial compression may be applied also to the buckling of a curved strip, long in the axial direction, subjected to axial compression, provided the curved width b is everywhere substituted for the length L. Substitution of the deflection

$$w=w_0\sin\frac{\pi x}{\lambda}\sin\frac{n\pi y}{b}$$

into equation (B2) (modified by substitution of b for L) gives

$$k_x=\frac{(n^2+\beta^2)^2}{\beta^2}+\frac{12Z^2\beta^2}{\pi^4(n^2+\beta^2)^2} \tag{B15}$$

where

$$\beta=\frac{b}{\lambda}$$

Equation (B15) is very similar to equation (B9) and each equation yields the same critical value for k_x at large values of Z. At small values of Z, the minimum value of k_x is found by taking $n=1$ in equation (B15) and minimizing with respect to β the resulting expression for k_x. The results are given in figure 10 together with results found by Leggett (reference 8).

APPENDIX C

PARAMETERS

It is shown that Donnell's equation implies that under certain limitations the buckling coefficient k, familiar from flat-plate theory, can be expressed in terms of the curvature parameter Z alone in the case of a complete cylinder or a curved rectangular panel of given length-width ratio.

Donnell's equation (A6) is (when p is constant or zero)

$$D\nabla^8 w+\frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4}+t\nabla^4\left(\sigma_x\frac{\partial^2 w}{\partial x^2}+2\tau\frac{\partial^2 w}{\partial x\partial y}+\sigma_y\frac{\partial^2 w}{\partial y^2}\right)=0 \quad \text{(C1)}$$

Let

$$\frac{x}{b}=\xi$$

$$\frac{y}{b}=\eta$$

and

$$\nabla_G{}^2=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}$$

Then

$$\nabla_G{}^2=b^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

Multiplication of equation (C1) by b^8 and substitution of the dimensionless coordinates ξ and η gives

$$D\nabla_G{}^8 w+\frac{Etb^4}{r^2}\frac{\partial^4 w}{\partial\xi^4}+b^2t\nabla_G{}^4\left(\sigma_x\frac{\partial^2 w}{\partial\xi^2}+2\tau\frac{\partial^2 w}{\partial\xi\partial\eta}+\sigma_y\frac{\partial^2 w}{\partial\eta^2}\right)=0$$

Division by D results in

$$\nabla_G{}^8 w+\frac{Etb^4}{Dr^2}\frac{\partial^4 w}{\partial\xi^4}+\frac{b^2t}{D}\nabla_G{}^4\left(\sigma_x\frac{\partial^2 w}{\partial\xi^2}+2\tau\frac{\partial^2 w}{\partial\xi\partial\eta}+\sigma_y\frac{\partial^2 w}{\partial\eta^2}\right)=0$$

or, since $D=\frac{Et^3}{12(1-\mu^2)}$,

$$\nabla_G{}^8 w+12Z^2\frac{\partial^4 w}{\partial\xi^4}+\pi^2\nabla_G{}^4\left(k_x\frac{\partial^2 w}{\partial\xi^2}+2k_s\frac{\partial^2 w}{\partial\xi\partial\eta}+k_y\frac{\partial^2 w}{\partial\eta^2}\right)=0 \quad \text{(C2)}$$

where

$$Z=\frac{b^2}{rt}\sqrt{1-\mu^2}$$

$$k_x=\frac{\sigma_x tb^2}{D\pi^2}$$

$$k_s=\frac{\tau tb^2}{D\pi^2}$$

$$k_y=\frac{\sigma_y tb^2}{D\pi^2}$$

Even without solving this equation it is clear that w must be a function of the independent variables ξ and η, and also the parameters Z, k_x, k_s, and k_y, and the derivatives of w will be functions of the same variables and parameters. Thus, if only one type of loading (represented by the buckling coefficient k) is present, equation (C2) may be written

$$f_1(\xi,\eta,Z,k)+12Z^2f_2(\xi,\eta,Z,k)+\pi^2kf_3(\xi,\eta,Z,k)=0 \quad \text{(C3)}$$

where f_1, f_2, and f_3 are definite, though unknown, functions. The variables ξ and η may now be eliminated by integration of both sides of this equation over the entire range of ξ and η. In the case of a curved panel of circumferential dimension a and axial dimension b the resulting equation is

$$\int_0^1 d\xi\int_0^{\frac{a}{b}} d\eta[f_1(\xi,\eta,Z,k)+12Z^2f_2(\xi,\eta,Z,k)+\pi^2kf_3(\xi,\eta,Z,k)]=0 \quad \text{(C4)}$$

The integrals of the functions f_1, f_2, and f_3 depend only upon Z, k, and the value of the ratio a/b. Accordingly, equation (C4) implies that a relationship of the following type exists:

$$f_4\left(k,Z,\frac{a}{b}\right)=0 \quad \text{(C5)}$$

Equation (C5) indicates that for any given value of the panel aspect ratio a/b, the critical-stress coefficient k depends only upon Z.

If a complete cylinder of length L rather than a panel of length b is under consideration, and the deflection w is periodic with wave length 2λ in the circumferential coordinate, the integration

$$\int_0^{\frac{a}{b}} d\eta$$

appearing in equation (C4) may be replaced by

$$\int_0^{\frac{2\lambda}{L}} d\eta$$

where ξ and η are now defined as x/L and y/L, respectively. The result then becomes

$$f_5\left(k,Z,\frac{2\lambda}{L}\right)=0$$

or

$$k=f_6\left(Z,\frac{2\lambda}{L}\right) \quad \text{(C6)}$$

The actual buckling stress is found by minimizing k with respect to $2\lambda/L$.

Theoretically, λ must satisfy the equation

$$\pi r = n\lambda \tag{C7}$$

where n is the number of circumferential waves and therefore an integer. When many circumferential waves are present, however, this restriction does not significantly affect the buckling stress, and the minimization of k with respect to $\frac{2\lambda}{b}$ (considered continuously variable) leads to the result

$$k = f_7(Z) \tag{C8}$$

Equation (C8) indicates that provided the number of circumferential waves is not too small the critical-stress coefficient for a cylinder depends for practical purposes only upon the curvature parameter Z.

When n is so small that its integral character must be taken into account, it appears from equations (C6) and (C7) that k depends upon both Z and r/L. Since, however,

$$\left(\frac{r}{L}\right)^2 = \frac{1}{Z}\frac{r}{t}\sqrt{1-\mu^2}$$

k for small values of n can alternatively be expressed in terms of Z and $\frac{r}{t}\sqrt{1-\mu^2}$, as in figures 1, 4, and 7.

By a similar analysis, it can be shown that when the buckling of a cylinder under hydrostatic pressure is represented by plotting the pressure coefficient C_p against Z, a single curve is obtained except where the small number of circumferential waves requires splitting the curve into a series of curves for different values of $\frac{r}{t}\sqrt{1-\mu^2}$.

APPENDIX D

BOUNDARY CONDITIONS ON EDGE DISPLACEMENTS WITHIN THE MEDIAN SURFACE

The solution of Donnell's eighth-order partial differential equation for the stability of cylindrical shells is not unique under the imposition of the ordinary boundary conditions for simply supported or clamped edges. Two more boundary conditions at each edge, for example, one condition for u and one for v, are required to define completely the physical problem and are therefore needed to make the solution unique. Because only w appears in the equation, boundary conditions on u and v cannot be imposed directly; they may, however, be implied by the method of solution. The purpose of this appendix is to show what boundary conditions on u and v are implied by the method of solution used in the present paper. In order to simplify the discussion, the analysis will first be made for the case when only axial compression is present and will then be extended to other cases.

When only axial stress is present, Donnell's equation (equation (A6)) becomes

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} + \sigma_x t \nabla^4 \frac{\partial^2 w}{\partial x^2} = 0$$

If the shell described by this equation is a curved panel with the origin of coordinates in one corner of the panel, a solution satisfying the usual boundary conditions for simple support is

$$w = w_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{D1}$$

where m and n are integers. This solution is also the solution to the problem of the buckling of an infinite two-dimensional array of panels identical to the one under consideration. (See fig. 16.) When such an array buckles, the displacements u, v, and w as well as the stresses, described by the stress function F, may be presumed to be periodic over the interval $2a$ in the axial direction and $2b$ in the circumferential direction.

Any function $u(x, y)$ that is periodic with a wave length $2a$ in the x-direction and with a wave length $2b$ in the y-direction may be expanded as follows (see, for example, reference 26):

$$\begin{aligned} u = &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} + \\ &\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} b_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} + \\ &\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} + \\ &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \end{aligned} \tag{D2}$$

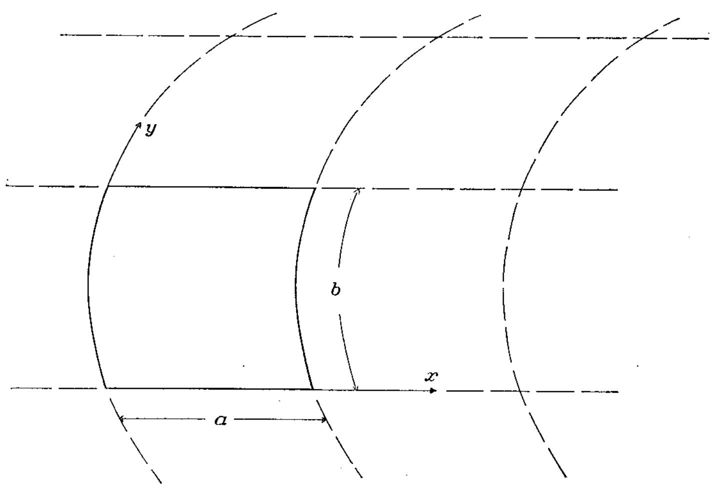


FIGURE 16.—Two-dimensional array of identical curved panels.

The relationship which must exist between u and w is (equation (A7))

$$r\nabla^4 u = -\mu \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}$$

Substitution into this equation of the expressions for u and w from equations (D2) and (D1), respectively, and use of the orthogonality of the functions in equation (D2) leads to the result

$$u = \frac{w_0 \left[\mu \left(\frac{m\pi}{a}\right)^3 - \frac{m\pi}{a}\left(\frac{n\pi}{b}\right)^2\right]}{r\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Accordingly, the boundary conditions on u are

$$u(x,0)=0 \tag{D3}$$

$$u(x,b)=0 \tag{D4}$$

$$\frac{\partial u}{\partial x}(0,y)=0 \tag{D5}$$

$$\frac{\partial u}{\partial x}(a,y)=0 \tag{D6}$$

Similarly by use of equation (A8) instead of equation (A7) it can be shown that the boundary conditions on v are

$$v(0,y)=0 \tag{D7}$$

$$v(a,y)=0 \tag{D8}$$

$$\frac{\partial v}{\partial y}(x,0)=0 \tag{D9}$$

$$\frac{\partial v}{\partial y}(x,b)=0 \tag{D10}$$

The boundary conditions of equations (D5), (D6), (D9), and (D10) may be combined to give four boundary conditions on the stresses induced by buckling. These boundary conditions, which are also derivable from equation (A4) by a method analogous to that just used to derive the conditions relating to u, are

$$\frac{\partial^2 F}{\partial y^2}(0,y)=0 \tag{D11}$$

$$\frac{\partial^2 F}{\partial y^2}(a,y)=0 \tag{D12}$$

$$\frac{\partial^2 F}{\partial x^2}(x,0)=0 \tag{D13}$$

$$\frac{\partial^2 F}{\partial x^2}(x,b)=0 \tag{D14}$$

where $\frac{\partial^2 F}{\partial y^2}$ and $\frac{\partial^2 F}{\partial x^2}$ are, respectively, the median-surface axial and circumferential stresses caused by buckling. The eight boundary conditions given by equations (D3), (D4), (D7), (D8), and equations (D11) to (D14), plus the eight boundary conditions on w for simple support of the four panel edges taken together uniquely determine the buckling stress.

Although the preceding discussion of boundary conditions started with the assumption of axial stress only, the only use made of this assumption was in obtaining equation (D1) as the solution for the buckling deformation. The same deformation, and hence the same arguments, apply when circumferential stress is present. When shear is present, a series of terms of the type in equation (D1) must be used to represent the deflection surface, and hence series of terms occur in the expressions for u, v, and F. Since the boundary conditions derived in the preceding analysis apply to each of the terms individually, by the principle of superposition they must also apply for the sum, so that equations (D11) to (D14) represent the boundary condition no matter what the applied stresses are.

In summary it may be stated that the substitution of one or more terms of a double-sine-series expansion for w into Donnell's equation and solution of the resulting equation for the buckling stress gives the solution corresponding to the following boundary conditions:

(1) Each edge of the panel (or cylinder) is simply supported; that is, the displacement normal to the surface of the panel and the applied moments are zero at the edges.

(2) Motion parallel to each edge during buckling is prevented entirely.

(3) Motion normal to each edge in the plane of the sheet occurs freely.

APPENDIX E

COMPARISON OF RESULTS OBTAINED BY USING DONNELL'S EQUATION AND THE MODIFIED EQUATION IN THE STABILITY ANALYSIS OF SIMPLY SUPPORTED CURVED PANELS

SOLUTION OF DONNELL'S EQUATION

Donnell's equation expressing the equilibrium of a curved panel under constant median-surface stresses can be written in general form as

$$D\nabla^8 w+\frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4}+\sigma_x t\nabla^4\frac{\partial^2 w}{\partial x^2}+2\tau t\nabla^4\frac{\partial^2 w}{\partial x\partial y}+\sigma_y t\nabla^4\frac{\partial^2 w}{\partial y^2}=0 \qquad \text{(E1)}$$

where x is the axial coordinate and y the circumferential coordinate. Division of equation (E1) by D and the introduction of the dimensionless stress coefficients k_x, k_y, and k_s, and the curvature parameter Z results in the following equation:

$$\nabla^8 w+\frac{12Z^2}{b^4}\frac{\partial^4 w}{\partial x^4}+k_x\frac{\pi^2}{b^2}\nabla^4\frac{\partial^2 w}{\partial x^2}+2k_s\frac{\pi^2}{b^2}\nabla^4\frac{\partial^2 w}{\partial x\partial y}+k_y\frac{\pi^2}{b^2}\nabla^4\frac{\partial^2 w}{\partial y^2}=0 \qquad \text{(E2)}$$

where

$$k_x=\sigma_x\frac{b^2 t}{\pi^2 D}$$

$$k_s=\tau\frac{b^2 t}{\pi^2 D}$$

$$k_y=\sigma_y\frac{b^2 t}{\pi^2 D}$$

and

$$Z=\frac{b^2}{rt}\sqrt{1-\mu^2}$$

Equation (E2) can be represented by

$$Q_1(w)=0 \qquad \text{(E3)}$$

where Q_1 is defined as the operator

$$\nabla^8+\frac{12Z^2}{b^4}\frac{\partial^4}{\partial x^4}+k_x\frac{\pi^2}{b^2}\nabla^4\frac{\partial^2}{\partial x^2}+2k_s\frac{\pi^2}{b^2}\nabla^4\frac{\partial^2}{\partial x\partial y}+k_y\frac{\pi^2}{b^2}\nabla^4\frac{\partial^2}{\partial y^2}$$

The equation of equilibrium (equation (E3)) is solved by using the Galerkin method as described in the section entitled "Theory." In applying this method the unknown deflection w is represented in terms of a set of functions (see equation (9)), each of which satisfies the boundary conditions but not in general the equation of equilibrium. A suitable set of functions of this type, which satisfies the boundary conditions for simple support, is

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \text{(E4)}$$

where the origin is taken at a corner of the plate. Substituting in equations (11) and (12)

$$f_m(x)=\sin\frac{m\pi x}{a}$$

$$g_n(y)=\sin\frac{n\pi y}{b}$$

and

$$Q=Q_1$$

and performing the integration over the whole plate (limits $x=0, a; y=0, b$) gives the set of equations

$$a_{mn}\left[\left(m^2+n^2\frac{a^2}{b^2}\right)^4+\frac{12Z^2m^4a^4}{\pi^4b^4}-k_xm^2\frac{a^2}{b^2}\left(m^2+n^2\frac{a^2}{b^2}\right)^2-k_yn^2\frac{a^4}{b^4}\left(m^2+n^2\frac{a^2}{b^2}\right)^2\right]+\frac{32k_s}{\pi^2}\frac{a^3}{b^3}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}a_{pq}\frac{\left(p^2+q^2\frac{a^2}{b^2}\right)^2 mnpq}{(m^2-p^2)\,(n^2-q^2)}=0 \qquad \text{(E5)}$$

where $m=1, 2, 3, \ldots, n=1, 2, 3, \ldots$, and p and q take only those values for which $m\pm p$ and $n\pm q$ are odd numbers.

Equation (E5) represents an infinite set of homogeneous linear equations involving the unknown deflection coefficients a_{ij}. In order for the deflection coefficients to have values other than zero, that is, in order for the panel to buckle, the determinant of the coefficients of the unknown deflection coefficients a_{ij} must vanish. This determinant can be factored into two subdeterminants, one involving the unknown deflection coefficients a_{ij} for which $i\pm j$ is odd and the other involving those coefficients for which $i\pm j$ is even. Buckling occurs, therefore, when either of the two subdeterminants vanishes. Only the buckling criterion involving the even subdeterminant is treated here. This criterion is

$$\begin{array}{l} \\ m=1,\ n=1 \\ m=1,\ n=3 \\ m=2,\ n=2 \\ m=3,\ n=1 \\ m=3,\ n=3 \\ \cdots \end{array}
\begin{array}{c} \begin{array}{cccccc} a_{11} & a_{13} & a_{22} & a_{31} & a_{33} & \cdots \end{array} \\
\left|\begin{array}{cccccc}
M_{11} & 0 & \frac{4}{9}\left(4+4\frac{a^2}{b^2}\right)^2 & 0 & 0 & \cdots \\
0 & M_{13} & -\frac{4}{5}\left(4+4\frac{a^2}{b^2}\right)^2 & 0 & 0 & \cdots \\
\frac{4}{9}\left(1+\frac{a^2}{b^2}\right)^2 & -\frac{4}{5}\left(1+9\frac{a^2}{b^2}\right)^2 & M_{22} & -\frac{4}{5}\left(9+\frac{a^2}{b^2}\right)^2 & \frac{36}{25}\left(9+9\frac{a^2}{b^2}\right)^2 & \cdots \\
0 & 0 & -\frac{4}{5}\left(4+4\frac{a^2}{b^2}\right)^2 & M_{31} & 0 & \cdots \\
0 & 0 & \frac{36}{25}\left(4+4\frac{a^2}{b^2}\right)^2 & 0 & M_{33} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots
\end{array}\right| \end{array} = 0 \tag{E6}$$

where

$$M_{mn}=\frac{\pi^2 b^3}{32k_s a^3}\left[\left(m^2+n^2\frac{a^2}{b^2}\right)^4+\frac{12Z^2m^4a^4}{\pi^4b^4}-k_x m^2\frac{a^2}{b^2}\left(m^2+n^2\frac{a^2}{b^2}\right)^2-k_y n^2\frac{a^4}{b^4}\left(m^2+n^2\frac{a^2}{b^2}\right)^2\right]$$

Division of each column of the determinant in equation (E6) by the proper

$$\left(i^2+j^2\frac{a^2}{b^2}\right)^2$$

gives the simplified equation

$$\begin{array}{c} \begin{array}{cccccc} a_{11}\left(1+\frac{a^2}{b^2}\right)^2 & a_{13}\left(1+9\frac{a^2}{b^2}\right)^2 & a_{22}\left(4+4\frac{a^2}{b^2}\right)^2 & a_{31}\left(9+\frac{a^2}{b^2}\right)^2 & a_{33}\left(9+9\frac{a^2}{b^2}\right)^2 & \cdots \end{array} \\
\left|\begin{array}{cccccc}
N_{11} & 0 & \frac{4}{9} & 0 & 0 & \cdots \\
0 & N_{13} & -\frac{4}{5} & 0 & 0 & \cdots \\
\frac{4}{9} & -\frac{4}{5} & N_{22} & -\frac{4}{5} & \frac{36}{25} & \cdots \\
0 & 0 & -\frac{4}{5} & N_{31} & 0 & \cdots \\
0 & 0 & \frac{36}{25} & 0 & N_{33} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots
\end{array}\right| \end{array} = 0 \tag{E7}$$

where

$$N_{mn}=\frac{\pi^2 b^3}{32k_s a^3}\left[\left(m^2+n^2\frac{a^2}{b^2}\right)^2+\frac{12Z^2m^4a^4}{\pi^4b^4\left(m^2+n^2\frac{a^2}{b^2}\right)^2}-k_x m^2\frac{a^2}{b^2}-k_y n^2\frac{a^4}{b^4}\right]$$

The vanishing of this determinant is the criterion for the symmetrical buckling of the shell. The same buckling criterion results from the use of the modified equation, as is shown in the following section.

SOLUTION OF MODIFIED EQUATION

The modified equation expressing the equilibrium of a curved panel under constant median-surface stresses in general form is

$$D\nabla^4 w+\frac{Et}{r^2}\nabla^{-4}\frac{\partial^4 w}{\partial x^4}+\sigma_x t\frac{\partial^2 w}{\partial x^2}+2\tau t\frac{\partial^2 w}{\partial x\,\partial y}+\sigma_y t\frac{\partial^2 w}{\partial y^2}=0 \tag{E8}$$

Division of equation (E8) by D and simplification of the result gives the following equation:

$$\nabla^4 w+\frac{12Z^2}{b^4}\nabla^{-4}\frac{\partial^4 w}{\partial x^4}+k_x\frac{\pi^2}{b^2}\frac{\partial^2 w}{\partial x^2}+2k_s\frac{\pi^2}{b^2}\frac{\partial^2 w}{\partial x\,\partial y}+k_y\frac{\pi^2}{b^2}\frac{\partial^2 w}{\partial y^2}=0 \tag{E9}$$

Equation (E9) can be represented by

$$Q_2(w)=0 \tag{E10}$$

where Q_2 is defined as the operator

$$\nabla^4+\frac{12Z^2}{b^4}\nabla^{-4}\frac{\partial^4}{\partial x^4}+k_x\frac{\pi^2}{b^2}\frac{\partial^2}{\partial x^2}+2k_s\frac{\pi^2}{b^2}\frac{\partial^2}{\partial x\,\partial y}+k_y\frac{\pi^2}{b^2}\frac{\partial^2}{\partial y^2}$$

By use of the Galerkin method and by use of the expression for w given in equation (E4), the following set of equations analogous to equations (E5) are obtained

$$a_{mn}\left[\left(m^2+n^2\frac{a^2}{b^2}\right)^2+\frac{12Z^2m^4a^4}{\pi^4b^4\left(m^2+n^2\frac{a^2}{b^2}\right)^2}-k_xm^2\frac{a^2}{b^2}-k_yn^2\frac{a^4}{b^4}\right]+$$

$$\frac{32k_sa^3}{\pi^2b^3}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}a_{pq}\frac{mnpq}{(m^2-p^2)(n^2-q^2)}=0 \qquad \text{(E11)}$$

where $m=1, 2, 3, \ldots, n=1, 2, 3, \ldots$, and p and q take only those values such that $m\pm p$ and $n\pm q$ are odd numbers.

As in the case of the solution of Donnell's equation, the stability determinant representing equations (E11) can be factored into an even and an odd subdeterminant. The even one is

$$\begin{array}{l} \\ m=1,\ n=1 \\ m=1,\ n=3 \\ m=2,\ n=2 \\ m=3,\ n=1 \\ m=3,\ n=3 \\ \cdots\cdots \end{array}
\begin{array}{c} \begin{array}{cccccc} a_{11} & a_{13} & a_{22} & a_{31} & a_{33} & \cdots \end{array} \\
\left|\begin{array}{cccccc}
N_{11} & 0 & \frac{4}{9} & 0 & 0 & \cdots \\
0 & N_{13} & -\frac{4}{5} & 0 & 0 & \cdots \\
\frac{4}{9} & -\frac{4}{5} & N_{22} & -\frac{4}{5} & \frac{36}{25} & \cdots \\
0 & 0 & -\frac{4}{5} & N_{31} & 0 & \cdots \\
0 & 0 & \frac{36}{25} & 0 & N_{33} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots
\end{array}\right| \end{array} =0 \qquad \text{(E12)}$$

The stability determinant (equation (E12)) obtained from the modified equation is identical with the simplified stability determinant (equation (E7)) obtained by use of Donnell's equation. This identity holds for the odd as well as the even determinants.

Although the stability determinants obtained by use of the two equations are identical and yield identical buckling loads, the determinant in equation (E7) consists of the coefficients of $a_{ij}\left(i^2+j^2\frac{a^2}{b^2}\right)^2$, whereas the determinant in equation (E12) consists of the coefficients of a_{ij}. Accordingly, although the buckling loads found by the two methods are the same, the buckle patterns are different. Of the two buckle patterns the one found by the use of the modified equation is believed to be correct. This conclusion has been verified for the limiting case of a flat plate ($Z=0$).

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