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A SIMPLIFIED SEMANTICS FOR MODAL LOGIC

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1. Until recently, philosophers could object to modal logic on the grounds that there were no known semantics for the many modal calculuses. To be sure, there were some isolated interpretations for individual systems, but there was no general theory which would apply to the many different systems, and which would give an account, from a semantical point of view, of the relations between them. With the work of Kripke, Hintikka, Kanger, Lemmon, and others, that has, of course, changed.

Kripke's semantics for modal logic begins with the idea of a *model structure*, $\langle G, K, R\rangle$, where K is a nonempty set (intuitively, of possible worlds), R is a relation on K (a relation of relative possibility or alternativeness), and $G \in K$. Truth-functional propositions are evaluated with respect to possible worlds (members of K) in the usual ways, and propositions of the sort, *necessarily B*, are said to be true in a world H in K if and only if B is true in every world, H', in K such that HRH'. A formula is valid if it is true in G for every appropriate model structure, $\langle G, K, R\rangle$. By stipulating different properties for R (e.g. reflexivity, reflexivity and transitivity, etc.) different model structures are defined which validate different classes of formulas corresponding to the different modal calculuses. Thus, e.g., all and only formulas provable in S4 are valid in all model structures in which R is both reflexive and transitive. With some minor modifications this account can be generalized to provide semantics for most all the standard systems of modal logic. (Cf. [11] and [12].) Similar moves are made by Hintikka in [8] and Kanger [9]. Lemmon develops analogous devices in his algebraic semantics for the systems in [13] and [14].

Nevertheless, while the mathematical problem of developing an adequate semantical theory of the modalities may have been solved in this way, philosophers critical of modal logic might still object that since these accounts depend on a notion of a possible world, which is an obscure as the original concepts of possibility and necessity, no real clarification of these modalities has been achieved by these interpretations. Moreover, these critics might also object to the introduction of the relations, R, between

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these supposed possible worlds. For, it might be argued, while the notion of a possible world itself is at least well entrenched in the history of philosophy, and maybe even common sense, the idea of a relation of 'relative possibility' (Kripke) or 'alternativeness' (Hintikka) between worlds is essentially new and has little or no intuitive appeal.[1]

Hintikka says, "The relation will be called the relation of alternativeness, and the sets [worlds] bearing it to some given set, μ, will be called the *alternatives* to μ. *Intuitively, they are* partial descriptions of *those states of affairs which could have been realized instead of* the one described by μ." ([8], p. 66; my italics.) But how should one understand this? What considerations should lead one to say that one world is an alternative to (could have been realized instead of) a given world, but that another is not?

In this paper I shall show a way in which these questions may be attacked, by developing a general semantical theory for modal logic which is simplified in one important direction. With this semantics one can see how, on the basis of one explication of a 'possible world', it is possible to extract definitions of all the relations and related concepts necessary to establish interpretations of the modalities which are adequate and valid for most of the standard calculuses. As a result, it is not necessary to posit relations of 'relative possibility' among the primitives of the semantics for modal logic.

Most of this paper will be devoted to defining the basic concepts of the simplified semantics and to establishing weak completeness theorems for a wide variety of systems. I shall not at this time argue that the conception of a possible world offered here is a natural one, nor shall I discuss philosophical applications of some of the interpretations defined, or reasons why one might prefer certain systems to others. In what follows I will limit my attention to propositional modal logics; I do not anticipate any new problems when quantifiers are added. Much of the ground to be covered here has been thoroughly explored by others. It is necessary for me to go over it once more in order to show that the present methods are as rich as those of the more familiar accounts. In the later sections I do propose semantics for some families of systems which, so far as I know, have not been discussed in the literature.

2. According to standard accounts, to say that a proposition, P, is possible or possibly true in a given world, is to say that P is true in some world which is a possible alternative to the given world. While this analysis may be substantially correct, as intimated above I believe it leaves some important questions unanswered. Let us instead say, roughly, that a

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1. From a mathematical point of view, of course, these criticisms are beside the point. K could be *any* set whatever, and R any relation on K satisfying the specified conditions.

proposition, P, is possible in a given world just in case *it is consistent with*, what I shall call, *the fundamental postulates of that world*. Similarly, one might call one world, w_j, a possible alternative to another, w_i, just in case all propositions true of w_j are consistent with the fundamental postulates of w_i. Thus, to paraphrase Hintikka, a possible alternative to a given world is a world which could, consistently, have been realized given the fundamental postulates of the given world. This idea forms the foundation of the semantical theory to follow.[2]

This proposal presupposes that, if we think of a world as constituted by a set of propositions, certain of these should be designated fundamental postulates. For example, if we think of a world as what is described by a general scientific theory together with a set of initial conditions, and their deductive consequences, one might associate the fundamental postulates of the world with the basic laws of the theory. Any alternative possible world would then have to satisfy at least these same laws, though it might differ in initial conditions. Or, to look at the matter differently, suppose one wants to interpret 'it is possible that P' to mean 'for all I know, P'; one might then identify the fundamental postulates (of one's own world) with 'all I know'. It is not necessary now to decide how the class of fundamental postulates should be defined, but we might observe that with different construals of 'fundamental postulate' different senses of possibility are thereby characterized.

Let us use 'P_i' to designate the set of funamental postulates of a world w_i, and 'B_i' to designate, roughly, the set of all truths of w_i. It is natural to suppose that well behaved worlds are all consistent, in the sense that all their fundamental postulates are true, i.e. that $P_i \subseteq B_i$. When this is the case, I will say that the world w_i is *normal*; for the moment I will confine my attention to normal worlds.

To say that a world, w_j, is possible relative to another, w_i, then is to say that all the members of B_j are consistent with the members of P_i. Since, we may suppose, B_j (and B_i as well, of course) is a complete and consistent set of propositions, this is equivalent to saying that $P_i \subseteq B_j$. Notice that if a world is normal, it is related to itself in this way. Thus, among normal worlds, the defined relation is reflexive. This relation provides the foundation for the semantics for the system T (or M) of Gödel-Feys-von Wright.

Alternatively, one might desire a stronger relation to hold between alternative possible worlds; one might, for example, require for w_j to be a possible alternative to w_i, not only that all the fundamental postulates of w_i are *true* in w_j, but also that they are all *fundamental postulates* of w_j, that $P_i \subseteq P_j$. This second relation is not only reflexive but transitive as well. It provides the basis for the semantics for S4. If one required yet a

2. This basic idea, and some of those derived from it, were originally developed in conversation with J. Michael Dunn; he preferred a slightly different version of a possible world to that which will be used here, however.

stronger relation, that P_i be the very same as P_j for w_i to be related to w_j, one obtains a relation which is reflexive, transitive and also symmetric, and thus is suited for S5. Finally, to round out the picture, the relation which obtains when both $P_i \subseteq B_j$ and $P_j \subseteq B_i$ is reflexive and symmetric and can be used in the semantics for the *Brouwerische* system, B, introduced by Kripke.

All of these notions will be put more precisely in the next section. The close connections between this approach and Kripke's should, however, already be apparent. There is, nevertheless, a fundamental difference in the two accounts, besides the fact that here we specify a certain sort of structure to count as a possible world, and then *define* the relations of relative possibility in such a way that the formal properties of the relations, which Kripke must assume, are consequences of the definitions. For, where Kripke must introduce a different *kind* of model structure, $\langle G, K, R\rangle$ for the semantics for each of the systems T, S4, B and S5, we can use just one basic kind of structure (defined below) for all of these systems. But, while Kripke can get by with one method for evaluating formulas, given any model structure, we must apply different evaluation functions, corresponding to the different relations, for each of the different systems.

3. In this section, the ideas introduced above are made more precise, and it is shown how they provide the basis of a semantics adequate for the normal Lewis modal systems T (M), S4, B, and S5.

It is supposed that these systems—and all other systems discussed below—are expressed in a language containing atomic formulas, p, q, r, etc., the usual truth-functional connectives, e.g., & (conjunction), v (disjunction), - (negation), etc., and the modal operators, N (necessity) and M (possibility), governed by the usual grammatical rules. So long as we are working with classically based systems, we may suppose that conjunction, negation and necessity are the only primitive connectives, the others being defined in terms of these three in the familiar ways. Letters 'A', 'B', 'C', etc. are used as variables for well-formed formulas. And Church's conventions for the elimination of parentheses are used throughout. The systems are generated from among the following axiom schemata:

A.0 *axioms which, together with R.1, are adequate for the classical propositional calculus.*

A.1 $NA \supset A$

A.2 $N(A \supset B) \supset .NA \supset NB$

A.3 $NA \supset NNA$

A.4 $A \supset N - N - A$

A.5 $-NA \supset N - NA$

with the rule schemata:

R.1 *modus ponens*: *from A and A* $\supset$ *B, to infer B*

R.2 *necessitation*: *if* $\vdash A$, *then* $\vdash NA$.

The system T is defined by *A.0*, *A.1*, and *A.2* with *R.1* and *R.2*. S4 is the result of adding *A.3* to T; B the result of adding *A.4* to T; and S5 is the result of adding *A.5* (or both *A.3* and *A.4*) to T.

To construct corresponding theories of necessity from a semantical point of view, we introduce the concept of a *world-frame* to answer to the informal notion of a possible world discussed in section **2**. A world-frame (or world, for short), w_i, is the pair $\langle P_i, B_i \rangle$, where P_i is a set of formulas (in the language assumed above), and B_i is a consistent and complete set of formulas (i.e., for every formula in the language, either it or its denial, but not both, is a member of B_i). Intuitively, P_i is the set of fundamental postulates of w_i, while B_i includes these and also all *contingent* atomic truths of w_i.

Other similar definitions of a world-frame could have been given. Thus, we could have dropped the requirement of completeness for B_i and instead imposed appropriate closure conditions on the sets; we would then have structures similar to Hintikka's model-sets. This would simplify the definition of ϕ given below. Or, we could let B_i and P_i be sets of atomic formulas or their denials, which would make a world-frame more like a Carnapian state-description. Equivalently, one could define a world-frame as a pair $\langle P, B \rangle$, where P and B are *functions* assigning truth values to atomic formulas, B a complete function and P a partial function. This is the approach preferred by Dunn; on its basis he first announced completeness results for the four systems T—S5, though not necessarily with proofs like those given below. However, the present proposal has the advantage of allowing more manipulation and hence greater generality. It will be seen later, for example, that in some cases it is desirable to allow P_i to be an inconsistent set. Such a world would be difficult to describe in the language of assignment functions.

A world, $w_i = \langle P_i, B_i \rangle$ is *normal* if and only if $P_i \subseteq B_i$. For the remainder of this section, it will be presumed that all worlds discussed are normal. As indicated in section **2**, we can define relations, R, of relative possibility between worlds on the basis of relations between their sets P and B. When $w_i = \langle P_i, B_i \rangle$ and $w_j = \langle P_j, B_j \rangle$, let us say:

$$w_i \; R_1 \; w_j \textit{ iff } P_i \subseteq B_j$$
$$w_i \; R_2 \; w_j \textit{ iff } P_i \subseteq P_j \textit{ and } P_i \subseteq B_j$$ [3]
$$w_i \; R_3 \; w_j \textit{ iff } P_i \subseteq B_j \textit{ and } P_j \subseteq B_i$$
$$w_i \; R_4 \; w_j \textit{ iff } P_i = P_j \textit{ and } P_i \subseteq B_j \textit{ and } P_j \subseteq B_i.$$ [3]

For normal worlds, obviously, R_1 is reflexive, R_2 is reflexive and transitive, R_3 is reflexive and symmetric, and R_4 is an equivalence relation.

Let us define a (normal) *model*, μ, as the pair $\langle w_0, W \rangle$, where W is a set of (normal) world-frames and $w_0 \in W$. Intuitively, 'w_0' may be thought

3. For normal worlds these last conjuncts are redundant; they become important later on.

to designate the actual world for μ. Given $\mu = \langle w_0, W\rangle$, the *evaluation function* on μ, $\phi_\mu(A, w_i)$, is defined as follows for formulas, A, and each world, w_i, in W:

(a) If A is an atomic formula, $\phi_\mu(A, w_i) = T$ iff $A \epsilon B_i$;

Supposing that $\phi_\mu(B, w_i)$ and $\phi_\mu(C, w_i)$ are defined for all $w_i \epsilon W$,

(b) $\phi_\mu(B \& C, w_i) = T$ iff $\phi_\mu(B, w_i) = T = \phi_\mu(C, w_i)$
(c) $\phi_\mu(-C, w_i) = T$ iff $\phi_\mu(C, w_i) \neq T$
(d) $\phi_\mu(NB, w_i) = T$ iff $\phi_\mu(B, w_j) = T$, for every w_j in W such that $w_i\ R\ w_j$.

Let $\phi_\mu(A, w_i) = F$ if $\phi_\mu(A, w_i) \neq T$ by (a)-(d). This definition of ϕ is, of course, ambiguous; what must actually be defined are four functions, ϕ^1, ϕ^2, ϕ^3, and ϕ^4, according as the relation R mentioned in clause (d) is R_1, R_2, R_3, or R_4. In what follows, I will continue to use 'R' (and 'ϕ') without indices when what is being said applies equally to all relations (or functions).

Given $\mu = \langle w_0, W\rangle$, a formula, A, is true on μ by R_1 (R_2, R_3, R_4) if and only if $\phi_\mu^{1(2,3,4)}(A, w_0) = T$. And A is T (S4, B, S5)-valid iff A is true on every (normal) model, μ, by R_1 (R_2, R_3, R_4). The semantical consistency of the systems T - S5 can easily be established on this basis.

Theorem 1. *If A is provable in* T (S4, B, S5), *then A is* T (S4, B, S5)-*valid.*

This follows as a consequence of Kripke's consistency theorems, since any $\mu(= \langle w_0, W\rangle)$ coupled with R_1 (R_2, R_3, R_4) is a T (S4, B, S5) model structure in Kripke's sense. It can also be easily established directly that all the axioms of these systems are valid in the present sense, and that the rules preserve this property.

To prove the completeness theorems, I shall adopt methods based on those of Henkin [6], thus avoiding the need for semantic tableaux. A similar proof is sketched by Kaplan [10], and Thomason [16] has applied related methods to Intuitionistic logic.

We begin by stating some general results which apply to all the systems considered in this paper.

Theorem 2. (Due to Kripke [12] p. 206) *If X is any calculus containing the theorem schemata $A \supset (A \& \ldots \& A)$ and $(A \& \ldots \& A) \supset A$, and if modus ponens is admissible in X, then X can be recursively axiomatized with modus ponens as the sole rule of inference.*

This theorem facilitates proofs of later lemmas. When appropriate it will, henceforth, be tacitly assumed that the systems discussed have been so re-axiomatized.

Lemma 1. *The well-formed formulas of a system X are enumerable.*
Lemma 2. *If X is any consistent system containing all the theorems of the classical (or intuitionistic) propositional calculus, then if $-A$ is not provable in X, then the system X', which results from the addition of A as an axiom to X, is consistent.* (See, e.g., Mendelson [15], p. 63).

As a corollary to lemma 2, we see that if X is a consistent extension of the classical propositional calculus, then if not $\vdash A$ in X then the result of adding $-A$ to X is likewise consistent.

Every consistent classically based system has a complete and consistent extension which, given some non-theorem, can be defined to determine a counter-model for that formula. (Cf. Mendelson [15], p. 64.) There is, however, a more general result which is applicable to the intuitionistically based systems of sections **7** and **8**, as well as to the classical systems now under consideration. Let us say, with Curry, that a system X has the *alternation property* if and only if whenever $\vdash C \vee D$ in X then $\vdash C$ in X or $\vdash D$ in X and conversely. This next lemma is similar to one of Thomason's [16]; it is here proved in a slightly different way.

Lemma 3. *If X is a consistent extension of the intuitionistic propositional calculus, and if not $\vdash A$ in X, then there is a system, Y, which is a consistent extension of X with the alternation property (or ACE of X, for short), and not $\vdash A$ in X.*

Proof: Let X be axiomatized as in theorem 2, and suppose that not $\vdash A$ in X. Let $B_1 \ldots . B_i, B_{i+1} \ldots$ be an enumeration of the well-formed formulas of X (lemma 1). We now define a sequence of systems $Y_0 \ldots Y_i \ldots$ inductively as follows: (a) $Y_0 = X$. Suppose that Y_i has been defined. (b) If $\vdash --B_{i+1}$ in Y_i, then $Y_{i+1} = Y_i$; otherwise, if not $\vdash --B_{i+1}$ in Y_i, let C be the first wff (in the enumeration) such that $\vdash B_{i+1} \vee C$ in Y_i; then if it is not the case that $B_{i+1} \vdash A$ in Y_i, let Y_{i+1} be the result of adding B_{i+1} as an axiom to Y_i. But, if $B_{i+1} \vdash A$ in Y_i, let Y_{i+1} be the result of adding C as an axiom to Y_i. Finally, let Y be the union of all these Y_i.

(1) It is obvious that Y is an extension of X. (2) To show that A is not provable in Y it suffices to show that A is not provable in any of the Y_i. A is not provable in Y_0 (=X) by hypothesis. Suppose that not $\vdash A$ in Y_i; then if $Y_{i+1} = Y_i$, then not $\vdash A$ in Y_{i+1}. If $Y_{i+1} = Y_i$ plus B_{i+1}, then not $B_{i+1} \vdash A$ in Y_i; but if A were provable in Y_{i+1}, then it would be that $B_{i+1} \vdash A$ in Y_i; so A is not provable in Y_{i+1}. On the other hand, if $B_{i+1} \vdash A$ in Y_i, and if $Y_{i+1} = Y_i$ plus C, where C is the first formula such that $\vdash B_{i+1} \vee C$ in Y_i, then if $\vdash A$ in Y_{i+1}, $C \vdash A$ in Y_i, from which it would follow that $\vdash A$ in Y_i (by the principle that if $\vdash B \vee C$ and $B \vdash A$ and $C \vdash A$, then $\vdash A$) contrary to the inductive hypothesis; hence A is not provable in Y_{i+1}. This completes the induction; A is provable in no Y_i, so A is not provable in Y itself.

(3) That Y is consistent follows from the fact that A is not provable in Y (since $\vdash B \supset . -B \supset A$); it could also be established directly by induction on i, as in (2).

To prove (4) that Y has the alternation property, suppose that $\vdash C \vee D$ in Y. $C = B_{i+1}$ and $D = B_{j+1}$, for some i and j. Since $\vdash B_{i+1} \vee B_{j+1}$ in Y, there is a least number k such that $\vdash B_{i+1} \vee B_{j+1}$ in Y_k. Let $m = \max(i, j, k)$; so $\vdash B_{i+1} \vee B_{j+1}$ in Y_m. Now, either $\vdash B_{i+1}$ in Y_{i+1} or $B_{i+1} \vdash A$ in Y_i. If $\vdash B_{i+1}$ in Y_{i+1}, then $\vdash B_{i+1}$ in Y, i.e. $\vdash C$ in Y, and we are done. Otherwise, $B_{i+1} \vdash A$ in Y_m. Similarly, either $\vdash B_{j+1}$ in Y_{j+1} or $B_{j+1} \vdash A$ in Y_j. If $\vdash B_{j+1}$ in Y_{j+1}, then $\vdash B_{j+1}$ in Y, i.e. $\vdash D$ in Y, and we are done. Otherwise, $B_{j+1} \vdash A$ in Y_m.

However, since $\vdash B_{i+1} \vee B_{j+1}$ in Y_m, it could not be that both $B_{i+1} \vdash A$ in Y_m and $B_{j+1} \vdash A$ in Y_m, since that would imply that $\vdash A$ in Y_m and hence $\vdash A$ in Y, contrary to what was established in (2) above. Therefore, either $\vdash C$ in Y or $\vdash D$ in Y.

Since the law of the excluded middle—$B \vee -B$, for all B—is provable in any extension of the classical propositional calculus, it follows as an immediate corollary to lemma 3, that if X is a consistent, classically based system in which A is not provable, there is a *complete and consistent extension* (CCE) of X in which A is not provable. We are now in a position to show how any non-theorem of any of these systems can be falsified.

Let K be any consistent extension of one of the systems T, S4, B or S5, (except where mentioned I shall consider these four cases as one), in which both *R.1* and *R.2* are admissible, and suppose that some formula, A, is not provable in K. We let K define a model which falsifies A as follows:

Let M be any ACE of K. Let the *world-frame determined by* M, w_M, be the pair $\langle P_M, B_M \rangle$, where P_M is the set of all formulas B such that $\vdash NB$ in M, and B_M is the set of all formulas B such that $\vdash B$ in M. Let L be an ACE of K in which (the above mentioned) A is not provable (lemma 3), with w_L the world-frame determined by L. Finally, let $\mu_K = \langle w_L, W \rangle$, where W is the set of all worlds determined by ACE's of K.

Since all these systems M are consistent, and since they all contain the law of the excluded middle, and so are complete (they are all CCE's of K), all the sets B_M are complete and consistent. Moreover, all members, w_M, of W are normal—$P_M \subseteq B_M$—since if $\vdash NB$ in M, then $\vdash B$ in M, by *A.1*. Clearly, $w_L \in W$. Hence, μ_K is an appropriate model by which to evaluate formulas in these systems. Before showing how μ_K must falsify A, we prove an intermediate lemma.

Lemma 4. *If M is any ACE of K, then there is a system N such that (a) $\vdash B$ in N if and only if $\vdash NB$ in M, for any B, (b) N is a consistent extension of K, and (c) the world w_{N^*} determined by any ACE, N^*, of N is such that $w_M \, R \, w_{N^*}$.*

Proof: Suppose that M is axiomatized as in theorem 2. Let the axioms of N be all and only those formulas, B, such that $\vdash NB$ in M. We note that modus ponens is admissible in N, since *A.2* is derivable in M. This suffices to establish (a). That N is an extension of K can be seen from the fact that if $\vdash B$ in K, then $\vdash NB$ in K, by *R.2*, and so $\vdash NB$ in M, which entails that $\vdash B$ in N. Moreover, if N were inconsistent then for some B, $\vdash NB$ in M and $\vdash N - B$ in M, which would imply that $\vdash B$ in M and $\vdash -B$ in M, contrary to the assumed consistency of M. This proves (b).

There are four cases necessary to prove (c) according as (i) R is R_1 and K is an extension of T; (ii) R is R_2 and K is an extension of S4; (iii) R is R_3 and K is an extension of B; and (iv) R is R_4 and K is an extension of S5.

(i) It is trivial that $P_M \subseteq B_{N^*}$, for if $B \in P_M$, $\vdash NB$ in M; so B in N and B in N^* which implies that $B \in B_{N^*}$. Therefore $w_M \, R_1 \, w_{N^*}$. (This argument applies to all the later cases as well.)

(ii) To see that $P_M \subseteq P_{N^*}$ (under the conditions of this case), observe that if $\vdash NB$ in M, $\vdash NNB$ in M (by $A.3$), so if $B \epsilon P_M$, $\vdash NB$ in N^* and $B \epsilon P_{N^*}$. Hence $w_M\ R_2\ w_{N^*}$.

(iii) Suppose that $B \epsilon P_{N^*}$, i.e., that $\vdash NB$ in N^*; it cannot be the case that $\vdash -NB$ in N (by the consistency of N^*), and so not $\vdash N - NB$ in M. But if not $\vdash N - NB$ in M, then $\vdash -N - NB$ in M (by $\vdash N - NB \vee -N - NB$ in M and the alternation property); but, in that case $\vdash B$ in M (by $A.4$) and so $B \epsilon B_M$. Therefore, $w_M\ R_3\ w_{N^*}$.

(iv) This case is proved by combining the arguments of (i)-(iii) and by applying $A.5$ to show that $P_{N^*} \subseteq P_M$.

Given μ_K as defined above we can now prove:

Lemma 5. *For all* $w_M \epsilon W$, $\phi_{\mu_K}(B, w_M) = \mathrm{T}$ *if and only if* $\vdash B$ *in* M.

Proof: By induction of B. (a) If B is an atomic formula, then $\vdash B$ in M iff $B \epsilon B_M$ (by definition) iff $\phi_{\mu_K}(B, w_M) = \mathrm{T}$ (by definition). Suppose then that the lemma holds for formulas C and D. (b) If $B = C\&D$, then $\vdash C\&D$ in M iff $\vdash C$ in M and $\vdash D$ in M, iff $\phi_{\mu_K}(C, w_M) = T$ and $\phi_{\mu_K}(D, w_M) = T$ (inductive hypothesis), iff $\phi_{\mu_K}(C\&D, w_M) = \mathrm{T}$. (c) If $B = -C$, then if $\vdash -C$ in M, then not $\vdash C$ in M (by the consistency of M), and so $\phi_{\mu_K}(C, w_M) \neq \mathrm{T}$ (inductive hypothesis), in which case $\phi_{\mu_K}(-C, w_M) = \mathrm{T}$. But if it be given that $\phi_{\mu_K}(-C, w_M) = \mathrm{T}$, then $\phi_{\mu_K}(C, w_M) \neq \mathrm{T}$, and so not $\vdash C$ in M. Now in this case if not $\vdash C$ in M, then $\vdash -C$ in M (by the completeness of M). (d) Suppose $B = NC$. If $\vdash NC$ in M, then $C \epsilon P_M$. Let N be any ACE of K (so $w_N \epsilon W$) such that $w_M\ R\ w_N$. Since $P_M \subseteq B_N$, $C \epsilon B_N$. Hence, $\vdash C$ in N and so, by the inductive hypothesis, $\phi_{\mu_K}(C, w_N) = \mathrm{T}$. This is sufficient to show that $\phi_{\mu_K}(NC, w_M) = \mathrm{T}$. If $\phi_{\mu_K}(NC, w_M) = \mathrm{T}$, then for all w_N in W such that $w_M\ R\ w_N$, $\phi_{\mu_K}(C, w_N) = \mathrm{T}$. But if NC were not provable in M, there would be a system N, as described in lemma 4, which is a consistent extension of K and not $\vdash C$ in N; let N^* be an ACE (CCE) of N such that not $\vdash C$ in N^* (lemma 3). $w_{N^*} \epsilon W$, obviously, and $w_M\ R\ w_{N^*}$ (lemma 4). If follows that if $\phi_{\mu_K}(NC, w_M) = \mathrm{T}$, then $\phi_{\mu_K}(C, w_{N^*}) = \mathrm{T}$, and so $\vdash C$ in N^*, contrary to the specification of N^*. Hence, if $\phi_{\mu_K}(NC, w_M) = \mathrm{T}$, then $\vdash NC$ in M. This completes the proof of lemma 5.

Semantical completeness theorems for the four normal systems now follow immediately from this lemma.

Theorem 3. *If* A *is* T (S4, B, S5)-*valid, then* A *is provable in* T (S4, B, S5).

For, if A were not provable, there would be a model $\mu_K = \langle w_L, W\rangle$, defined as for lemma 5 letting K be T (S4, B, S5), such that A is not true in μ_K by R_1 (R_2, R_3, R_4), since it is stipulated that A is not provable in L and so $\phi_{\mu_K}(A, w_L) \neq \mathrm{T}$ (lemma 5).

4. We now begin to examine some non-normal systems of modal logic, that is, systems which are not closed under $R.2$. This section is devoted to the, so called, *epistemic systems*, in which no formulas of the form NA are provable. The formulations of the systems below are those of Lemmon [14] or very close to his.

The system E2 is defined by the principles $A.0$, $A.1$, $A.2$, and $R.1$ of section **3**, but in place of $R.2$ it has

R.2' If $\vdash A \supset B$, then $\vdash NA \supset NB$.

The system E3 is formed by adding the axiom schema

A.2' $N(A \supset B) \supset N(NA \supset NB)$

to E2. If E2 is extended by the axiom

A.6 $NT \supset NNT$,

where T is some designated tautology, such as $(p \supset p)$, the system ET results. To form E4 add *A.6* to E3; one could also add *A.3* to E2 to form an equivalent system. The epistemic version of B, EB, is formed by adding

A.4' $NB \supset .A \supset N - N - A$

to ET; while if this axiom be added to E4, the system E5 is obtained. Equivalently, one could define E5 by extending ET with the axiom schema

A.5' $NB \supset .-NA \supset N - NA$,

or by extending E2 with the schema

A.5'' $NB \supset N(-NA \supset N - NA)$.

In an important sense, each of these systems is an epistemic analogue of one of the normal systems discussed in section **3**. No formula of the form NA is provable in any of these systems, but if the formula $N(p \supset p)$ is added as an axiom to ET (E4, EB, E5), the system so formed is equivalent to T (S4, B, S5). Similarly, if $N(p \supset p)$ be added to E2 (E3) and the extended system is closed under *R.2'*, the system so formed is also equivalent to T (S4). If, however, *R.2'* is restricted to formulas provable in the unextended system—that is, if E2 (E3) is re-axiomatized with modus ponens as the sole rule of inference (theorem 2), and then extended by the addition of $N(p \supset p)$ as an axiom—the resultant system is equivalent to S2 (S3) discussed below.

It is easily shown that if a formula, A, is provable in one of these six E-systems, then NA is provable in the corresponding S-system. Hence, of course, each E-system is properly contained in the corresponding S-system. It is also known that if A is provable in one of the systems S2-S5, then $N(p \supset p) \supset A$ is provable in E2-E5. (Cf. Lemmon [14], p. 200, 213.)

When trying to develop semantics for these systems E2-E5, along the lines presented in section **3**, keeping all basic semantical concepts the same as defined there, a complication arises. As mentioned, in the epistemic systems no formula of the form NA is provable. Accordingly, no such formula should be valid on interpretation. Certain formulas, however, such as $p \supset p$, being tautologies, are true in every world, and so would be true in every world related by R to w_0 in any model $\mu = \langle w_0, W\rangle$. Following the definition of ϕ presented above, this would make, e.g., $N(p \supset p)$ true in such a w_0 and hence valid.

The key to avoiding this problem, as observed by Kripke, lies in deploying the notion of the normality of worlds. A non-normal world—a world, w_i, in which $P_i \not\subset B_i$—is, so to speak, an impossible possible world.

In such a world it is natural to think that *no* proposition whatever is necessary. Or, to say the same thing, we should require that for a proposition of the form *NA* to be true in a world, that that world must be normal.

Accordingly, we modify clause (d) in the definition of the evaluation function ϕ on μ to read:

$$\phi_\mu(NB, w_i) = \text{T iff } w_i \text{ is normal and } \phi_\mu(B, w_j) = \text{T}, \text{ for all } w_j \in W \text{ such that } w_i \, R \, w_j.$$

This emmedation is redundant for the sorts of models considered in section **3**, for it was supposed there that all worlds were normal. But we now freely allow for models $\mu = \langle w_0, W \rangle$ in which some or all members of W are non-normal. A model, μ, in which every world in W is normal will be called a *normal model*. One in which at least w_0 is normal will be called a *semi-normal model*. (These become important later.) Theorems 1 and 3 of section **3** should now be understood as saying that a formula is provable if and only if it is true in all normal models.

Definitions of truth in a model and validity are as before. For E2 and E3 we evaluate formulas on arbitrary models using R_1 and R_2. Thus, we should say that A is E2 (E3)-valid iff A is true on all (arbitrary) models, μ, by R_1 (R_2). Notice, in this regard, that R_2 remains transitive and both relations are reflexive on the set of normal worlds; i.e. $w_i \, R \, w_i$ iff w_i is normal.

To interpret formulas in ET-E5 some additions are necessary. In order to validate *A.6*, $NT \supset NNT$, it is necessary to insure that if a normal world, w_i, is related to w_j by R, then w_j is also normal. We can accomplish this in two ways. We could introduce the notion of an *E-normal model* as a model, $\mu = \langle w_0, W \rangle$, for which every member of W, except for perhaps w_0, is normal, and then use R_1 - R_4 as before. Or we could simply stipulate the normality of w_j when $w_i \, R \, w_j$. To simplify some later proofs, I shall now adopt the latter course. (Later we shall see how the former can be applied.) Thus, let us say:

$w_i \, R_5 \, w_j$ *iff* $w_i \, R_1 \, w_j$ *and* w_j *is normal*
$w_i \, R_6 \, w_j$ *iff* $w_i \, R_2 \, w_j$ *and* w_j *is normal*
$w_i \, R_7 \, w_j$ *iff* $w_i \, R_3 \, w_j$ *and* w_j *is normal*
$w_i \, R_8 \, w_j$ *iff* $w_i \, R_4 \, w_j$ *and* w_j *is normal.*

A formula, A, is ET (E4, EB, E5)-valid if and only if A is true on all (arbitrary) models, μ, by R_5 (R_6, R_7, R_8). Given these bases we can now extend the semantical consistency and completeness theorems of section **3** to all the E-systems:

Theorem 4. *If A is provable in* E2 (E3, ET, E4, EB, E5), *then A is* E2 (E3, ET, E4, EB, E5)*-valid.*

I leave to the reader the task of verifying that all the axioms of these systems and *R.1* are valid, in the appropriate senses. To see that *R.2′* is also valid, suppose that $A \supset B$ is valid, but that $NA \supset NB$ is not—i.e., that there is a $\mu = \langle w_0, W \rangle$, such that $\phi_\mu(NA \supset NB, w_0) \neq$ T. If $\phi_\mu(NA \supset NB, w_0) \neq$ T, $\phi_\mu(NA, w_0) =$ T and so w_0 is normal, and $\phi_\mu(NB, w_0) \neq$ T. Since w_0 is normal, there must be a world $w_i \epsilon W$ for which $w_0 R w_i$ and $\phi_\mu(B, w_i) \neq$ T. Since $\phi_\mu(NA, w_0) =$ T and $w_0 R w_i$, $\phi_\mu(A, w_i) =$ T. It follows that $\phi_\mu(A \supset B, w_i) \neq$ T. Let μ^* be the model $\langle w_i, W \rangle$ with the same W as for μ. By the

Lemma. *If* $\mu = \langle w_0, W \rangle$ *and* $\mu^* = \langle w_i, W \rangle$, *then* $\phi_\mu(A, w_j) = \phi_{\mu^*}(A, w_j)$

it follows that $\phi_{\mu^*}(A \supset B, w_i) \neq$ T, contrary to the assumption that $A \supset B$ was valid. (Cf. Kripke [12], p. 214.) A similar argument validates *R.2* on all normal models.

To establish the semantical completeness of these epistemic systems, it is necessary to modify slightly the methods of section **3**. (Theorem 2 and lemmas 1, 2, and 3, however, carry over without alteration.) Let K be any extension of one of the systems E2-E5 which is closed under *R.1* and *R.2*,′ and let L, M, N, etc. be ACE's of K. We define w_M, the world-frame determined by M, now as the pair $\langle P_M, B_M \rangle$, where B_M is, as before, the set of formulas provable in M, and P_M is the set of formulas, B, such that $N(p \supset p) \supset NB$ is provable in M.

If A is a formula not provable in K, we want a model which will falsify A. Let μ_K be $\langle w_L, W \rangle$, as before, where W is the set of all world-frames determined by ACE's of K and w_L is a world determined by an ACE of K, L, in which A is not provable (lemma 3). With μ_K defined, lemma 5 can now be proved. However, to establish case (d), two more small lemmas and a minor modification of lemma 4 are required.

Lemma 6. *If there is any formula, B, such that* $\vdash NB$ *in M, then w_M is normal.*

Proof: Suppose that $\vdash NB$ in M, and suppose that $C \epsilon P_M$. $NB \supset N(p \supset p)$ is a theorem of K and so a theorem of M. Hence, $N(p \supset p)$ is provable in M. If $C \epsilon P_M$, then $\vdash N(p \supset p) \supset NC$ in M; hence $\vdash NC$ in M. But then $\vdash C$ in M, and so $C \epsilon B_M$. This shows that $P_M \subseteq B_M$ and w_M is normal.

Lemma 7. *If w_M is normal, then* $\vdash N(p \supset p)$ *in M.*

Proof: Suppose that w_M is normal, but that $N(p \supset p)$ is not provable in M. By the completeness of M, it follows that $-N(p \supset p)$ is provable in M. $-N(p \supset p) \supset . N(p \supset p) \supset NB$ and $-N(p \supset p) \supset . N(p \supset p) \supset N - B$ are also both provable in M. Consequently, $\vdash N(p \supset p) \supset NB$ in M and $\vdash N(p \supset p) \supset N - B$ in M. So $B \epsilon P_M$ and $-B \epsilon P_M$. Since, by hypothesis, $P_M \subseteq B_M$, $B \epsilon B_M$ and $-B \epsilon P_M$, which is to say $\vdash B$ in M and $\vdash -B$ in M, contrary to the consistency of M. Hence, $N(p \supset p)$ must be provable in M.

In order to apply lemma 4 to these systems, with the revised definition of P_M, it is necessary to add to it the condition that $N(p \supset p)$ be provable in M; thus lemma 4 should now read:

If M is an ACE of K and if $\vdash N(p \supset p)$ *in M, then there is a system N, such that . . . . etc.*

This added condition has, by lemma 6, the effect of requiring w_M to be normal. Under this condition the set P_M defined as above is the same as the set of formulas, B, such that $\vdash NB$ in M, which was how P_M was defined in section **3**. The proof of lemma 4 can now go through as before, but with these modifications to part (c):

(1) Case (i) in which R is R_1 and K is an extension of E2 is unchanged.

(2) For case (ii) in which R is R_2 and K is an extension of E3, we cannot appeal to *A.3* as before. However, to show that $P_M \subseteq P_{N^*}$, it is sufficient to show that if $\vdash NC$ in M, then $\vdash N(p \supset p) \supset NC$ in N^*. $NC \supset N(N(p \supset p) \supset NC)$ is a theorem of E3; so $N(N(p \supset p) \supset NC)$ is provable in M, from which it follows that $N(p \supset p) \supset NC$ is provable in N and thus in N^*.

(3) For case (v) ((vi), (vii), (viii)) in which R is R_5 (R_6, R_7, R_8) and K is from ET (E4, EB, E5), it is necessary to know that w_{N^*} is normal (in addition to knowing that the requisite relations between P_M and P_{N^*} and B_{N^*} obtain). This follows by lemma 6 and the fact that $\vdash N(p \supset p)$ in N^*, since $\vdash N(p \supset p) \supset NN(p \supset p)$ in M, so $\vdash NN(p \supset p)$ in M which implies that $\vdash N(p \supset p)$ in N^*.

The proof of lemma 5, that $\phi_{\mu_K}(B, w_M) = \text{T}$ iff $\vdash B$ in M, now proceeds as before. For case (d), in which B has the form NC, if $\vdash NC$ in M, then w_M is normal, by lemma 6; we then argue as in section **3** to show that $\phi_{\mu_K}(NC, w_M) = \text{T}$. If, on the other hand, it is given that $\phi_{\mu_K}(NC, w_M) = \text{T}$, then w_M must be normal. So, $\vdash N(p \supset p)$ in M, by lemma 7. Lemma 4 is then applicable, and it can be shown that NC must be provable in M as in section **3**. This guarantees the semantical completeness of the systems E2-E5:

Theorem 5. *If A is* E2 (E3, ET, E4, EB, E5)*-valid then A is provable in* E2 (E3, ET, E4, EB, E5).

5. Lewis' own favorite systems of modal logic, S2 and S3, are like the epistemic systems just discussed in failing to satisfy *R.2*. They do have theorems of the form *NA*, but none of the form *NNA*, however. The systems S2 and S3 are defined (following Lemmon [14]) by taking *strict* forms of the axioms of E2 and E3 respectively. Thus, given the axiom *A.1*, for example, let *NA.1* be the result of prefixing it with *N*; similarly for the other postulates. The axioms of S2 are then *NA.0*, *NA.1*, *NA.2*, and, of course, *A.1* itself. *R.1* continues to hold, as does this strict version of *R.2'*:

R.2'' *If* $\vdash N(A \supset B)$, *then* $\vdash N(NA \supset NB)$.

S3 results from the addition of *NA.2'* to S2. (*R.2''* is thus redundant in S3.)

As mentioned earlier, if a formula, A, is provable in E2 (E3), then NA is provable in S2 (S3). This is readily proved by induction on the proof of A in E2 (E3). Hence E2 (E3) is contained in S2 (S3). S2 and S3 are themselves, obviously, contained in the systems T and S4 respectively. Furmore, just as the addition of $N(p \supset p)$ to E2 and E3 produces S2 and S3, so the addition of $NN(p \supset p)$ to S2 and S3 results in T and S4, if the extended systems are closed under *R.2''*.

To develop S2 and S3 from a semantical point of view, we employ the class of semi-normal models mentioned in the preceding section. A model, $\mu = \langle w_0, W\rangle$, is semi-normal if and only if w_0 is normal (though some, or all, other members of W might be non-normal). Using the same definitions of R_1 and R_2 and of ϕ as were used in section **4**, let us say that a formula, A, is S2 (S3)-valid if and only if A is true on every semi-normal model, μ, by R_1 (R_2).

Theorem 6. *If A is provable in* S2 (S3), *then A is* S2 (S3)-*valid.*

The reader may verify for himself the validity of all the axioms and rules of these systems.

Rather than trying to establish the completeness of these systems directly as was done in the preceding sections, we can reduce the problem to the results already established for E2 and E3.

Lemma 8. *If A is* S2 (S3)-*valid, then* $N(p \supset p) \supset A$ is E2 (E3)-*valid.*

Proof: Suppose that A is S2 (S3)-valid, i.e. that A is true on all semi-normal models by R_1 (R_2). We show that for any arbitrary model, $\mu = \langle w_0, W\rangle$, that if $\phi_\mu(N(p \supset p), w_0) = T$, then $\phi_\mu(A, w_0) = T$. If $\phi_\mu(N(p \supset p), w_0) = T$, then w_0 must be normal. So this μ is semi-normal. So $\phi_\mu(A, w_0) = T$, from the supposition of S2 (S3)-validity of A. From this it follows that:

Theorem 7. *If A is* S2 (S3)-*valid, then A is provable in* S2 (S3).

For, if A is true on all semi-normal models, then $N(p \supset p) \supset A$ is true on all (arbitrary) models, by lemma 8. By theorem 5, this implies that $N(p \supset p) \supset A$ is provable in E2 (E3). Since E2 is contained in S2 (and E3 contained in S3), $N(p \supset p) \supset A$ is thus provable in S2 (S3). $N(p \supset p)$ is provable in these systems; hence, A is provable in S2 (S3).

This argument also suggests an alternate characterization of the systems T, S4, B and S5. For we could evaluate formulas in these systems using semi-normal models and R_5, R_6, R_7, and R_8 respectively. An argument similar to the above would then prove:

Theorem 8. *A is true on all semi-normal models by* R_5 (R_6, R_7, R_8) *if and only if A is provable in* T (S4, B, S5).

Four related systems should be mentioned at this point. These are the systems S2.B and S3.5 and their epistemic analogues E2.B and E3.5. S2.B and E2.B are defined by the addition of *A.4'*: $NB \supset . A \supset N - N - A$, to S2 and E2 respectively. S3.5 and E3.5 are formed by adding *A.5'*: $NB \supset . -NA \supset N - NA$, to S3 and E3. S2.B and E2.B were first introduced by Lemmon in [14]. (He called them S2(S) and E2(S).) S3.5 is due to Åqvist [1] and was discussed from a semantical point of view by Cresswell [3], using Kripkean model structures, as well as by Lemmon [14] who also considered E3.5 there. These systems are interesting in that they stand to S2 (E2) and S3 (E3) as B (EB) and S5 (E5) stand to T (ET) and S4 (E4). Thus, for example, S2.B and S3.5 become B and S5 when closed under *R.2*; E2.B and E3.5 become EB and E5 when extended by the addition of *A.6*.

Thus far, these four systems have proved recalcitrant when their semantics is approached with the present methods. It would be natural to think that for S2.B we could evaluate formulas using R_3 on semi-normal models and for S3.5 using R_4 on semi-normal models. (Similarly for E2.B and E3.5 on arbitrary models.) However, the proof of lemma 4 breaks down for the case with R_3 when we cannot appeal to the derivability of $NN(p \supset p)$ in the system M when $N(p \supset p)$ is derivable. R_4 as defined is identical with R_8, so that manner of evaluation validates all formulas of S5 (E5). I leave it an open problem to develop an adequate semantics for these systems within the present framework.

6. In this section, the basic ideas of the semantics so far developed are adapted to apply to *deontic* varients of all the systems discussed above. These systems are characterized by the failure of *A.1*, $NA \supset A$, to be derivable. If N is interpreted now not as necessity but as obligation, it would be inappropriate to infer from the fact that something was obligatory (NA) that it occur (A). It would, however, be in order to infer that it was permissible, or that its denial was not obligatory ($-N - A$). Accordingly, in place of *A.1*, the deontic systems have

A.1' $NA \supset -N - A$.

Where X is any of the modal systems E2-S5 considered in sections **3**, **4** and **5**, let its deontic counterpart, DX, be the result of replacing *A.1* with *A.1'*. In addition it is necessary to postulate for DS2 the strict version of *A.1'*, *NA.1'*, i.e., $N(NA \supset -N - A)$, in place of *NA.1*, and also the 'material' axioms *A.0* and *A.2*, since these are no longer derivable from *NA.0* and *NA.2* through *A.1* and *R.1*. DS3 results from the addition of *A.2'* and *NA.2'* to DS2.

One curious fact results from this way of defining the deontic systems. DB is not contained in DS5 as one would expect. For, $A \supset N - N - A$ is derivable in DB, but not in DS5. If it were derivable in DS5, then *A.1* would be a theorem of that system.[4] $N(A \supset N - N - A)$ is, however, provable in DS5 without trouble. (Similar remarks apply to the systems DEB and DE5.) This disparity between DB and DS5 is unfortunate, but unavoidable given the present approach. Shortly I will present another system, closely related to DB which is properly contained in DS5.

Most of these deontic systems, DX, are new. DE2 was examined by Lemmon in [13] under the name 'D2'. DT, DS4, and DB are not the same as Hanson's systems bearing the same names [5], for his systems all contain the thesis $N(NA \supset A)$, which is not provable in the systems defined here. DT is the same as Hanson's D, and, since $N(NA \supset A)$ is derivable in DS5,

4. Thus:

1.	$\vdash A \supset N{-}N{-}A$	hypothesis
2.	$\vdash -N{-}NA \supset A$	from 1, substituting $-A$ for A
3.	$\vdash NA \supset NNA$	*A.3*
4.	$\vdash NNA \supset -N{-}NA$	*A.1'*
5.	$\vdash NA \supset -N{-}NA$	3, 4 transitivity
6.	$\vdash NA \supset A$	2, 5 transitivity

that system is the same as his DS5. (Shortly I will consider some other systems in which this formula, $N(NA \supset A)$, is a theorem.) DT, DS4 and DS5 are the same as Fitch's DM, DS4 and DS5 in [4]; however, in his DB $A \supset N - N - A$ is not provable, although $N(A \supset N - N - A)$ is, as is $N(NA \supset A)$.

The basic change required for the semantics of these systems is in the notion of normality. If a proposition, P, is necessary (obligatory) in a world iff P is true in every world related by R to the given world, one should not want to demand that that world be related to itself lest *A.1* be validated. Rather, one should require only that there be *some* world related to the given world. This ensures the validity of *A.1′*. Let us, therefore, say that for $\mu = \langle w_0, W\rangle$, a world, w_i, in W is *d-normal in* μ if and only if there is a world, w_j, in W such that $w_i \; R \; w_j$. (Notice that, unlike the original notion of normality, d-normality is only defined with respect to a model μ.) For these deontic systems, the definition of the evaluation function ϕ on μ should now be modified so that clause (d) reads:

(d′) $\phi_\mu(NB, w_i) = \mathrm{T}$ iff w_i is d-normal in μ and $\phi_\mu(B, w_j) = \mathrm{T}$ for every $w_j \in W$ such that $w_i \; R \; w_j$.

Let us say that a model, $\mu = \langle w_0, W\rangle$, is d-normal iff every member of W is d-normal in μ; μ is d-semi-normal iff w_0 is d-normal in μ. A formula, A, is DT (DS4, DB, DS5)-valid iff A is true on all d-normal models by R_1 (R_2, R_3, R_4). A is DS2 (DS3)-valid iff A is true on all d-semi-normal models by R_1 (R_2). A is DE2 (DE3)-valid iff A is true on all (arbitrary) models by R_1 (R_2). Validity for DET-DE5 requires special attention and will be considered momentarily.

Theorem 9. *A is DX-valid iff A is provable in DX (where X is one of* E2, E3, S2-S5).

Semantical consistency can be proved directly without difficulty. To prove completeness we must adapt the arguments of the preceding sections to suit the deontic systems.

If A is a non-theorem of K (K any extension on any of DE2, DE3, DT-DS5, closed under the rules), the model μ_K to falsify A is defined exactly as in section **4**. Lemma 4 holds without modification. The analogue of lemma 7, that if w_M is d-normal in μ_K, then $\vdash N(p \supset p)$ in M, can be proved in much the same way as lemma 7 was originally proved. It is of course trivial for extensions of DT-DS5; hence, we need only consider the case in which K is an extension of DE2 or DE3. Suppose that w_M is d-normal in μ_K. This means that there is a world, w_N, such that $w_M \; R \; w_N$, which entails *inter alia* that for such an N, $P_M \subseteq B_N$. So, suppose that not $\vdash N(p \supset p)$ in M; then $\vdash -N(p \supset p)$ in M by the completeness of M. It follows (by $-A \supset . A \supset B$) that $\vdash N(p \supset p) \supset NB$ in M and $\vdash N(p \supset p) \supset N - B$ in M, so that $B \in P_M$ and $-B \in P_M$. But then $B \in B_N$ and $-B \in B_N$, which is to say, $\vdash B$ in N and $\vdash -B$ in N, contrary to the consistency of N. Hence, $\vdash N(p \supset p)$ in M.

The analogue of lemma 6, that if $\vdash NB$ in M, for any B, then w_M is d-normal in μ_K, cannot be proved directly as it was in section **4**, for that argument depended on the derivability of *A.1*. However, this lemma does follow immediately from lemma 4. If $\vdash N(p \supset p)$ in M, then there is a system N^* which is an ACE of K such that $w_M\ R\ w_{N^*}$. Hence, if $\vdash N(p \supset p)$ in M, then w_M must be d-normal in μ_K. Since $NB \supset N(p \supset p)$ is provable in M, if $\vdash NB$ in M, then $\vdash N(p \supset p)$ in M, which assures the lemma. This also guarantees that if K is an extension of DT, DS4, DB or DS5, then μ_K is a d-normal model as required.

Given these versions of lemmas 4, 6 and 7, the argument for lemma 5, that $\phi_{\mu_K}(B, w_M) = \text{T}$ iff $\vdash B$ in M, proceeds exactly as in section **4**. This is sufficient to prove the semantical completeness of the systems DE2, DE3, DT-DS5.

The reduction of the completeness problem for DS2 and DS3 to that of DE2 and DE3 follows the argument of section **5**. Lemma 8 carries over unmodified. It is only necessary to show that DE2 $\subseteq$ DS2 and DE3 $\subseteq$ DS3. This follows from

Lemma 9. *If B is provable in* DE2 (DE3), *then both B and NB are provable in* DS2 (DS3).

The proof is left to the reader.

I have deliberately excluded the systems DET-DE5 from the preceding discussion, for these systems demand more modification of the methods previously developed. In section **4** formulas in ET-E5 were evaluated on arbitrary models using R_5-R_8. These relations contain aspects of normality, as is appropriate for those systems. For their deontic counterparts, it would be natural to replace regular normality with d-normality in the definitions of the R, so that we would say, for example, that $w_i R_5 w_j$ iff $w_i R_1 w_j$ and w_j is d-normal in μ. However, since the definition of d-normality presupposes antecedent definitions of the relations R, this cannot be done. In place of using relations like R_5-R_8 for formulas in DET-DE5, we can, however, use R_1-R_4 as for DT-DS5, applying them to, what I shall call, *d-E-normal models*. A model, $\mu = \langle w_0, W\rangle$, is d-E-normal iff every member of W, except for perhaps w_0, is d-normal in μ. A formula is DET (DE4, DEB, DE5)-valid iff A is true on all d-E-normal models by R_1 (R_2, R_3, R_4). This allows us to state

Theorem 10. *A is* DET (DE4, DEB, DE5)-*valid if and only if A is provable in* DET (DE4, DEB, DE5).

Proving the consistency of these systems presents no new problems. Semantical completeness does present one additional difficulty. To prove the falsifiability of any non-theorem—through lemma 5—it is necessary to define a model μ_K in such a way as to guarantee that it is a d-E-normal model.

Suppose that A is the formula not provable in K (from DET-DE5) which we seek to falsify. Select a system L which is an ACE of K such that A is not provable in L (lemma 3). Then if $N(p \supset p)$ is not provable in L, let $\mu_K = \langle w_L, \{w_L\}\rangle$. Otherwise, if $\vdash N(p \supset p)$ in L, then the system K' obtained by adding $N(p \supset p)$ as an axiom to K is consistent; so let $\mu_K = \langle w_L, W\rangle$, where W is the set of all world-frames determined by ACE's of K', including w_L. Every member of this set is also a world determined by an ACE of K. If μ_K is defined by the first procedure, it is normal d-E-even though w_L itself is not d-normal in μ_K. If μ_K is defined by the second method, then $N(p \supset p)$ is provable in every system M such that w_M is in W; hence, every w_M is d-normal in μ_K (lemma 6) and μ_K is d-E-normal; indeed, μ_K is d-normal.

The proofs of lemmas 4, 6, and 7 are unaffected by these modifications in the way of defining μ_K. In the proof of lemma 5, one must consider the two ways of defining μ_K. If μ_K is defined by the second procedure, then μ_K is d-normal and one would argue for lemma 5 as one argued it for the systems DT-DS5. (Note that K' is an extension of DT-DS5 as K is an extension of DET-DE5). If, on the other hand, μ_K is defined by the first of the two methods, so that $\mu_K = \langle w_L, \{w_L\}\rangle$, then no world in the 'universe' of μ_K is d-normal in μ_K (by lemma 7 and the fact that $N(p \supset p)$ is not provable in L), so no proposition of the form NC could be true in any such world. By the same token, since $N(p \supset p)$ is not provable in any system determining worlds in this universe, no proposition of the form NC is provable. Consequently, case (d) in the proof of lemma 5 becomes inapplicable when μ_K is defined in this way. Hence, the proof of lemma 5 can go through as before. Thus the semantical completeness of the systems DET-DE5 is guaranteed.

A similar argument could be applied to show that we can adequately evaluate formulas in ET-E5 using R_1-R_4 on E-normal models (models, $\mu = \langle w_0, W\rangle$, in which every member of W, except for perhaps w_0, is normal), thus obviating the need for R_5-R_8 altogether. R_5-R_8 are, however, helpful in evaluating formulas in a family of systems I shall call D$^+$-systems. When DX is one of the deontic systems discussed above, but especially one of DET-DE5 or DT-DS5, let DX^+ be the result of adding

A.7 $NB \supset . N(NA \supset A)$

as an axiom to DX. And for DB$^+$ and DEB$^+$ let the axiom *A.4* or *A.4'* be replaced by $NB \supset . N(A \supset N - N - A)$.

The systems DT$^+$, DS4$^+$, DB$^+$ and DS5$^+$ are equivalent to Hanson's systems DM, DS4, DB and DS5 [5]. DET$^+$-DE5$^+$ are, to my knowledge, new. *A.7* is redundant in DS5$^+$ and DE5$^+$ so these systems are equivalent to DS5 and DE5 respectively. We should also note that DB$^+ \subseteq$ DS5$^+$ and DEB$^+ \subseteq$ DE5$^+$. *A.7* says, in effect, that $NA \supset A$ holds in every world related to a d-normal (actual) world, so that every such world is itself *normal*, even though the actual world might not be. If we were to speak of alternate permissible worlds (instead of alternate possible worlds), this says that every alternate permissible world to our actual world is a perfect world, a world in which everything which ought to happen, does happen.

The effect of this condition on the relation of 'alternate permissibility' is achieved through the use of R_5-R_8 as they were defined in section **4**. Thus, we may say that a formula, A, is DET$^+$ (DE4$^+$, DEB$^+$, DE5$^+$)-valid if and only if A is true on all (arbitrary) models by R_5 (R_6, R_7, R_8), using ϕ as defined for the deontic systems above, and that a formula, A, is DT$^+$ (DS4$^+$, DB$^+$, DS5$^+$)-valid iff A is true on all d-semi-normal models by R_5 (R_6, R_7, R_8). The proofs of all earlier lemmas go through unimpeded for these systems, so we conclude:

Theorem 11. *A is* DET$^+$ (DE4$^+$, DEB$^+$, DE5$^+$)-*valid if and only if A is provable in* DET$^+$ (DE4$^+$, DEB$^+$, DE5$^+$);

Theorem 12. *A is* DT$^+$ (DS4$^+$, DB$^+$, DS5$^+$)-*valid iff A is provable in* DT$^+$ (DS4$^+$, DB$^+$, DS5$^+$).

(We note that the equivalence of DS5 and DS5$^+$ and of DE5 and DE5$^+$ is reflected in the identity of R_4 and R_8.)

Another approach to the D$^+$-systems, more in line with Hanson's account, would be to define a class of d$^+$-models, $\mu = \langle w_0, W\rangle$, for which every member of W, except for perhaps w_0 is normal, and d$^+$-normal models as those d$^+$-models, μ, for which w_0 is d-normal in μ. We could then evaluate formulas in DET$^+$-DE5$^+$ using R_1-R_4 on all d$^+$-models, and evaluate formulas in DT$^+$-DS5$^+$ using R_1-R_4 on all d$^+$-normal models. (Unhappily, $A \supset N$-N-A, and $NB \supset . A \supset N$-N-A become valid for DB$^+$(DEB$^+$) when they are approached in this way. Since these formulas are not derivable in DB$^+$ and DEB$^+$ as originally defined, they would have to be reintroduced as postulates if these systems are to prove complete on this interpretation.)

7. Exploiting known analogies between intuitionistic logic and Lewis' system S4, Kripke has shown how a semantics for intuitionistic logic can be constructed on the basis of what are essentially S4 model structures $\langle G, K, R\rangle$. In this section I show how a similar course can be followed within the present framework to determine a semantics for the intuitionistic propositional calculus, IPC. In the next section, I show how this account can be extended to some modal extensions of this system.

I shall presume some standard axiomatization of the intuitionistic propositional calculus, such as given by Heyting [7], expressed in the same propositional language as before. Now, however, the connectives, $\supset$ and $\vee$, must be regarded as primitives along with & and -. The outstanding difference between IPC and the classical propositional calculus is, of course, the failure of the law of the excluded middle, $A \vee -A$, to be derivable in the former system. Semantically this difference may be expressed by weakening the definition of a world so that the set B, while consistent, need not be complete. (Since we can ignore the role of P in a world-frame for now, I will now change terminology slightly in order to avoid confusion.)

Let a model, ι, be a pair $\langle d_0, D\rangle$, where D is a set of sets of formulas such that for each d_i in D, d_i is a consistent, but not necessarily complete, set of formulas, and $d_0 \in D$.

Given such a model, $\iota = \langle d_0, D\rangle$, we can define an evaluation function, $\phi_\iota(A, d_i)$ for each formula A and each $d_i \in D$.

(a) If A is an atomic formula, then $\phi_\iota(A, d_i)$ = T iff $A \in d_i$.

Suppose that $\phi_\iota(B, d_i)$ and $\phi_\iota(C, d_i)$ are defined for every $d_i \in D$.

(b) $\phi_\iota(B \& C, d_i) = \mathrm{T}$ iff $\phi_\iota(B, d_i) = \mathrm{T} = \phi_\iota(C, d_i)$

(c) $\phi_\iota(B \vee C, d_i) = \mathrm{T}$ iff $\phi_\iota(B, d_i) = \mathrm{T}$ or $\phi_\iota(C, d_i) = \mathrm{T}$;

(d) $\phi_\iota(B \supset C, d_i) = \mathrm{T}$ iff for every $d_j \in D$ such that $d_i \subseteq d_j$, if $\phi_\iota(B, d_j) = \mathrm{T}$, then $\phi_\iota(C, d_j) = \mathrm{T}$;

(e) $\phi_\iota(-B, d_i) = \mathrm{T}$ iff for every $d_j \in D$ such that $d_i \subseteq d_j$, $\phi_\iota(B, d_j) \neq \mathrm{T}$.

And, if $\phi_\iota(A, d_i) \neq \mathrm{T}$ by (a)-(e), let us say that $\phi_\iota(A, d_i) = \mathrm{F}$. What is distinctive in this manner of evaluating formulas can, of course, be seen in clauses (d) and (e). A formula, A, is IPC-*valid* iff $\phi_\iota(A, d_0) = \mathrm{T}$, for all models $\iota = \langle d_0, D \rangle$.

Theorem 13. *A is* IPC-*valid if and only if A is provable in* IPC.

Proof of semantical consistency is left to the reader.

Enough machinery has already been mobilized, in section **3**, to prove the semantical completeness of the calculus. Suppose that K is any consistent extension of IPC and suppose that A is some formula not provable in K which we want to falsify. Let a model ι_K be defined as the pair $\langle d_L, D \rangle$, where D is the set of all theorems of all systems, M, which are consistent extensions of K having the alternation property (ACE's of K); d_L is the set of theorems of L, an ACE of K in which the given formula, A, is not provable (lemma 3).

We now establish the principal lemma leading toward completeness, the analogue of lemma 5, that $\phi_{\iota_K}(B, d_M) = \mathrm{T}$ iff $\vdash B$ in M, for every system M such that $d_M \in D$. The proof is by induction of B. The cases in which B is an atomic formula or of the form $C \& D$ or $C \vee D$ are trivial. Supposing that the lemma holds for formulas C and D, we show it for $C \supset D$ and $-C$.

If $\vdash C \supset D$ in M, then $C \supset D \in d_M$, by definition; hence, for any N such that $d_M \subseteq d_N$, $C \supset D \in d_N$, which is to say, $\vdash C \supset D$ in N. For such an N, if $\phi_{\iota_K}(C, d_N) = \mathrm{T}$, given the inductive hypothesis, $\vdash C$ in N; so $\vdash D$ in N, by modus ponens, and $\phi_{\iota_K}(D, d_N) = \mathrm{T}$, by the inductive hypothesis again. This shows that $\phi_{\iota_K}(C \supset D, d_M) = \mathrm{T}$. Suppose it be given that $\phi_{\iota_K}(C \supset D, d_M) = \mathrm{T}$, but suppose not $\vdash C \supset D$ in M. This implies that not $C \vdash D$ in M. Now either $\vdash -C$ in M or not $\vdash -C$ in M. If $\vdash -C$ in M, then by $-A \supset . A \supset B$, $\vdash C \supset D$ in M contrary to the assumption; so not $\vdash -C$ in M. The system, N, obtained by adding C as an axiom to M is therefore consistent (lemma 2). N is an extension of K. D is not provable in N, for otherwise D would be derivable from C in M. Hence there is a system, N^*, which is an ACE of N is which D is not provable (lemma 3). Let d_{N^*} be the set of theorems of N^*; $d_{N^*} \in D$, since N^* is an ACE of an extension of K. $d_M \subseteq d_{N^*}$, since N^* is an extension of an extension of M. Given that $\phi_{\iota_K}(C \supset D, d_M) = \mathrm{T}$, if $\phi_{\iota_K}(C, d_{N^*}) = \mathrm{T}$, then $\phi_{\iota_K}(D, d_{N^*}) = \mathrm{T}$. Since $\vdash C$ in N^*, by the inductive hypothesis, $\phi_{\iota_K}(C, d_{N^*}) = \mathrm{T}$; so, $\phi_{\iota_K}(D, d_{N^*}) = \mathrm{T}$. But then, again by the inductive hypothesis, $\vdash D$ in N^*, contrary to the specification of N^*. So we must conclude that $\vdash C \supset D$ in M.

If $\vdash -C$ in M, then $\vdash -C$ in N for any N such that $d_N \in D$ and $d_M \subseteq d_N$. Hence, not $\vdash C$ in N, by the consistency of N, so $\phi_{\iota_K}(C, d_N) \neq \mathrm{T}$, for any such N. Therefore, $\phi_{\iota_K}(-C, d_M) = \mathrm{T}$. Suppose now that $\phi_{\iota_K}(-C, d_M) = \mathrm{T}$, but that not $\vdash$-C in M. It follows that the system N obtained by adding C as an axiom to M is consistent (lemma 2). Hence, N has an ACE, N^*, (lemma 3) such that $d_{N^*} \in D$ and $d_M \subseteq d_{N^*}$. Given that $\phi_{\iota_K}(-C, d_M) = T$, $\phi_{\iota_K}(C, d_{N^*}) \neq T$. It follows, by the inductive hypothesis, that not $\vdash C$ in N^*. But C is an axiom of N^*. So $-C$ must be provable in M.

This establishes the lemma; from it completeness follows directly. Suppose that A is some formula not provable in IPC. Let K be IPC. $\phi_{\iota_K}(A, d_L) \neq T$, since by stipulation A is not provable in L, an ACE of K. Therefore, if A is not provable in IPC, A is not true on ι_K. If A is IPC-valid, A must be provable in IPC.

8. We can now combine this approach to intuitionistic logic with the preceding account of the modalities to provide semnatics for some intuitionistically based modal systems. These systems are developed in a direction suggested by a system of Bull [2], which is, in a sense, an intuitionistic counterpart to S5.

One of the prime advantages of Lemmon's formulations of the modal calculuses, which have been followed in this paper, is that they make manifest how the modal systems can be based on a foundation of classical logic. We obtain intuitionistic analogues of these systems by changing that basis: Where X is any of the modal systems discussed above, let IX be the result of replacing *A.0*, the set of axioms for classical logic, by *A.0'* a set of axioms which, with *R.1*, generates the intuitionistic propositional calculus, IPC. In addition, we postulate for these systems the axiom schema

A.8 $NA \vee -NA$.

This postulate is somewhat unfaithful to the spirit of the intuitionistic enterprise. With it, the systems say, in effect, that while propositions in general may obey intuitionistic principles, necessitative propositions—those provably equivalent to propositions of the form NB—obey classical laws. However, the semantical principles given below seem to require the presence of this postulate.[5]

Semantics for these systems, IX, can be developed with only minor adaptations of methods proposed for X. Let a world, w_i, be the pair $\langle P_i, B_i\rangle$ as before, except now B_i need not be a complete set. Thus the B_i will work like the d_i of section 7. As before, a model, μ, will be a pair $\langle w_0, W\rangle$ where W is a set of worlds, w_i, as described, and $w_0 \in W$. Let R_1, R_2, R_3, and R_4 be defined as in section 3. (We shall ignore R_5-R_8; for the systems IET-IE5, we shall instead make use of E-normal$_I$ models along the lines

5. One consequence of having *A.8* derivable in IX, which simplifies discussion, is that it allows us to define possibility in terms of necessity and negation. From a more genuinely intuitionistic point of view, the possibility operator should be treated as a primitive, not equivalent to $-N-$, just as the existential quantifier cannot be defined in terms of the universal quantifier and negation in the intuitionistic predicate calculus.

suggested in section **6**.) For the evaluation of formulas of the sort $-A$ and $A \supset B$, we introduce a relation, S, such that

$w_i \; S \; w_j$ iff $P_i = P_j$, and $B_i \subseteq B_j$.

This is just an elaboration on the relation, $\subseteq$, as used in section **7**.

Let us say that a world, w_i, is *normal$_I$ in* μ ($\mu = \langle w_0, W\rangle$) if and only if $P_j \subseteq B_j$, for every world, w_j, in W such that $w_j \; S \; w_i$. This is a strong condition; in effect, it requires that for a world to be normal not only must it be related by R to itself, but also that every world 'contained' in it must be related by R to itself. (For classical worlds which were normal in the original sense, this condition was automatically met, since the only world w_j such that $w_j \; S \; w_i$ was w_i itself.) For deontic systems this is weakened, as it was in section **6**, to d-normality$_I$: a world, w_i, is *d-normal$_I$ in* μ iff for every w_j in W such that $w_j \; S \; w_i$, there is a world, w_k, in W such that $w_j \; R \; w_k$.

A model, $\mu = \langle w_0, W\rangle$, is normal$_I$ iff every world in W is normal$_I$ in μ. μ is semi-normal$_I$ iff w_0 is normal$_I$ in μ. μ is E-normal$_I$ iff every world in W, except for perhaps w_0, is normal$_I$ in μ. Similarly, for d-normal$_I$, d-semi-normal$_I$, d-E-normal$_I$ models, but with d-normality$_I$ replacing normal normality$_I$. A model, μ, is d$^+$-normal$_I$ iff μ is E-normal$_I$ and w_0 is d-normal$_I$ in μ. (We could also speak of d$^+$-E-normal$_I$ models, but these would just be E-normal$_I$ models.)

Given a model, $\mu = \langle w_0, W\rangle$, we define the evaluation function, ϕ_μ as before. If A is an atomic formula or of the form $B \& C$ or $B \vee C$, $\phi_\mu(A, w_i)$ is defined exactly as in section **7**, with B_i in place of d_i. If A has the form $B \supset C$, then $\phi_\mu(A, w_i) = \mathrm{T}$ iff for every w_j in W such that $w_i \; S \; w_j$, if $\phi_\mu(B, w_j) = \mathrm{T}$, then $\phi_\mu(C, w_j) = \mathrm{T}$. Similarly, if A has the form $-B$, $\phi_\mu(A, w_i) = \mathrm{T}$ iff $\phi_\mu(B, w_j) \neq \mathrm{T}$, for every w_j in W such that $w_i \; S \; w_j$. If A has the form NB, then $\phi_\mu(A, w_i) = \mathrm{T}$ iff w_i is normal$_I$ in μ and $\phi_\mu(B, w_j) = \mathrm{T}$, for every w_j in W such that $w_i \; R \; w_j$. (It is understood that for deontic systems normality$_I$ in this definition is to be replaced by d-normality$_I$.)

A formula, A is IT (IS4, IB, IS5)-valid iff A is true on every normal$_I$ model by R_1 (R_2, R_3, R_4). A is IS2 (IS3)-valid iff A is true on every semi-normal$_I$ model by R_1 (R_2). A is IE2 (IE3)-valid iff A is true on every model by R_1 (R_2), and finally, A is IET (IE4, IEB, IE5)-valid iff A is true on every E-normal$_I$ model by R_1 (R_2, R_3, R_4). We define validity for the ID and ID$^+$ systems similarly, replacing each type of model by its d or d$^+$ counterpart, and using the definition of ϕ adapted for deontic interpretations.

Theorem 14. *A formula, A, is IX-valid iff A is provable in IX (where IX is any of the above mentioned systems).*

The semantical consistency of these systems can be shown without difficulty. It is, however, helpful to know these facts:

(1) If w_i is normal$_I$ (d-normal$_I$) in μ and $w_j \; S \; w_i$, then w_j is normal$_I$ (d-normal$_I$) in μ.

(2) If w_i is normal$_I$ (d-normal$_I$) in μ and $w_j \; S \; w_i$ and $\phi_\mu(A, w_j) = \mathrm{T}$, then $\phi_\mu(A, w_i) = \mathrm{T}$.

(3) If $\mu = \langle w_0, W\rangle$ and $\mu^* = \langle w_i, W\rangle$ (same W), then $\phi_\mu(A, w_j) = \phi_{\mu^*}(A, w_j)$, for every w_j in W.

To prove the semantical completeness of these systems, *IX*, let μ_K be defined as in section **4** (with the modifications of section **6** for IET-IE5 and their deontic variants). Then the proof of lemma 5 that $\phi_{\mu_K}(B, w_M) = T$ iff $\vdash B$ in M, where M is any ACE of K, can go through as in sections **4** and **8**. However, to establish the cases in which B is of the form $C \supset D$ or $-C$, it is necessary to have

Lemma 10. *If M and N are ACE's of K and if N is an extension of M, then $w_M\ S\ w_N$.*

Proof: That $B_M \subseteq B_N$ is trivial. Similarly, if $C \epsilon P_M$, then $C \epsilon P_N$. Suppose that $C \epsilon P_N$; i.e. $\vdash N(p \supset p) \supset NC$ in N. Since M has the alternation property and $\vdash N(p \supset p) \vee -N(p \supset p)$ in M, $\vdash N(p \supset p)$ in M or $\vdash -N(p \supset p)$ in M. If $\vdash N(p \supset p)$ in M, then $\vdash N(p \supset p)$ in N, so $\vdash NC$ in N. Then, if not $\vdash NC$ in M, again with *A.8* and the alternation property, $\vdash -NC$ in M, in which case $\vdash -NC$ in N, violating the consistency of N. Hence $\vdash NC$ in M, so $\vdash N(p \supset p) \supset NC$ in M and $C \epsilon P_M$. If, on the other hand, $\vdash -N(p \supset p)$ in M, then $\vdash N(p \supset p) \supset NC$ in M (by $-A \supset . A \supset B$) and $C \epsilon P_M$. Thus, $P_N \subseteq P_M$, and so $P_M = P_N$. This shows that $w_M\ S\ w_N$. This lemma is necessary to guarantee that the worlds determined by the systems described for these cases in section **7** stand in the requisite relation S.

The proof of lemma 7 goes through without modification for these intuitionistically based systems, given the reflexivity of S and given *A.8* and the fact that the systems determining worlds in W all have the alternation property.

To re-establish lemma 6, that if $\vdash NB$ in M, then w_M is normal$_I$ in μ_K, we must show that if $\vdash NB$ in M, then for every w_N such that $w_N\ S\ w_M$, $P_N \subseteq B_N$. Suppose that $\vdash NB$ in M; it follows that $\vdash N(p \supset p)$ in M. It also follows that, if $w_N\ S\ w_M$, $\vdash N(p \supset p)$ in N; for if not $\vdash N(p \supset p)$ in N, then $\vdash -N(p \supset p)$ in N, by *A.8* and the alternation property, which would mean that $\vdash -N(p \supset p)$ in M, contrary to the consistency of that system. So, for any C, if $C \epsilon P_N$, i.e. if $\vdash N(p \supset p) \supset NC$ in N, then $\vdash NC$ in N. So $\vdash C$ in N, by *A.1* and thus $C \epsilon B_N$. Hence, $P_N \subseteq B_N$. (For the deontic systems one would argue as above that if $\vdash NB$ in M, then $\vdash N(p \supset p)$ in N, after which lemma 4 can be brought to bear, as in section **6**, to show that there is a world, w_P such that $w_N\ R\ w_P$.)

With these variations on lemmas 6 and 7, the proof of lemma 5 for the case in which B has the form NC can be carried out exactly as in section **4**. It follows that the systems *IX* are all semantically complete with respect to the specified interpretations.[6]

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6. The role played by *A.8* in the foregoing discussion is so central that, at this time, I see no way of dispensing with this postulate to form modal extensions of IPC which are truer to the intuitionistic point of view while remaining within the present framework.

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