A SINGLE-SAMPLE MULTIPLE-DECISION PROCEDURE FOR SELECTING THE MULTINOMIAL EVENT WHICH HAS THE HIGHEST PROBABILITY[1]

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Summary. The problem of selecting the multinomial event which has the highest probability is formulated as a multiple-decision selection problem. Before experimentation starts the experimenter must specify two constants (θ^*, P^*) which are incorporated into the requirement: "The probability of a correct selection is to be equal to or greater than P^* whenever the true (but unknown) ratio of the largest to the second largest of the population probabilities is equal to or greater than θ^*." A single-sample procedure which meets the requirement is proposed. The heart of the procedure is the proper choice of N, the number of trials. Two methods of determining N are described: the first is exact and is to be used when N is small; the second is approximate and is to be used when N is large. Tables and sample calculations are provided.

1. Introduction. We are concerned in this paper with the multiple-decision problem which arises when one attempts to answer questions such as the following:

(a) Which of the six faces of a loaded die has the largest probability of landing face up?

(b) Which of the thirty-six "bettable" numbers on an unbalanced roulette wheel has the largest probability associated with it?

(c) Which of the k television programs available to a given TV audience in a certain locale can claim the largest proportion of the total audience as listeners?

The multinomial distribution provides a statistical model for dealing with each of these questions. In the following sections it is shown how such questions as these can be formulated as *multiple-decision selection problems*. A single-sample procedure is proposed which provides a solution to these problems.

2. Statistical assumptions, and definitions. Let $\mathbf{X}_j = (X_{1j}, X_{2j}, \cdots, X_{kj})$ be independent vector-observations from the same multinomial population with a common unknown probability vector $\mathbf{p} = (p_1, p_2, \cdots, p_k)$; here p_i is the probability of the event $E_i (0 \leqq p_i \leqq 1, \sum_{i=1}^k p_i = 1)$ and $X_{ij} = 1$ or 0 according as E_i does or does not occur on the jth observation $(i = 1, 2, \cdots, k; j = 1, 2, \cdots)$. Let $p_{[1]} \leqq p_{[2]} \leqq \cdots \leqq p_{[k]}$ denote the ranked probabilities. It is assumed

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that the experimenter has no a priori knowledge which would rule out any of the $k!$ possible pairings of the $p_{[i]}$ with the $E_i (i = 1, 2, \cdots, k)$. Let $\theta_{i,j} = p_{[i]}/p_{[j]}$ $(i \geqq j;\ i, j = 1, 2, \cdots, k)$. Let N be the number of vector observations; let $Y_{iN} = \sum_{j=1}^{N} X_{ij} (i = 1, 2, \cdots, k)$, and let $y_{[1]N} \leqq y_{[2]N} \leqq \cdots \leqq y_{[k]N}$ denote their ranked values. Let $E_{[i]N}$ denote the event associated with $y_{[i]N}$. Let $Y_{(i)N}$ be that one of the Y_{iN} which is associated with the event having probability $p_{[i]}$ $(i = 1, 2, \cdots, k)$.

3. Goal, specification, and requirement. We now state the experimenter's goal, specification, and requirement:

Goal. The experimenter's goal is to select the event associated with $p_{[k]}$.

The statistical formulation of the problem for this goal involves the true ratio $\theta_{k,k-1} = \theta$ (say) and the true probability P of a correct selection. It is assumed that *before experimentation starts* the experimenter can specify a pair of constants (θ^*, P^*) with $1 < \theta^* < \infty$ and $1/k < P^* < 1$ as described below.

Specification. The experimenter specifies:

(a) The smallest value, θ^*, of the ratio θ that is worth detecting, and

(b) The smallest acceptable value, P^*, of the probability P of achieving the above goal when $\theta \geqq \theta^*$.

The specification above is summarized in the following:

Requirement. The experimenter requires that the procedure to be used guarantee that

$$\text{Probability }\{\text{Correct selection} \mid \theta_{k,k-1} \geqq \theta^*\} \geqq P^*.$$

That is, the probability of a correct selection is to be equal to or greater than P^* whenever the true (but unknown) ratio of the largest to the second largest of the population probabilities p_i is equal to or greater than θ^*.

4. Procedure. We propose a single-sample procedure that will guarantee the requirement. It is similar to ones described in [1], [2], and [6] which are used to solve selection problems involving parameters associated with other basic distributions. (A *sequential* procedure which will guarantee the same requirement (above) was reported on in [3]; the theory underlying the sequential procedure will be given in a paper with the same title as [3] which is being prepared by the authors of that abstract. Either the sequential procedure reported on in [3] or the single-sample procedure described in the present paper can also be used to solve problems of the type posed in [4].)

Our single-sample procedure takes the following form:

Procedure.

(a) Select a random sample of N vector-observations.

(b) Compute the $y_{[i]N}$ $(i = 1, 2, \cdots, k)$.

(c) If *exactly* s $(s = 1, 2, \cdots, k)$ of the $y_{[i]N}$ are tied for largest, $y_{[k-s+1]N} = y_{[k-s+2]N} = \cdots = y_{[k]N}$, select as the event associated with $p_{[k]}$, one of $E_{[k-s+1]N}$, $E_{[k-s+2]N}, \cdots, E_{[k]N}$ using a random device which assigns probability $1/s$ to each of them. In particular, if there is a *single* largest one, *i.e.*, if $s = 1$, select $E_{[k]N}$.

The heart of the problem in terms of designing the experiment is the proper choice of N; it must be chosen just large enough to guarantee the requirement, i.e., P^* must be achieved for all possible $\mathbf{p} = (p_1, p_2, \cdots, p_k)$ for which $\theta_{k,k-1} \geqq \theta^*$. In order to accomplish this we consider the *least favorable configuration* (l.f.c.) of the $p_{[i]}$'s the definition of which follows.

We define the l.f.c. of the $p_{[i]}$'s as that probability vector $\mathbf{p}$ which for any given N, k, θ^* minimizes the probability of a correct selection when $\theta_{k,k-1} \geqq \theta^*$. It is proved in [7] that:

(a) The l.f.c. is independent of N.

(b) The l.f.c. is given by

$$\theta_{k,i} = \theta^* \qquad (i = 1, 2, \cdots, k-1). \tag{1}$$

Since $\sum_{i=1}^{k} p_{[i]} = 1$, this implies

$$\begin{aligned} p_{[1]} &= p_{[2]} = \cdots = p_{[k-1]} = 1/(\theta^* + k - 1) \\ p_{[k]} &= \theta^*/(\theta^* + k - 1). \end{aligned} \tag{2}$$

It is intuitively clear that for any $\mathbf{p}$ for which the p_i are not all equal, the probability of a correct selection increases with N. Hence if N is chosen large enough to guarantee a specified probability of a correct selection when the configuration is least favorable, it will guarantee *at least* that probability for *any* configuration of $p_{[i]}$'s with $\theta_{k,k-1} \geqq \theta^*$, i.e., it will be large enough to guarantee the requirement. We shall denote by N^* the smallest value of N which will guarantee the requirement.

In the next section we shall show how to compute the *exact* probability of a correct selection for any k, N, and $\mathbf{p}$; this method is appropriate when N is small. Methods of approximating these probabilities when N is not too small are given in sections 6 and 7. Using these methods it is possible to compute tables from which N^* can be determined for any specification. Several such tables are given at the end of this paper.

5. Exact probability of a correct selection. For any fixed k and N and any probability vector $\mathbf{p}$ with $p_{[k-1]} < p_{[k]}$, the probability of a correct selection is given by

$$\phi_k = \phi(p_{[1]}, \cdots, p_{[k]}) = \sum \frac{1}{s} \cdot \frac{N!}{y_{(1)N}! \cdots y_{(k)N}!} p_{[1]}^{y_{(1)N}} \cdots p_{[k]}^{y_{(k)N}} \tag{3}$$

where the summation is over all vectors $\mathbf{y}_N = (y_{(1)N}, y_{(2)N}, \cdots, y_{(k)N})$ such that $\sum_{i=1}^{k} y_{(i)N} = N$ and $y_{(k)N} \geqq y_{(i)N}$ $(i = 1, 2, \cdots, k-1)$, and s, which is a function of $\mathbf{y}_N$, is the number of $y_{(i)N}$'s tied for largest. (Note: If *exactly* s ($s = 1, 2, \cdots, k$) of the $p_{[i]}$ are tied for largest, then $p_{[k-s+1]} = p_{[k-s+2]} = \cdots = p_{[k]}$, and we consider the selection of any one of the associated s events as a correct selection.)

For fixed k and N it is straightforward (but tedious) to express ϕ_k in terms of the $p_{[i]} (i = 1, 2, \cdots, k)$; for example, if $k = 3$, $N = 4$ and $p_{[3]} > p_{[2]}$, it is easy to verify that

$$(4) \quad \phi(p_{[1]}, p_{[2]}, p_{[3]}) = p_{[3]}^4 + 4p_{[3]}^3(1 - p_{[3]}) + 3p_{[3]}^2(p_{[2]}^2 + 4p_{[2]}p_{[1]} + p_{[1]}^2)$$

and if the $p_{[i]}$ $(i = 1, 2, 3)$ are in the l.f.c. (2), this reduces to

$$(5) \qquad \phi\left(\frac{1}{\theta^* + 2}, \frac{1}{\theta^* + 2}, \frac{\theta^*}{\theta^* + 2}\right) = \frac{(\theta^*)^2}{(\theta^* + 2)^4}\{(\theta^*)^2 + 8\theta^* + 18\}$$

which involves only θ^*. Substituting the numerical value of the specified θ^* in (5) gives the *exact* probability of a correct selection for $k = 3$ and $N = 4$ when the $p_{[i]}$ $(i = 1, 2, 3)$ are in the l.f.c. (2). General expressions of the type of (4) can be obtained for arbitrary k and N, and using (2) these can be reduced to expressions of the type of (5) which involve only θ^*. The exact probabilities of a correct selection which are listed in Tables A-2, A-3, and A-4 are all associated with the l.f.c. and were computed using expressions of the type of (5). It should be emphasized, however, that it can be extremely laborious to express (3) in the form (5), especially when N is moderately large. For example, for $k = 4$ and $N = 20$ the expression analogous to the term in braces in (5) is a fifteenth-degree polynomial in which five of the coefficients are eleven digit integers. The evaluation of such expressions for the specified θ^* is an additional problem (although this can be done expeditiously on a high-speed electronic computer). It thus is obvious that some large-sample approximation for (3) would be desirable.[2] We consider this problem in the next sections.

6. Large-sample approximations to the probability of a correct selection. First we shall consider the case $k = 3$ to see what might be involved in making a large-sample approximation. We have

$$(6) \qquad \begin{aligned} \phi_3 = {} & \Pr\{Y_{(3)N} - Y_{(1)N} > 0, Y_{(3)N} - Y_{(2)N} > 0\} \\ & + \tfrac{1}{2}\Pr\{Y_{(3)N} - Y_{(1)N} = 0, Y_{(3)N} - Y_{(2)N} > 0\} \\ & + \tfrac{1}{2}\Pr\{Y_{(3)N} - Y_{(1)N} > 0, Y_{(3)N} - Y_{(2)N} = 0\} \\ & + \tfrac{1}{3}\Pr\{Y_{(3)N} - Y_{(1)N} = 0, Y_{(3)N} - Y_{(2)N} = 0\}. \end{aligned}$$

Because of the equality signs, the last three terms become negligible for large N; hence, for large N we shall approximate ϕ_3 by

$$(7) \qquad \bar{\phi}_3 = \Pr\{Y_{(3)N} - Y_{(1)N} \geqq 0, Y_{(3)N} - Y_{(2)N} \geqq 0\}.$$

[2] The referee has called our attention to an unpublished Stanford technical report, "A procedure for determining the loaded face of a die," 1952, written by S. G. Allen, Jr., which reports some results of H. Rubin and the late M. A. Girshick. They were concerned with obtaining a large-sample approximation for (3) when (1) holds. The approximation that they propose is different from ours and appears to be much more tedious to apply. We have not attempted to determine which approximation is the better.

For arbitrary k we let

$$W_i = \frac{Y_{(k)N}}{N} - \frac{Y_{(i)N}}{N} \qquad (i = 1, 2, \cdots, k-1) \tag{8}$$

and approximate (3) by

$$\phi_k = \Pr\{W_1 \geqq 0, W_2 \geqq 0, \cdots, W_{k-1} \geqq 0\}. \tag{9}$$

TABLE A-2

Exact Probability of a Correct Selection for $k = 2$ and selected θ^* and N when $p_{[2]}/p_{[1]} = \theta^*$

N	θ^*									
	1.02	1.04	1.06	1.08	1.10	1.20	1.30	1.40	1.50	1.60
1	.504951	.509804	.514563	.519231	.523810	.545455	.565217	.583333	.600000	.615385
2	.504951	.509804	.514563	.519231	.523810	.545455	.565217	.583333	.600000	.615385
3	.507426	.514704	.521838	.528832	.535687	.567994	.597271	.623843	.648000	.670005
4	.507426	.514704	.521838	.528832	.535687	.567994	.597271	.623843	.648000	.670005
5	.509282	.518378	.527290	.536022	.544575	.584759	.620903	.653381	.682560	.708788
6	.509282	.518378	.527290	.536022	.544575	.584759	.620903	.653381	.682560	.708788
7	.510828	.521438	.531830	.542005	.551965	.598614	.640261	.677312	.710208	.739386
8	.510828	.521438	.531830	.542005	.551965	.598614	.640261	.677312	.710208	.739386
9	.512181	.524114	.535798	.547232	.558417	.610637	.656910	.697670	.733432	.764734
10	.512181	.524114	.535798	.547232	.558417	.610637	.656910	.697670	.733432	.764734
11	.513399	.526522	.539367	.551930	.564210	.621369	.671640	.715483	.753498	.786332
12	.513399	.526522	.539367	.551930	.564210	.621369	.671640	.715483	.753498	.786332
13	.514515	.528729	.542636	.556230	.569509	.631125	.684913	.731359	.771156	.805076
14	.514515	.528729	.542636	.556230	.569509	.631125	.684913	.731359	.771156	.805076
15	.515551	.530777	.545668	.560217	.574417	.640109	.697028	.745691	.786897	.821554
16	.515551	.530777	.545668	.560217	.574417	.640109	.697028	.745691	.786897	.821554
17	.516523	.532697	.548509	.563949	.579009	.648462	.708193	.758754	.801064	.836180
18	.516523	.532697	.548509	.563949	.579009	.648462	.708193	.758754	.801064	.836180
19	.517440	.534509	.551189	.567468	.583336	.656286	.718558	.770748	.813908	.849257
20	.517440	.534509	.551189	.567468	.583336	.656286	.718558	.770748	.813908	.849257
21	.518312	.536229	.553734	.570807	.587437	.663657	.728237	.781826	.825622	.861019
22	.518312	.536229	.553734	.570807	.587437	.663657	.728237	.781826	.825622	.861019
23	.519143	.537871	.556160	.573989	.591342	.670635	.737319	.792107	.836357	.871648
24	.519143	.537871	.556160	.573989	.591342	.670635	.737319	.792107	.836357	.871648
25	.519940	.539444	.558484	.577034	.595077	.677267	.745874	.801686	.846232	.881292
26	.519940	.539444	.558484	.577034	.595077	.677267	.745874	.801686	.846232	.881292
27	.520706	.540956	.560716	.579958	.598660	.683590	.753961	.810641	.855348	.890072
28	.520706	.540956	.560716	.579958	.598660	.683590	.753961	.810641	.855348	.890072
29	.521445	.542413	.562867	.582773	.602107	.689639	.761626	.819036	.863787	.898087
30	.521445	.542413	.562867	.582773	.602107	.689639	.761626	.819036	.863787	.898087

TABLE A-2 (*Continued*)

	1.70	1.80	1.90	2.00	2.20	2.40	2.60	2.80	3.00	10.00
1	.629630	.642857	.655172	.666667	.687500	.705882	.722222	.736842	.750000	.909091
2	.629630	.642857	.655172	.666667	.687500	.705882	.722222	.736842	.750000	.909091
3	.690088	.708455	.725286	.740741	.768066	.791370	.811385	.828692	.843750	.976709
4	.690088	.708455	.725286	.740741	.768066	.791370	.811385	.828692	.843750	.976709
5	.732384	.753637	.772806	.790123	.819994	.844615	.865048	.882123	.896484	.993474
6	.732384	.753637	.772806	.790123	.819994	.844615	.865048	.882123	.896484	.993474
7	.765261	.788215	.808593	.826703	.857182	.881462	.900934	.916658	.929443	.998093
8	.765261	.788215	.808593	.826703	.857182	.881462	.900934	.916658	.929443	.998093
9	.792095	.816001	.836890	.855154	.885145	.908237	.926132	.940096	.951073	.999428
10	.792095	.816001	.836890	.855154	.885145	.908237	.926132	.940096	.951073	.999428
11	.814623	.838967	.859905	.877915	.906773	.928249	.944330	.956457	.965673	.999826
12	.814623	.838967	.859905	.877915	.906773	.928249	.944330	.956457	.965673	.999826
13	.833885	.858301	.878969	.896461	.923811	.943483	.957717	.968090	.975710	.999946
14	.833885	.858301	.878969	.896461	.923811	.943483	.957717	.968090	.975710	.999946
15	.850569	.874788	.894967	.911768	.937407	.955231	.967692	.976468	.982700	.999983
16	.850569	.874788	.894967	.911768	.937407	.955231	.967692	.976468	.982700	.999983
17	.865159	.888983	.908521	.924525	.948360	.964376	.975196	.982560	.987615	.999995
18	.865159	.888983	.908521	.924525	.948360	.964376	.975196	.982560	.987615	.999995
19	.878012	.901294	.920089	.935234	.957251	.971550	.980883	.987023	.991097	.999998
20	.878012	.901294	.920089	.935234	.957251	.971550	.980883	.987023	.991097	.999998
21	.889402	.912036	.930019	.944277	.964509	.977209	.985219	.990311	.993577	.999999
22	.889402	.912036	.930019	.944277	.964509	.977209	.985219	.990311	.993577	.999999
23	.899543	.921452	.938586	.951950	.970463	.981695	.988540	.992745	.995353	1.000000
24	.899543	.921452	.938586	.951950	.970463	.981695	.988540	.992745	.995353	1.000000
25	.908609	.929740	.946005	.958486	.975366	.985265	.991094	.994555	.996630	1.000000
26	.908609	.929740	.946005	.958486	.975366	.985265	.991094	.994555	.996630	1.000000
27	.916740	.937058	.952450	.964073	.979418	.988116	.993066	.995904	.997549	1.000000
28	.916740	.937058	.952450	.964073	.979418	.988116	.993066	.995904	.997549	1.000000
29	.924053	.943538	.958069	.968861	.982776	.990399	.994590	.996914	.998216	1.000000
30	.924053	.943538	.958069	.968861	.982776	.990399	.994590	.996914	.998216	1.000000

6.1 The normal approximation. In the least favorable configuration the chance variables W_i $(i = 1, 2, \cdots, k - 1)$ have a $(k - 1)$-variate distribution with

$$E\{W_i\} = \frac{\theta^* - 1}{\theta^* + k - 1} \tag{10a}$$

$$\text{Var}\,\{W_i\} = \frac{(k + 2)\theta^* + (k - 2)}{N(\theta^* + k - 1)^2} \tag{10b}$$

$$\underset{i \neq j}{\text{Cov}\,\{W_i, W_j\}} = \frac{(k + 1)\theta^* - 1}{N(\theta^* + k - 1)^2} \tag{10c}$$

(10d)
$$\operatorname{Corr}\{W_i, W_j\}_{i \neq j} = \frac{(k+1)\theta^* - 1}{(k+2)\theta^* + (k-2)};$$

it is to be noted that the W_i have a common mean and a common variance, and all pairs (W_i, W_j) with $i \neq j$ have a common correlation. The standardized variables obtained by subtracting the common mean from each W_i and dividing the differences by the common standard deviation have zero mean, unit variance and correlation (10d). As N approaches infinity it can be shown that their joint

TABLE A-3

Exact Probability of a Correct Selection for $k = 3$ and selected θ^* and N when $p_{[3]}/p_{[2]} = p_{[3]}/p_{[1]} = \theta^*$

N	θ^*									
	1.02	1.04	1.06	1.08	1.10	1.20	1.30	1.40	1.50	1.60
1	.337748	.342105	.346405	.350649	.354839	.375000	.393939	.411765	.428571	.444444
2	.337748	.342105	.346405	.350649	.354839	.375000	.393939	.411765	.428571	.444444
3	.339230	.345067	.350845	.356563	.362223	.389648	.415644	.440261	.463557	.485597
4	.340211	.347015	.353746	.360404	.366988	.398804	.428798	.457023	.483549	.508459
5	.340708	.348015	.355254	.362424	.369524	.403954	.436571	.467376	.496400	.523701
6	.341530	.349659	.357717	.365703	.373614	.412000	.448349	.482601	.514760	.544870
7	.342261	.351116	.359896	.368599	.377220	.419041	.458579	.495737	.530497	.562902
8	.342690	.351980	.361198	.370341	.379405	.423441	.465140	.504341	.540988	.575098
9	.343331	.353264	.363125	.372909	.382612	.429767	.474385	.516236	.555226	.591360
10	.343924	.354448	.364899	.375270	.385556	.435526	.482730	.526881	.567859	.605664
11	.344311	.355228	.366075	.376845	.387531	.439500	.488625	.534549	.577108	.616279
12	.344852	.356312	.367703	.379016	.390244	.444851	.496410	.544485	.588874	.629544
13	.345363	.357335	.369237	.381060	.392794	.449841	.503615	.553613	.599602	.641549
14	.345718	.358049	.370316	.382506	.394608	.453486	.508992	.560545	.607866	.650901
15	.346193	.359004	.371751	.384422	.397003	.458204	.515819	.569185	.617984	.662159
16	.346649	.359916	.373121	.386247	.399282	.462662	.522228	.577239	.627350	.672506
17	.346978	.360581	.374125	.387593	.400971	.466050	.527195	.583582	.634822	.680846
18	.347408	.361444	.375422	.389326	.403137	.470311	.533327	.591273	.643725	.690617
19	.347822	.362275	.376671	.390991	.405217	.474375	.539137	.598513	.652051	.699693
20	.348131	.362899	.377614	.392256	.406805	.477552	.543766	.604368	.658865	.707190
21	.348525	.363692	.378808	.393850	.408798	.481465	.549362	.611321	.666818	.715798
22	.348908	.364459	.379962	.395391	.410723	.485220	.554699	.617912	.674310	.723854
23	.349200	.365050	.380855	.396589	.412226	.488221	.559043	.623353	.680564	.730636
24	.349566	.365787	.381966	.398073	.414083	.491856	.564207	.629708	.687746	.738300
25	.349923	.366505	.383045	.399514	.415882	.495360	.569158	.635764	.694548	.745512
26	.350202	.367067	.383896	.400655	.417315	.498211	.573255	.640847	.700315	.751678
27	.350546	.367759	.384939	.402049	.419059	.501616	.578062	.646703	.706852	.758554
28	.350882	.368435	.385956	.403407	.420755	.504912	.582687	.652307	.713070	.765054
29	.351147	.368973	.386770	.404499	.422126	.507632	.586569	.657072	.718410	.770679
30	.351472	.369627	.387756	.405818	.423776	.510844	.591073	.662503	.724397	.776883

TABLE A-3 (*Continued*)

	1.70	1.80	1.90	2.00	2.20	2.40	2.60	2.80	3.00	10.00
1	.459459	.473684	.487179	.500000	.523810	.545455	.565217	.583333	.600000	.833333
2	.459459	.473684	.487179	.500000	.523810	.545455	.565217	.583333	.600000	.833333
3	.506446	.526170	.544834	.562500	.595076	.624343	.650694	.647479	.696000	.937500
4	.531844	.553794	.574400	.593750	.629013	.660201	.687858	.712457	.734400	.954861
5	.549348	.573422	.596006	.617188	.655677	.689539	.719367	.745687	.768960	.976562
6	.573002	.599248	.623707	.646484	.687420	.722879	.753615	.780302	.803520	.985605
7	.593033	.620996	.646908	.670898	.713621	.750156	.781411	.808186	.831168	.992135
8	.606741	.636019	.663056	.687988	.732096	.769443	.801050	.827817	.850522	.995066
9	.624709	.655391	.383552	.709351	.754530	.792237	.823677	.849902	.871811	.997263
10	.640379	.672142	.701122	.727509	.773277	.810967	.841960	.867454	.888455	.998342
11	.652138	.684827	.714528	.741447	.787793	.825541	.856218	.881142	.901414	.999029
12	.666581	.700148	.730459	.757750	.804259	.841598	.871496	.895425	.914596	.999425
13	.679553	.713808	.744556	.772069	.818512	.855299	.884350	.907277	.925387	.999662
14	.689754	.724627	.755785	.783525	.829974	.866334	.894690	.916778	.933995	.999797
15	.701829	.737229	.768662	.796463	.842547	.878110	.905440	.926415	.942521	.999881
16	.712849	.748649	.780247	.808021	.853620	.888334	.914642	.934550	.949621	.999929
17	.721803	.757985	.789763	.817546	.862776	.896789	.922234	.941231	.955417	.999957
18	.732124	.768579	.800397	.828033	.872575	.905597	.929943	.947853	.961031	.999975
19	.741646	.778285	.810073	.837510	.881309	.913341	.936627	.953517	.965769	.999985
20	.749569	.786405	.818200	.845490	.888677	.919865	.942239	.958246	.969696	.999991
21	.758521	.795439	.827106	.854109	.896418	.926542	.947841	.962856	.973441	.999995
22	.766844	.803783	.835277	.861964	.903378	.932464	.952742	.966836	.976630	.999997
23	.773897	.810887	.842256	.868687	.909339	.937525	.956912	.970200	.979306	.999998
24	.781741	.818669	.849790	.875841	.915513	.942633	.961019	.973439	.981825	.999999
25	.789077	.825899	.856745	.882401	.921100	.947194	.964638	.976255	.983987	.999999
26	.795385	.832141	.862765	.888089	.925944	.951138	.967749	.978658	.985817	1.000000
27	.802311	.838895	.869184	.894068	.930902	.955073	.970782	.980949	.987524	1.000000
28	.808819	.845199	.875139	.899580	.935412	.958605	.973468	.982950	.988996	1.000000
29	.814481	.850706	.880351	.904409	.939360	.961685	.975795	.984672	.990250	1.000000
30	.820632	.856601	.885853	.909437	.943363	.964731	.978045	.986299	.991412	1.000000

distribution approaches a $(k - 1)$-variate normal distribution. Thus, for large N the probability of a correct selection in the least favorable configuration can be approximated by the volume under a $(k - 1)$-variate normal surface. However for $k \geqq 4$ such volumes are tabulated (see Table 1 of [1], Table A1 of [5], and [9]) only for the particular correlation matrix $\{\rho_{ij}\}$ where

$$\rho_{ij} = \begin{cases} 1 & \text{if } i = j \\ \frac{1}{2} & \text{if } i \neq j. \end{cases} \tag{11}$$

Although the matrix of Corr $\{W_i, W_j\}$ *is* of this *form*, the correlation coefficient when $i \neq j$ is *not* $\frac{1}{2}$; in fact, as k approaches infinity this correlation coefficient approaches $\theta^*/(\theta^* + 1)$. Because of the lack of appropriate tables this approach was abandoned. The approach described in the next section yielded the desired results.

6.2. The arcsin transformation and the normal approximation.

6.2.1. Derivation of formulae for the approximation.

We next consider the chance variables

TABLE A-4

Exact Probability of a Correct Selection for $k = 4$ and selected θ^* and N when $p_{[4]}/p_{[3]} = p_{[4]}/p_{[2]} = p_{[4]}/p_{[1]} = \theta^*$

N	θ^*									
	1.02	1.04	1.06	1.08	1.10	1.20	1.30	1.40	1.50	1.60
1	.253731	.257426	.261084	.264706	.268293	.285714	.302326	.318182	.333333	.347826
2	.253731	.257426	.261084	.264706	.268293	.285714	.302326	.318182	.333333	.347826
3	.254673	.259318	.263935	.268522	.273081	.295432	.317041	.337904	.358025	.377414
4	.255611	.261192	.266744	.272264	.277752	.304688	.330730	.355833	.379973	.403143
5	.256139	.262250	.268331	.274382	.280401	.309977	.338609	.366224	.392775	.418245
6	.256583	.263145	.269686	.276203	.282694	.314699	.345828	.375948	.404969	.432836
7	.257172	.264331	.271472	.278595	.285694	.320770	.354949	.388029	.419872	.450388
8	.257687	.265365	.273030	.280677	.288304	.326017	.362783	.398342	.432516	.465190
9	.258099	.266195	.274282	.282357	.290415	.330318	.369274	.406967	.443171	.477740
10	.258492	.266989	.275485	.283975	.292454	.334516	.375660	.415493	.453734	.490198
11	.258928	.267868	.276814	.285758	.294695	.339075	.382512	.424536	.464814	.503121
12	.259327	.268673	.278029	.287388	.296744	.343240	.388764	.432773	.474882	.514835
13	.259681	.269389	.279113	.288846	.298580	.347007	.394460	.440317	.484138	.525629
14	.260022	.270078	.280158	.290253	.300355	.350670	.400013	.447681	.493169	.536145
15	.260383	.270808	.281264	.291741	.302228	.354499	.405770	.455249	.502373	.546771
16	.260724	.271497	.282308	.293144	.303995	.358115	.411204	.462384	.511035	.556749
17	.261038	.272133	.283272	.294443	305634	.361490	.416297	.469092	.519189	.566146
18	.261340	.272747	.284205	.295701	.307222	.364774	.421263	.475631	.527129	.575274
19	.261657	.273387	.285177	.297011	.308874	.368164	.426353	.482287	.535152	.584432
20	.261959	.274001	.286109	.298266	.310457	.371415	.431231	.488657	.542817	.593158
21	.262243	.274578	.286985	.299447	.311949	.374495	.435869	.494723	.550117	.601462
22	.262519	.275137	.287836	.300597	.313402	.377504	.440402	.500650	.557238	.609543
23	.262804	.275716	.288716	.301784	.314900	.380587	.445020	.506648	.564399	.617614
24	.263079	.276276	.289567	.302931	.316349	.383569	.449483	.512437	.571294	.625364
25	.263341	.276807	.290375	.304023	.317730	.386423	.453764	.517995	.577911	.632792
26	.263595	.277324	.291164	.305090	.319079	.389219	.457960	.523439	.584380	.640033
27	.263857	.277857	.291975	.306185	.320463	.392071	.462216	.528926	.590860	.647242
28	.264112	.278375	.292763	.307249	.321808	.394844	.466352	.534250	.597133	.654198
29	.264355	.278870	.293518	.308270	.323100	.397516	.470344	.539390	.603185	.660898
30	.264592	.279355	.294257	.309271	.324366	.400140	.474265	.544434	.609112	.667440

TABLE A-4 (*Continued*)

	1.70	1.80	1.90	2.00	2.20	2.40	2.60	2.80	3.00	10.00
1	.361702	.375000	.387755	.400000	.423077	.444444	.464286	.482759	.500000	.769231
2	.361702	.375000	.387755	.400000	.423077	.444444	.464286	.482759	.500000	.769231
3	.396088	.414063	.431359	.448000	.479404	.508459	.535350	.560253	.583333	.892126
4	.425352	.446615	.466955	.486400	.522732	.555877	.586110	.613697	.638889	.929939
5	.442630	.465942	.488203	.509440	.548979	.584855	.617382	.646869	.673611	.956118
6	.459521	.485016	.509329	.532480	.575418	.614126	.648945	.680232	.708333	.972900
7	.479526	.507268	.533616	.558592	.604569	.645564	.682011	.714362	.743056	.983657
8	.496297	.525812	.553736	.580096	.628297	.670860	.708314	.741205	.770062	.989681
9	.510588	.541676	.571001	.598589	.648751	.692668	.730951	.764235	.793130	.993584
10	.524771	.557400	.588076	.616825	.668762	.713796	.752642	.786038	.814697	.996024
11	.539326	.573366	.605234	.634964	.688278	.734014	.773022	.806167	.834270	.997522
12	.552481	.587752	.620642	.651193	.705602	.751811	.790802	.823564	.851026	.998450
13	.564618	.601028	.634853	.666144	.721496	.768040	.806895	.839178	.865925	.999033
14	.576412	.613886	.648563	.680503	.736604	.783293	.821837	.853493	.879409	.999395
15	.588232	.626666	.662079	.694543	.751145	.797744	.835780	.866653	.891625	.999622
16	.599300	.638596	.674651	.707554	.764506	.810902	.848351	.878395	.902410	.999763
17	.609715	.649804	.686436	.719716	.776908	.823012	.859809	.888988	.912033	.999851
18	.619802	.660620	.697760	.731348	.788646	.834343	.870404	.898662	.920712	.999907
19	.629849	.671313	.708875	.742683	.799919	.845071	.880292	.907565	.928587	.999942
20	.639392	.681436	.719356	.753325	.810407	.854951	.889302	.915588	.935605	.999963
21	.648459	.691030	.729261	.763347	.820203	.864092	.897552	.922853	.941887	.999977
22	.657254	.700300	.738788	.772941	.829482	.872652	.905187	.929494	.947559	.999986
23	.665978	.709434	.748111	.782265	.838375	.880744	.912303	.935600	.952702	.999991
24	.674328	.718142	.756962	.791078	.846700	.888239	.918823	.941130	.957307	.999994
25	.682313	.726446	.765375	.799421	.854511	.895201	.924814	.946155	.961441	.999996
26	.690070	.734481	.773477	.807418	.861918	.901727	.930363	.950752	.965176	.999997
27	.697743	.742376	.781387	.815172	.869005	.907887	.935530	.954974	.968560	.999998
28	.705121	.749939	.788930	.822533	.875664	.913614	.940279	.958810	.971600	.999999
29	.712210	.757181	.796128	.829528	.881935	.918950	.944656	.962304	.974335	1.000000
30	.719106	.764196	.803068	.836237	.887883	.923956	.948710	.965501	.976808	1.000000

$$(12) \qquad Z_i = 2 \arcsin \sqrt{\frac{Y_{(k)N}}{N}} - 2 \arcsin \sqrt{\frac{Y_{(i)N}}{N}} \qquad (i = 1, 2, \cdots, k-1),$$

and for arbitrary k we shall write (9) as

$$(13) \qquad \tilde{\phi}_k = \Pr\{Z_1 \geqq 0, Z_2 \geqq 0, \cdots, Z_{k-1} \geqq 0\}.$$

The chance variables Z_i $(i = 1, 2, \cdots, k-1)$ also have a $(k-1)$-variate distribution, with means, variances, and covariances which can be expressed as power series in $1/N$. In the l.f.c. these power series involve only k, N and θ^*. They are found in the following way. We expand $2 \arcsin (Y_{(i)N}/N)^{1/2}$, considered as a function of $Y_{(i)N}/N$, in a Taylor series around the point $Y_{(i)N}/N = p_{[i]}$ obtaining

$$
\begin{aligned}
2 \arcsin \sqrt{\frac{Y_{(i)N}}{N}} &= f(p_{[i]}) + f'(p_{[i]}) \left(\frac{Y_{(i)N}}{N} - p_{[i]}\right) \\
&\quad + \frac{1}{2} f''(p_{[i]}) \left(\frac{Y_{(i)N}}{N} - p_{[i]}\right)^2 + \frac{1}{6} f'''(p_{[i]}) \left(\frac{Y_{(i)N}}{N} - p_{[i]}\right)^3 \\
&\quad + \frac{1}{24} f^{\text{iv}}(p_{[i]}) \left(\frac{Y_{(i)N}}{N} - p_{[i]}\right)^4 + O\left(\frac{1}{N^5}\right)
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
f(p_{[i]}) &= 2 \arcsin \sqrt{p_{[i]}} \\
f'(p_{[i]}) &= \frac{1}{\sqrt{p_{[i]}(1 - p_{[i]})}} \\
f''(p_{[i]}) &= \frac{2p_{[i]} - 1}{2\sqrt{p_{[i]}^3(1 - p_{[i]})^3}} \\
f'''(p_{[i]}) &= \frac{8p_{[i]}^2 - 8p_{[i]} + 3}{4\sqrt{p_{[i]}^5(1 - p_{[i]})^5}} \\
f^{\text{iv}}(p_{[i]}) &= \frac{3(2p_{[i]} - 1)(8p_{[i]}^2 - 8p_{[i]} + 5)}{8\sqrt{p_{[i]}^7(1 - p_{[i]})^7}}.
\end{aligned}
\tag{15}
$$

Now $E\{Z_i\}$, Var $\{Z_i\}$, and Cov $\{Z_i, Z_j\}$ all can be expressed in terms of $E\{2 \arcsin (Y_{(i)N}/N)^{1/2}\}$, $E\{[2 \arcsin (Y_{(i)N}/N)^{1/2}]^2\}$ and $E\{[2 \arcsin (Y_{(i)N}/N)^{1/2}] [2 \arcsin (Y_{(j)N}/N)^{1/2}]\}$, and these latter involve the moments of multinomially distributed variables which are well known. Thus, for example,

$$
\begin{aligned}
E\{2 \arcsin \sqrt{Y_{(i)N}/N}\} &= f(p_{[i]}) + f''(p_{[i]}) \frac{p_{[i]}(1 - p_{[i]})}{2N} \\
&\quad + f'''(p_{[i]}) \frac{p_{[i]}(1 - p_{[i]})(1 - 2p_{[i]})}{6N^2} \\
&\quad + f^{\text{iv}}(p_{[i]}) \left[\frac{p_{[i]}^2(1 - p_{[i]})^2}{8N^2} + \frac{p_{[i]}(1 - p_{[i]})(1 - 6p_{[i]} + 6p_{[i]}^2)}{24N^3}\right] \\
&\quad + O(1/N^4) \qquad (i = 1, 2, \cdots, k - 1)
\end{aligned}
\tag{16}
$$

which immediately yields $E\{Z_i\}$ up to terms of order $1/N^4$.

In the l.f.c. we use (2) and obtain after simplifications

$$
E\{Z_i\} = a_1 + \frac{a_2}{N} + O(1/N^2) \tag{17a}
$$

$$
\text{Var}\{Z_i\} = \frac{b_1}{N} + \frac{b_2}{N^2} + O(1/N^3) \tag{17b}
$$

$(i = 1, 2, \cdots, k - 1)$

where

$$(18a)\quad a_1 = 2 \arcsin \sqrt{\frac{\theta^*}{\theta^* + k - 1}} - 2 \arcsin \sqrt{\frac{1}{\theta^* + k - 1}}$$

$$(18b)\quad a_2 = \frac{\theta^* + k - 3}{4\sqrt{\theta^* + k - 2}} + \frac{\theta^* - k + 1}{4\sqrt{(k-1)\theta^*}}$$

$$(18c)\quad b_1 = 2 + 2\sqrt{\frac{\theta^*}{(k-1)(\theta^* + k - 2)}}$$

$$(18d)\quad b_2 = \frac{(\theta^* + k - 1)^2(k\theta^* + k - 2) - \theta^*(k - \theta^* - 1)(\theta^* + k - 3)}{4\sqrt{\theta^*(k-1)^3(\theta^* + k - 2)^3}} + \frac{3(\theta^* + k - 1)^2(k\theta^* + k - 2)}{8\theta^*(k-1)(\theta^* + k - 2)} - 1.$$

In addition,

$$(19)\quad \operatorname{Corr}_{i \neq j}\{Z_i, Z_j\} = \frac{1 + 2\sqrt{\dfrac{\theta^*}{(k-1)(\theta^* + k - 2)}} - \dfrac{1}{\theta^* + k - 2} + \dfrac{c}{N} + O\left(\dfrac{1}{N^2}\right)}{2 + 2\sqrt{\dfrac{\theta^*}{(k-1)(\theta^* + k - 2)}} + \dfrac{b_2}{N} + O\left(\dfrac{1}{N^2}\right)}$$

where

$$(20)\quad c = \frac{(\theta^* + k - 1)^2(k\theta^* + k - 2) + \theta^*(\theta^* - k + 1)(\theta^* + k - 3)}{4\sqrt{\theta^*(k-1)^3(\theta^* + k - 2)^3}} - \frac{2(\theta^* + k - 1)^2(\theta^* + k - 2) - (\theta^* + k - 3)^2}{8(\theta^* + k - 2)^3}.$$

The following table shows how (19) varies with k and θ^* when N is large. (In these computations, N in (19) is assumed to be infinite.)

θ^*	k					
	3	4	5	6	10	36
1.10	0.512	0.508	0.506	0.505	0.503	0.501
1.25	0.527	0.519	0.514	0.511	0.506	0.502
1.50	0.548	0.533	0.526	0.521	0.512	0.503
1.75	0.564	0.546	0.536	0.529	0.517	0.505
2.00	0.577	0.556	0.544	0.536	0.521	0.506
10.00	0.674	0.645	0.626	0.611	0.577	0.527

It is seen from (19) with $N = +\infty$ (and from the above table) that as θ^* approaches unity, (19) approaches $\frac{1}{2}$ for any k; also, as k approaches infinity, (19) approaches $\frac{1}{2}$ for any θ^*.

As with the W_i, when (1) holds the Z_i have a common mean and a common variance, and all pairs (Z_i, Z_j) with $i \neq j$ have a common correlation. Also as N approaches infinity it can be shown that the joint distribution of the standardized variables approaches a $(k-1)$-variate normal distribution with zero means, unit variances, and a correlation matrix of the form of (11); *and the common correlation coefficient* (19) *for* $i \neq j$ *is approximately* $\frac{1}{2}$. Thus, the tables in [1], [5], and [9] can be used for finding the volume under the $(k-1)$-variate normal surface, and hence an approximation to the probability of a correct selection can be obtained.

6.2.2. The approximation. To obtain the approximation we let

$$A = a_1 + \frac{a_2}{N} \tag{21a}$$

and

$$B = \frac{b_1}{N} + \frac{b_2}{N^2}. \tag{21b}$$

Then

$$\tilde{\phi}_k \sim \int_{-\frac{A}{\sqrt{B}}}^{\infty} \int_{-\frac{A}{\sqrt{B}}}^{\infty} \cdots \int_{-\frac{A}{\sqrt{B}}}^{\infty} g(t_1, t_2, \cdots, t_{k-1})\, dt_1\, dt_2 \cdots dt_{k-1}, \tag{22}$$

where $g(t_1, t_2, \cdots, t_{k-1})$ is the $(k-1)$-variate normal density function with zero means, unit variances, and correlation matrix (11). Since [9] gives $P_k(\Lambda)$ as a function of Λ where

$$P_k(\Lambda) = \int_{-\frac{\Lambda}{\sqrt{2}}}^{\infty} \int_{-\frac{\Lambda}{\sqrt{2}}}^{\infty} \cdots \int_{-\frac{\Lambda}{\sqrt{2}}}^{\infty} g(t_1, t_2, \cdots, t_{k-1})\, dt_1\, dt_2, \cdots, dt_{k-1}, \tag{23}$$

we obtain

$$\tilde{\phi}_k \sim P_k\left(A\sqrt{\frac{2}{B}}\right). \tag{24}$$

6.2.3. Tables based on the approximation. Some computations were made to indicate the goodness of the approximation. These approximate probabilities are listed along with the corresponding exact probabilities (which were extracted from Tables A-2, A-3, and A-4) in Tables B-2, B-3, and B-4. They were computed as follows: The quantities a_1, a_2, b_1, and b_2 were computed for each k and θ^* using (18a), (18b), (18c), and (18d), respectively; then A and B were computed for each N using (21a) and (21b), respectively; then $\Lambda = A(2/B)^{1/2}$

TABLE B-2

Exact[1] and Approximate[1] Probability of a Correct Selection for k = 2 when $p_{[2]}/p_{[1]} = \theta^*$

θ^*	$N = 10$	$N = 15$	$N = 20$	$N = 25$	$N = 30$
1.02	.51250	.51530	.51767	.51974	.52163
	.512181	.515551	.517440	.519940	.521445
1.10	.55993	.57322	.58437	.59414	.60293
	.558417	.574417	.583336	.595077	.602107
1.50	.73764	.78204	.81587	.84279	.86481
	.733432	.786897	.813908	.846232	.863787
2.00	.85744	.90506	.93711	.95494	.96836
	.855154	.911768	.935234	.958486	.968861
3.00	.94834	.97750	.98985	.99533	.99781
	.951073	.982700	.991097	.996630	.998216
10.00	.99700	.99973	.99998	1.00000	1.00000
	.999428	.999983	.999998	1.000000	1.000000

[1] Exact probabilities are given to six decimal places; approximate probabilities are given to five decimal places.

TABLE B-3

Exact[1] and Approximate[1] Probability of a Correct Selection for k = 3 when $p_{[3]}/p_{[2]} = p_{[3]}/p_{[1]} = \theta^*$

θ^*	$N = 10$	$N = 15$	$N = 20$	$N = 25$	$N = 30$
1.02	.34361	.34593	.34789	.34963	.35120
	.343924	.346193	.348131	.349923	.351472
1.10	.38408	.39577	.40571	.41451	.42251
	.385556	.397003	.406805	.415882	.423776
1.50	.56255	.61374	.65541	.69061	.72097
	.567859	.617984	.658865	.694548	.724398
2.00	.71977	.79077	.84134	.87854	.90642
	.727509	.796463	.845490	.882401	.909437
3.00	.87858	.93681	.96644	.98196	.99022
	.888455	.942521	.969696	.983987	.991412
10.00	.99450	.99958	.99995	1.00000	1.00000
	.998342	.999881	.999991	.999999	1.000000

[1] Exact probabilities are given to six decimal places; approximate probabilities are given to five decimal places.

was computed; then [9] was entered with Λ and k, and $P_k(\Lambda)$ was read out. (See section 6.2.4 for a description of the tables in [1], [5], and [9].) Comparison of the approximate and exact probabilities in Tables B-2, B-3, and B-4 indicates that the approximation is excellent even when N is only moderately large. Of course, the approximation breaks down if θ^* is large and k is small for then (19) differs too much from $\frac{1}{2}$.

TABLE B-4

Exact[1] *and Approximate*[1] *Probability of a Correct Selection for* $k = 4$ *when* $p_{[4]}/p_{[3]} = p_{[4]}/p_{[2]} = p_{[4]}/p_{[1]} = \theta^*$

θ^*	$N = 10$	$N = 15$	$N = 20$	$N = 25$	$N = 30$
1.02	.25812	.25997	.26154	.26292	.26315
	.258492	.260383	.261959	.263341	.264592
1.10	.29066	.30022	.30839	.31567	.32232
	.292454	.302228	.310457	.317730	.324366
1.50	.44645	.49462	.53529	.57078	.60235
	.453734	.502373	.542817	.577911	.609112
2.00	.60566	.68421	.74450	.79203	.83003
	.616825	.694543	.753325	.799421	.836237
3.00	.80045	.88202	.92943	.95751	.97430
	.814697	.891625	.935605	.961441	.976808
10.00	.98975	.99878	.99986	1.00000	1.00000
	.996024	.999622	.999963	.999997	1.000000

[1] Exact probabilities are given to six decimal places; approximate probabilities are given to five decimal places.

The problem of including a correction for continuity in the approximation for small N was considered, but the authors were not successful in finding one which would give uniform improvements over the approximation finally adopted.

6.2.4. Reference tables. Since [9], Table A1 of [5], and Table 1 of [1] employ different notations, some comments about these tables might be appropriate.

In [9], $P_k(\Lambda)$ is tabulated as a function of Λ for various k. In the notation of these tables, $\Lambda = x$ and $P_k(\Lambda) = P(1, k)$ which is a function of Λ. The tabulations are for $\Lambda = 0.00\ (0.01)\ 6.09$ and $k = 2(1)10$. These tables were originally computed for the purpose of preparing Table 1 of [1].

In Table 1 of [1], Λ is tabulated as a function of k and $P_k(\Lambda)$. In the notation of these tables, $\Lambda = N^{1/2}\lambda$, and the columns to be entered are those headed $t = 1$. The tabulations are for $P_k(\Lambda) = 0.10\ (0.05)\ 0.80\ (0.02)\ 0.90\ (0.01)\ 0.99, 0.9950, 0.9990, 0.9995$ and $k = 2(1)10$.

In Table A1 of [5], Λ is tabulated as a function of k and $P_k(\Lambda)$. In the notation of these tables $\Lambda = u_\alpha(n)$, $k = n + 1$, and $P_k(\Lambda) = 1 - \alpha$. The tabulations are for $P_k(\Lambda) = 0.75, 0.90, 0.95, 0.975, 0.99$ and $k = 2(1)51$.

7. Choice of N to meet the requirement. We now shall show how to determine N^*, the smallest N which will guarantee the requirement. To guarantee the requirement in the least favorable configuration we must have

$$\int_{-\frac{A}{\sqrt{B}}}^{\infty} \int_{-\frac{A}{\sqrt{B}}}^{\infty} \cdots \int_{-\frac{A}{\sqrt{B}}}^{\infty} g(t_1, t_2, \cdots, t_{k-1})\, dt_1\, dt_2 \cdots dt_{k-1} \geq P^*. \tag{25}$$

Hence, N must be chosen large enough to make

(26) $$\frac{A}{\sqrt{B}} = \frac{a_1 + a_2/N}{\sqrt{b_1/N + b_2/N^2}} \geqq \frac{\Lambda}{\sqrt{2}}$$

where Λ is determined from [1], [5], or [9] to satisfy the equation

(27) $$P_k(\Lambda) = P^*.$$

We consider the inequality in (26) and note that the middle expression approaches infinity as N grows large. Therefore, since a_1 is positive there exists (for any a_1, a_2, b_1, b_2, and Λ) a smallest integer, which we denote by N^*, with the property that the inequality (26) holds. When N is large, the terms in (26) involving a_2 and b_2 can be ignored, and N^* is approximately the smallest integer equal to or greater than

(28) $$\frac{\Lambda^2 b_1}{2a_1^2}.$$

A tendency for the approximation (22) to underestimate the exact value of (3) (except for $k = 2$ when N is *even*) is evident from examination of Tables B-1, B-2, and B-3. Hence, one usually can expect the value of N^* obtained by the methods described above to err on the conservative side, i.e., to be somewhat larger than the exact value of N^* which is required.

8. Numerical example. Suppose that one were interested in selecting that one of the thirty-six bettable numbers on an unbalanced roulette wheel which has the largest probability associated with it. Suppose further that he specifies that if this probability is at least 10% larger than the second-largest probability, he wishes to make a correct selection with probability at least 0.90. Then we have $k = 36$, $\theta^* = 1.10$, and $P^* = 0.90$; the least favorable configuration is

(29) $$p_{[1]} = \cdots = p_{[35]} = \frac{1}{36.1} = 0.02770083$$
$$p_{[36]} = \frac{1.1}{36.1} = 0.03047091.$$

We can anticipate here that N^* will be large, and hence we need compute only

(30a) $$a_1 = 2 \arcsin\sqrt{0.03047091} - 2 \arcsin\sqrt{0.02770083} = 0.0164885$$

and

(30b) $$b_1 = 2 + 2\sqrt{1.1/35(35.1)} = 2.05985\,.$$

(Note: A comprehensive set of tables of arcsin x is found in [8].) Table A1 of [5] (with $n = 35$, $\alpha = 0.10$ in the notation of that table) yields $\Lambda = 3.5351$. (From the table in section 6.2.1 we see that (19) is approximately 0.501, and we would expect the approximation to be a good one.) Using (28) we compute N^* as the smallest integer equal to or greater than $(3.5351)^2(2.05985)/2(0.0164885)^2 = 47{,}341.7$. Thus in order to guarantee the requirement, one must take at least

47,342 observations, i.e., 47,342 spins of the wheel. (Of course, the last digits in N are not accurate, but they were retained to indicate the method.)

For illustrative purposes we have computed the following table which gives N^* for selected values of k and P^* when $\theta^* = 1.1$ is specified.

P^*	k			
	2	3	6	36
0.75	201	669	2,475	30,775
0.90	724	1,618	4,698	47,342
0.95	1,193	2,389	6,383	59,080
0.99	2,385	4,255	10,303	84,952

9. Generalizations. Thus far in this paper we have considered only Goal 1: "To select the event associated with $p_{[k]}$." However, the same approach can be used in connection with different goals, or more general goals. For example, we might consider Goal 2: "To select the event associated with $p_{[1]}$," or Goal 3: "To select the events associated with $p_{[k-t+2]}, \cdots, p_{[k]}$ *without* regard to order" for any $1 \leqq t \leqq k - 1$. Clearly, Goals 1 and 2 are special cases of Goal 3 since the selection of the $k - 1$ largest is equivalent to the selection of the one smallest. Table A1 of [5], Table 1 of [1], and [9] all provide the constants necessary to deal with Goal 2, but only the two latter tables provide the constants necessary to deal with Goal 3.

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