A SINGLE SERVER BULK INPUT QUEUE WITH RANDOM FAILURES AND TWO PHASE REPAIRS WITH DELAY

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ABSTRACT
We study a batch arrival single server queueing system, where the server (service channel) provides one by one general service to customers. It is assumed that the service channel goes through random failures from time to time. As the result of a breakdown, the service of a customer in service is suspended, the service channel waits for the repairs to start and this waiting time termed as 'delay time' is assumed to be general. Further, the repair process involves two phases of repairs with different general repair time distributions. We derive the queue size distribution as well as mean number of customers in the system at a random epoch under the steady state conditions. In addition, we discuss some particular cases and derive some known results known earlier.

KEYWORDS: batch arrivals, single server queue, random time-homogeneous failures, delay time, two-phase repairs, queue size distribution, mean number of customers in the system.

MSC: 60K25; 60J15; 90B10.

RESUMEN
Estudiamos el arribo de paquetes a un servidor de colas simple, donde el servidor (canal de servicio) provee de un servicio general a los clientes del tipo uno-a-uno. Se asume que el canal de servicio transita por fallos aleatorios de ocasión en ocasión como resultado de las interrupciones. Como resultado de las interrupciones, el servicio al cliente se suspende, el canal de servicio se pone en espera de la reparación y el tiempo de espera, llamado tiempo de atraso, se asume como general. Además, el proceso de reparación contiene dos fases de reparación con diferentes distribuciones de los tiempos de reparación. Derivamos la distribución del tamaño de la cola así como el número medio de clientes en el sistema en un periodo aleatorio bajo condiciones de fijas de estado. Adicionalmente , discutimos algunos casos particulares y derivamos algunos resultados conocidos previamente.

1. INTRODUCTION

Sudden failure or a breakdown of a system or the service channel is common in many queueing situations. As a result of a sudden breakdown, the service of a customer or a unit undergoing service has to be suspended and the customers have to wait till the server returns to the system or the system becomes operable again. Consequently, such failures have a definite effect on the system, particularly on the queue length and customers' waiting time in the system.

Among some earlier papers on service interruptions, we refer the reader to Gaver [3], Avi-Itzhak and Naor [1], Thiruvengadan [11] and Madan [6]. Li et al. [5], Sengupta [9], Takine and Sengupta [10] and Towsley and Tripathy [12] have studied some queueing systems with service interruptions and Madan [8] has studied a queueing system with time-homogeneous server breakdowns and deterministic repair time. Dorda [2] has studied a finite single-server queueing system subjected to breakdowns where customers' interarrival and service times follow the Erlang distribution defined with certain fixed parameters and the times of failures and repairs are exponentially distributed.

Most of these and other systems assume single (one by one) arrivals and they further assume that as soon as the service channel fails, the repairs start instantly. We further assume that the repairs on the service channel do not start immediately after a breakdown. Rather, the service channel has to wait for the repairs to start, which is a much more realistic assumption in many real-life queueing situations. This delay in starting repairs may occur due to the non-availability of the repairmen or the necessary apparatus needed for the repairs. This type of delay time was earlier introduced by Madan [7] in an M/M/1 queue with random breakdowns, general delay time and exponential repair time. Choudhury

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and Tadj [13] investigated an M/G/1 queuing system with Poisson arrival process, general service times and a second optional service channel subjected to general random breakdowns and general delayed repair times. Wang [14] studied an M/G/1 queue with an exponentially distributed second optional service and random breakdowns with generally distributed repair time but without delay time. However, in the present paper we attempt a wider generalization of the problem in which we assume that the system receives input of customers in batches of variable size and the service times as well as the delay times have general distributions. Recently Khalaf et al. [4] have studied such a system with general delay times and general repair times. However, the present paper is different from Khalaf et al. [4] in the scene that we consider two phases of repairs with different general repair time distributions and the most important new assumption is the time-homogeneous failures which means that the service channel can fail not only while working, it may also fail even when it is in the idle state. For such time-homogeneous failure the reader is referred to Madan [8]. Thus all the four random variables namely, the service times, the delay times, repair times of phase 1 and the repair times of phase 2 follow general probability distributions. Further we denote our model as $M^X/G(RB)/G(D)/G(R_1,R_2)/1$ queue, where $G(RB)$ denotes general service with random breakdowns, $G(D)$ denotes general delay time and $G(R_1,R_2)$ denotes the two phases of different general repair times.

2. DESCRIPTION OF THE MATHEMATICAL MODEL

We consider a batch arrival queueing system, where arrivals occur according to a compound Poisson process with the batch size random variable X. The server provides one by one service to customers on a first-come first-served basis, and the service time random variable S of a customer follows a general probability law with distribution function $S(x)$, Laplace-Stieltjes Transform $S^*(\theta)$ and finite moments $E(S^k)$, $k \geq 1$. It is further assumed that the server is subject to random breakdowns such that $\alpha\, dt$ is the first order probability that the service channel will fail during the short interval of time $(t, t+dt]$. As soon as the server breaks down, it has to wait for the repairs to start. We define this waiting time as the *delay time* and assume that the delay time random variable D follows a general probability law with distribution function $D(x)$, Laplace-Stieltjes Transform $D^*(\theta)$ and finite moments $E(D^k)$, $k \geq 1$. Next, we assume that the repair process comprises of two phases of repairs, the first phase followed by the second phase. Let the repair time random variables R_1 and R_2 of the two phases of repairs follow different general probability law with distribution functions $R_1(x)$ and $R_2(x)$, Laplace-Stieltjes transforms $R_1^*(\theta)$ and $R_2^*(\theta)$ and finite moments $E(R_1^k)$ and $E(R_2^k)$, respectively, $k \geq 1$. As soon as the second phase of repairs is complete, the server begins serving units, starting with the unit whose service was interrupted due to the breakdown. Further, it is assumed that the inter-arrival time, the service time, the delay time, the first phase repair time and the second phase repair time are all mutually independent of each other.

3. STEADY STATE EQUATIONS GOVERNING THE SYSTEM

In this section, we first set up the system state equations for the distribution of the queue size (the number of customers in the queue including the one being served, if any) at a random epoch by treating the elapsed service time, the elapsed delay time, the elapsed repair time of phase 1 and the elapsed repair time of phase 2 as supplementary variables. Then we solve these equations and derive the probability generating function (PGF) of the queue size. Assuming that the system is in the steady state, we define the following:

$\lambda =$ batch arrival rate, $X =$ batch size (a random variable), $a_k =$ Pr[$X = k$], $X(z) = \sum_{k=1}^{\infty} z^k a_k$, the PGF of *X*, and $E[X_{[k]}] = E[X(X-1)...(X-k+1)]$, the *k*-th factorial moment of *X*.

Further, it may be noted that since $S(x)$, $D(x)$, $R_1(x)$ and $R_2(x)$ are distribution functions, we have $S(0)=0$, $S(\infty)=1$, $D(0)=0$, $D(\infty)=1$, $R_1(0)=0$, and $R_1(\infty)=1$. Moreover, since

$S(x)$, $D(x)$, $R_1(x)$ and $R_2(x)$ are continuous at *x*=0, we see that $\mu(x)dx = \dfrac{d\,S(x)}{1-S(x)}$,

$\beta(x)dx = \dfrac{d\,D(x)}{1-D(x)}$, $\gamma_1(x)dx = \dfrac{dR_1(x)}{1-R_1(x)}$ and $\gamma_2(x)dx = \dfrac{d\,R_2(x)}{1-R_2(x)}$ are the first order differential functions (hazard rates) of $S(x)$, $D(x)$, $R_1(x)$ and $R_2(x)$, respectively.

At any time $'t'$, let $N_Q(t)$ be the queue size (including a customer in service, if any), $S_1{}^0(t)$ be the elapsed service time at time $'t'$, $D^0(t)$ be the elapsed delay time, $R_1^0(t)$ be the elapsed repair time of phase 1 and $R_2^0(t)$ be the elapsed repair time of phase 2. For further development of this model, we use these supplementary variables to obtain bivariate Markov process $\{N_Q(t), L(t)\}$, where $L(t)$ is defined as follows:

$$L(t) = \begin{cases} 0, \textit{ if the server is idle at time } 't', \\ S^0(t), \textit{ if the server is busy providing service at time } 't', \\ D^0(t), \textit{ if the server is waiting for repairs to start at time } 't', \\ R_1^0(t), \textit{ if the server is under repairs at time } 't'. \\ R_2^0(t), \textit{ if the server is under repairs at time } 't'. \end{cases}$$

For $x > 0$, define the following

$$Q(t) = \Pr[\,N_Q(t) = 0,\ L(t) = 0\,],$$
$$W_n(x;t)dx = \Pr[\,N_Q(t) = n,\ L(t) = S^0(t);\ x < S^0(t) \le x + dx\,],\ n \ge 1,$$
$$F_n^D(x;t)dx = \Pr[\,N_Q(t) = n,\ L(t) = D^0(t);\ x < D^0(t) \le x + dx\,],\ n \ge 0,$$
$$F_n^{R_1}(x;t)dx = \Pr[\,N_Q(t) = n,\ L(t) = R_1{}^0(t);\ x < R_1{}^0(t) \le x + dx\,],\ n \ge 0,$$
$$F_n^{R_2}(x;t)dx = \Pr[\,N_Q(t) = n,\ L(t) = R_2{}^0(t);\ x < R_2{}^0(t) \le x + dx\,],\ n \ge 0.$$

Now, to perform the analysis of the limiting behaviour of this queueing process at a random epoch with the help of Kolmogorov forward equations we assume that the following steady state probabilities exist and are independent of the initial state:

$$Q = \underset{t\to\infty}{Lim}\, Q(t),\ W_n(x)dx = \underset{t\to\infty}{Lim}\, W_n(x;t)dx,\ x > 0,\ n \ge 1,\ W_n = \int_0^\infty W_n(x)dx,$$

$$F_n^D(x)dx = \underset{t\to\infty}{Lim}\, F_n^D(x;t)dx,\ x > 0,\ n \ge 0,\ F_n^D = \int_0^\infty F_n^D(x)dx,$$

$$F_n^{R_1}(x)dx = \underset{t\to\infty}{Lim}\, F_n^{R_1}(x;t)dx,\ x > 0,\ n \ge 0,\ F_n^{R_1} = \int_0^\infty F_n^{R_1}(x)dx,$$

$$F_n^{R_2}(x)dx = \underset{t\to\infty}{Lim}\, F_n^{R_2}(x;t)dx,\ x > 0,\ n \ge 0,\ F_n^{R_2} = \int_0^\infty F_n^{R_2}(x)dx.$$

Then using the usual arguments for various transition probabilities of the system, we have the following set of Kolmogorov forward equations under the steady state conditions:

$$\frac{d}{dx}W_n(x)+[\lambda+\mu(x)+\alpha]W_n(x)=\lambda\sum_{k=1}^{n}a_k W_{n-k}(x),\ \ x>0,\ \ n\geq 1, \tag{3.1}$$

$$\frac{d}{dx}F_n^D(x)+[\lambda+\beta(x)]F_n^D(x)=\lambda\sum_{k=1}^{n}a_k\ F_{n-k}^D(x),\ \ x>0,\ \ n\geq 1, \tag{3.2}$$

$$\frac{d}{dx}F_0^D(x)+[\lambda+\beta(x)]F_0^D(x)=0\,,\ x>0, \tag{3.3}$$

$$\frac{d}{dx}F_n^{R_1}(x)+[\lambda+\gamma_1(x)]F_n^{R_1}(x)=\lambda\sum_{k=1}^{n}a_k\ F_{n-k}^{R_1}(x),\ \ x>0,\ \ n\geq 1, \tag{3.4}$$

$$\frac{d}{dx}F_0^{R_1}(x)+[\lambda+\gamma_1(x)]F_0^{R_1}(x)=0\,,\ x>0, \tag{3.5}$$

$$\frac{d}{dx}F_n^{R_2}(x)+[\lambda+\gamma_2(x)]F_n^{R_2}(x)=\lambda\sum_{k=1}^{n}a_k\ F_{n-k}^{R_2}(x),\ \ x>0,\ \ n\geq 1, \tag{3.6}$$

$$\frac{d}{dx}F_0^{R_2}(x)+[\lambda+\gamma_2(x)]F_0^{R_2}(x)=0\,,\ x>0, \tag{3.7}$$

$$(\lambda+\alpha)Q=\int_0^\infty W_1(x)\mu(x)dx+\int_0^\infty F_0^{R_2}(x)\gamma_2(x)dx\,, \tag{3.8}$$

where $W_0(x)=0$ occurring in equation (3.1).

The above set of equations is to be solved under the following boundary conditions at x = 0:

$$W_n(0)=\lambda a_n Q+\int_0^\infty W_{n+1}(x)\mu(x)dx+\int_0^\infty F_n^{R_2}(x)\gamma_2(x)dx,\ \ n\geq 1, \tag{3.9}$$

$$F_n^D(0)=\alpha W_n,\ \ n\geq 1,\ \text{where } W_n=\int_0^\infty W_n(x)dx \tag{3.10}$$

$$F_0^D(0)=\alpha Q\,, \tag{3.11}$$

$$F_n^{R_1}(0)=\int_0^\infty F_n^D(x)\beta(x)dx,\ \ n\geq 0\ , \tag{3.12}$$

$$F_n^{R_2}(0)=\int_0^\infty F_n^{R_1}(x)\gamma_1(x)dx,\ \ n\geq 0\ , \tag{3.13}$$

and the normalizing condition

$$Q+\sum_{n=1}^\infty\int_0^\infty W_n(x)dx+\sum_{n=0}^\infty\int_0^\infty F_n^D(x)dx+\sum_{n=0}^\infty\int_0^\infty F_n^{R_1}(x)dx+\sum_{n=0}^\infty\int_0^\infty F_n^{R_2}(x)dx=1, \tag{3.14}$$

where $W_n=\int_0^\infty W_n(x)dx$.

4. QUEUE SIZE DISTRIBUTION AT A RANDOM EPOCH

Next, we define the following probability generating functions:

$$W(x;z)=\sum_{n=1}^\infty z^n W_n(x),\ \ x>0,\ W(0;z)=\sum_{n=1}^\infty z^n W_n(0),\ \ |\,z\,|<1, \tag{4.1}$$

$$F^{D}(x;z)=\sum_{n=0}^{\infty}z^{n}F_{n}^{D}(x),\ x>0,\ F^{D}(0;z)=\sum_{n=0}^{\infty}z^{n}F_{n}^{D}(0),\ \ |z|<1, \tag{4.2}$$

$$F^{R_1}(x;z)=\sum_{n=0}^{\infty}z^{n}F_{n}^{R_1}(x), x>0,\ F^{R_1}(0;z)=\sum_{n=0}^{\infty}z^{n}F_{n}^{R_1}(0),\ \ |z|<1, \tag{4.3}$$

$$F^{R_2}(x;z)=\sum_{n=0}^{\infty}z^{n}F_{n}^{R_2}(x), x>0,\ F^{R_2}(0;z)=\sum_{n=0}^{\infty}z^{n}F_{n}^{R_2}(0),\ \ |z|<1, \tag{4.4}$$

$$W(z)=\sum_{n=1}^{\infty}z^{n}W_{n}\,, \tag{4.5}$$

$$F^{D}(z)=\int_{0}^{\infty}F^{D}(x,z)\,dx\,, \tag{4.6}$$

$$F^{R_1}(z)=\int_{0}^{\infty}F^{R_1}(x,z)\,dx, \tag{4.7}$$

$$F^{R_2}(z)=\int_{0}^{\infty}F^{R_2}(x,z)\,dx. \tag{4.8}$$

We multiply equations (3.1) - (3.7) and (3.9) – (3.13) by suitable powers of z, use equations (4.1) through (4.4) and (3.8) and simplify. Thus, we obtain

$$W(x,z)+[\lambda-\lambda X(z)+\mu(x)+\alpha]W(x,z)=0, \tag{4.9}$$

$$F^{D}(x)+[\lambda-\lambda X(z)+\beta(x)]F^{D}(x,z)=0, \tag{4.10}$$

$$F^{R_1}(x)+[\lambda-\lambda X(z)+\gamma_1(x)]F^{R_1}(x,z)=0, \tag{4.11}$$

$$F^{R_2}(x)+[\lambda-\lambda X(z)+\gamma_2(x)]F^{R_2}(x,z)=0, \tag{4.12}$$

$$zW(0,z)=\left(\lambda\left(X(z)-1\right)-\alpha\right)zQ+\int_{0}^{\infty}W(x,z)\mu(x)dx+\ z\int_{0}^{\infty}F^{R_2}(x,z)\gamma_2(x)dx, \tag{4.13}$$

$$F^{D}(0,z)=\alpha Q+\alpha W(z), \tag{4.12}$$

$$F^{R_1}(0,z)=\int_{0}^{\infty}F^{D}(x,z)\beta(x)dx,\ \ n\geq 0\ , \tag{4.15}$$

$$F^{R_2}(0,z)=\int_{0}^{\infty}F^{R_1}(x)\gamma_1(x)dx,\ \ n\geq 0. \tag{4.16}$$

Next, we integrate equations (4.9) – (4.12) between the limits 0 and *x* and obtain

$$W(x,z)=W(0,z)\exp\left[-\left(\lambda-\lambda X(z)+\alpha\right)x-\int_{0}^{x}\mu(t)dt\right], \tag{4.17}$$

$$F^{D}(x,z)=F^{D}(0,z)\exp\left[-\left(\lambda-\lambda X(z)\right)x-\int_{0}^{x}\beta(t)dt\right], \tag{4.18}$$

$$F^{R_1}(x,z)=F^{R_1}(0,z)\exp\left[-\left(\lambda-\lambda X(z)\right)x-\int_{0}^{x}\gamma_1(t)dt\right], \tag{4.19}$$

$$F^{R_2}(x,z) = F^{R_2}(0,z)\exp\left[-(\lambda - \lambda X(z))x - \int_0^x \gamma_2(t)dt\right]. \tag{4.20}$$

We again integrate equations (4.17) to (4.20) with respect to *x* by parts and get

$$W(z) = W(0,z)\left(\frac{1 - S^*(\lambda - \lambda X(z) + \alpha)}{\lambda - \lambda X(z) + \alpha}\right), \tag{4.21}$$

where $S^*(\lambda - \lambda X(z) + \alpha) = \int_0^\infty e^{-(\lambda - \lambda X(z) + \alpha)x} dS(x)$ is the Laplace–Steiltjes is transform of the service time,

$$F^D(z) = F^D(0,z)\left(\frac{1 - D^*(\lambda - \lambda X(z))}{\lambda - \lambda X(z)}\right), \tag{4.22}$$

where $D^*(\lambda - \lambda X(z)) = \int_0^\infty e^{-(\lambda - \lambda X(z))x} d\, D(x)$ is the Laplace –Steiltjes transform of the delay times,

$$F^{R_1}(z) = F^D(0,z)\left(\frac{1 - R_1^*(\lambda - \lambda X(z))}{\lambda - \lambda X(z)}\right), \tag{4.23}$$

where $R_1^*(\lambda - \lambda X(z)) = \int_0^\infty e^{-(\lambda - \lambda X(z))x} dR_1(x)$ is the Laplace –Steiltjes transform of the phase 1 repair time, and

$$F^{R_2}(z) = F^{R_2}(0,z)\left(\frac{1 - R_2^*(\lambda - \lambda X(z))}{\lambda - \lambda X(z)}\right), \tag{4.24}$$

where $R_2^*(\lambda - \lambda X(z)) = \int_0^\infty e^{-(\lambda - \lambda X(z))x} d\, R_2(x)$ is the Laplace –Steiltjes transform of the phase 2 repair time.

Now, we shall determine the integrals $\int_0^\infty W(x,z)\,\mu(x)dx$, $\int_0^\infty F^D(x,z)\,\beta(x)dx$, $\int_0^\infty F^{R_1}(x,z)\gamma_1(x)dx$, and $\int_0^\infty F^{R_2}(x,z)\gamma_2(x)dx$ appearing in the right side of equations (4.13), (4.15) and (4.16). For this purpose, we multiply equations (4.17) to (4.20) by $\mu(x)$, $\beta(x)$, $\gamma_1(x)$ and $\gamma_2(x)$ respectively, integrate with respect to *x* and obtain

$$\int_0^\infty W(x,z)\,\mu(x)dx = W(0,z)\,S^*(\lambda - \lambda X(z) + \alpha), \tag{4.25}$$

$$\int_0^\infty F^D(x,z)\,\beta(x)dx = F^D(0,z)\,D^*(\lambda - \lambda X(z)), \tag{4.26}$$

$$\int_0^\infty F^{R_1}(x,z)\gamma_1(x)dx = F^{R}\ (0,z)R_1^{\ *}\left(\lambda-\lambda X(z)\right), \tag{4.27}$$

$$\int_0^\infty F^{R_2}(x,z)\gamma_2(x)dx = F^{R_2}(0,z)R_2^{\ *}\left(\lambda-\lambda X(z)\right). \tag{4.28}$$

Now using equations (4.25) to (4.28) in equations (4.13) to (4.16), we get on simplifying

$$\left(z-S^*\left(\lambda-\lambda X(z)+\alpha\right)\right)W(0,z)=\left(\lambda X(z)-\lambda-\alpha\right)zQ+zF^{R_2}(0,z)R_2^*\left(\lambda-\lambda X(z)\right), \tag{4.29}$$

$$F^{R_1}(0,z)=F^{D}(0,z)D^*\left(\lambda-\lambda X(z)\right), \tag{4.30}$$

$$F^{R_2}(0,z)=F^{R_1}(0,z)\ R_1^*\left(\lambda-\lambda X(z)\right). \tag{4.31}$$

Next we use $F^{D}(0,z)=\alpha Q+\alpha W(z)$ from (4.14) into (4.30) and obtain

$$F^{R_1}(0,z)=\left(\alpha Q+\alpha W(z)\right)D^*\left(\lambda-\lambda X(z)\right). \tag{4.32}$$

Then using $F^{R_1}(0,z)=\left(\alpha Q+\alpha W(z)\right)D^*\left(\lambda-\lambda X(z)\right)$ from (4.32) into (4.31) we get

$$F^{R_2}(0,z)=\left(\alpha Q+\alpha W(z)\right)D^*\left(\lambda-\lambda X(z)\right)\ R_1^*\left(\lambda-\lambda X(z)\right). \tag{4.33}$$

Next substituting the value of $F^{R_2}(0,z)$ from (4.33) into equation (4.29) and obtain on simplifying

$$W(0,z)=\frac{\left(\lambda X(z)-\lambda-\alpha\right)zQ+\left(\alpha Q+\alpha W(z)\right)zD^*\left(\lambda-\lambda X(z)\right)R_1^*\left(\lambda-\lambda X(z)\right)R_2^*\left(\lambda-\lambda X(z)\right)}{z-S^*\left(\lambda-\lambda X(z)+\alpha\right)}. \tag{4.34}$$

Next, we use equation (4.34) in (4.21) and simplify. Thus we obtain

$$W(z)=\frac{\left[\left(\lambda X(Z)-\lambda-\alpha\right)zQ+\alpha QzD^*\left(\lambda-\lambda X(z)\right)R_1^*\left(\lambda-\lambda X(z)\right)R_2^*\left(\lambda-\lambda X(z)\right)\right]\left[\dfrac{1-S^*\left(\lambda-\lambda X(z)+\alpha\right)}{\lambda-\lambda X(z)+\alpha}\right]}{\left[z-S^*\left(\lambda-\lambda X(z)+\alpha\right)\right]-\left(\dfrac{1-S^*\left(\lambda-\lambda X(z)+\alpha\right)}{\lambda-\lambda X(z)+\alpha}\right)\alpha zD^*\left(\lambda-\lambda X(z)\right)R_1^*\left(\lambda-\lambda X(z)\right)R_2^*\left(\lambda-\lambda X(z)\right)} \tag{4.35}$$

We note that (4.35) gives the PGF of the probability that the server is busy providing service.
Then we substitute $F^{D}(0,z)=\alpha Q+\alpha W(z)$ from (4.14) into (4.22) and get

$$F^{D}(z)=\left[\alpha Q+\alpha W(z)\right]\left(\frac{1-D^*\left(\lambda-\lambda X(z)\right)}{\lambda-\lambda X(z)}\right), \tag{4.36}$$

where W (z) is given by (4.35). Note that (4.36) gives the PGF for the probability that the server is down and waiting for repairs to start.
Next we use $F^{R_1}(0,z)=\left(\alpha Q+\alpha W(z)\right)D^*\left(\lambda-\lambda X(z)\right)$ from equation (4.32) into equation (4.23), we get

$$F^{R_1}(z)=\left(\alpha Q+\alpha W(z)\right)D^*\left(\lambda-\lambda X(z)\right)\left(\frac{1-R_1^{\ *}\left(\lambda-\lambda X(z)\right)}{\lambda-\lambda X(z)}\right), \tag{4.37}$$

where W (z) is given by (4.35). Note that this gives the PGF for the probability that the server is in phase 1 repairs.
Then we use $F^{R_2}(0,z) = (\alpha Q + \alpha W(z)) D^*(\lambda - \lambda X(z))\, R_1^*(\lambda - \lambda X(z))$ from equation (4.33) into equation (4.24) we get

$$F^{R_2}(z) = (\alpha Q + \alpha W(z)) D^*(\lambda - \lambda X(z))\, R_1^*(\lambda - \lambda X(z)) \left(\frac{1 - R_2^*(\lambda - \lambda X(z))}{\lambda - \lambda X(z)} \right), \qquad (4.38)$$

where W (z) is given by (4.35). Note that this is the PGF for the probability that server is in phase 2 repairs.
In order to determine the only unknown quantity Q, we would use the normalizing condition (3.14), which is equivalent to $Q + W(1) + F^D(1) + F^{R_1}(1) + F^{R_2}(1) = 1$.
Now from (4.35) we obtain

$$W(1) = \lim_{z \to 1} W(z)$$

$$= \frac{\left(\frac{1 - S^*(\alpha)}{\alpha} \right) [\lambda E(X) + \alpha\lambda E(X)(E(D) + E(R_1) + E(R_2))] Q}{1 - \left(\frac{1 - S^*(\alpha)}{\alpha} \right) [\alpha + \lambda E(X) + \alpha\lambda E(X)(E(D) + E(R_1) + E(R_2))]}, \qquad (4.39)$$

Using (4.39) into (4.36) to (4.38) we obtain

$$F^D(1) = \lim_{z \to 1} F^D(z)$$

$$= \frac{\alpha S^*(\alpha) E(D) Q}{1 - \left(\frac{1 - S^*(\alpha)}{\alpha} \right) [\alpha + \lambda E(X) + \alpha\lambda E(X)(E(D) + E(R_1) + E(R_2))]}, \qquad (4.40)$$

$$F^{R_1}(1) = \lim_{z \to 1} F^{R_1}(z)$$

$$= \frac{\alpha S^*(\alpha) E(R_1) Q}{1 - \left(\frac{1 - S^*(\alpha)}{\alpha} \right) [\alpha + \lambda E(X) + \alpha\lambda E(X)(E(D) + E(R_1) + E(R_2))]}, \qquad (4.41)$$

$$F^{R_2}(1) = \lim_{z \to 1} F^{R_2}(z)$$

$$= \frac{\alpha S^*(\alpha) E(R_2) Q}{1 - \left(\frac{1 - S^*(\alpha)}{\alpha} \right) [\alpha + \lambda E(X) + \alpha\lambda E(X)(E(D) + E(R_1) + E(R_2))]}. \qquad (4.42)$$

Next, using (4.39) to (4.42) into the normalizing condition
$Q + W(1) + F^D(1) + F^{R_1}(1) + F^{R_2}(1) = 1$, and simplifying we obtain

$$Q = \frac{1 - \left(\frac{1 - S^*(\alpha)}{\alpha} \right) [\alpha + \lambda E(X) + \alpha\lambda E(X)(E(D) + E(R_1) + E(R_2))]}{S^*(\alpha)[1 + \alpha(E(D) + E(R_1) + E(R_2))]}, \qquad (4.43)$$

where

$$\left(\frac{1-S^*(\alpha)}{\alpha}\right)\left[\alpha+\lambda E(X)+\alpha\lambda E(X)\big(E(D)+E(R_1)+E(R_2)\big)\right]<1. \tag{4.44}$$

Note that (4.44 a) is the stability condition under which the steady state solution exists.
Next, if we substitute for Q from (4.44) into (4.39) – (4.42), we obtain the following probabilities.
The probability that the service channel is in the operating state,

$$W(1)=\left(\frac{1-S^*(\alpha)}{\alpha S^*(\alpha)}\right)\lambda E(X)=\rho\cdot \tag{4.45}$$

Note that (4.45) also gives the utilization factor of the system.
The probability that the service channel is in the failed state, waiting for repairs to start,

$$F^D(1)=\frac{\alpha E(D)}{\left[1+\alpha\big(E(D)+E(R_1)+E(R_2)\big)\right]}. \tag{4.46}$$

The probability that the service channel is in the failed state and is under phase 1 repairs,

$$F^{R_1}(1)=\frac{\alpha E(R_1)}{\left[1+\alpha\big(E(D)+E(R_1)+E(R_2)\big)\right]}, \tag{4.47}$$

The probability that the service channel is in the failed state and is under phases 2 repairs

$$F^{R_2}(1)=\frac{\alpha E(R_2)}{\left[1+\alpha\big(E(D)+E(R_1)+E(R_2)\big)\right]}. \tag{4.48}$$

Now, we define the probability generating function of the queue size distribution at a random epoch irrespective of the state of the system as follows:

$$P(z)=Q+W(z)+F^D(z)+F^{R_1}(z)+F^{R_2}(z), \tag{4.49}$$

which can be obtained by adding equations (4.35) to (4.38) and (4.43) and simplifying.

5. THE AVERAGE SYSTEM SIZE

Let L denote the mean system size at a random epoch. Then using the PGF $P(z)$ in equation (4.49) and after somewhat heavy algebra and simplification, we obtain

$$L=\frac{d}{dz}P(z)\Big|_{z=1}=\rho+\frac{\lambda^2\big(E(X)\big)^2}{(1-\rho)}\left\{\frac{-\alpha E\big(S\,e^{-s\alpha}\big)+\big(1-S^*(\alpha)\big)}{\alpha^2}\right\}$$
$$+\frac{\alpha\lambda^2\big(E(X)\big)^2}{2(1-\rho)}\left\{E\big(D^2\big)+E\big(R_1^{\,2}\big)+E\big(R_2^2\big)+2E(D)\big(E(R_1)+E(R_2)\big)\right\}+\frac{\rho E(X_R)}{(1-\rho)}, \tag{5.1}$$

where $E(X_R)=\dfrac{E\big(X(X-1)\big)}{2E(X)}$ is the mean residual batch size.

6. SOME PARTICULAR CASES

Case 1: One by one arrivals, exponential service times, exponential delay times, exponential repair times of phase 1 and exponential repair times of phase 2.

$\left(M/M(RB)/M(D)/M(R_1,R_2)/1\right)$

The results for this case can be obtained from the main results obtained in sections 4 and 5 above by substituting $E(X)=1, X(z)=z$, $E(X_R)=\dfrac{E\left(X(X-1)\right)}{2E(X)}=0$, $E(D)=\dfrac{1}{\beta}$, $E(R_1)=\dfrac{1}{\gamma_1}$, $E(R_2)=\dfrac{1}{\gamma_2}$, $E(D^2)=\dfrac{2}{\beta^2}$ and $E(R_1^2)=\dfrac{2}{\gamma_1^2}$, $E(R_2^2)=\dfrac{2}{\gamma_2^2}$, $S^*(\alpha)=\dfrac{\mu}{\mu+\alpha}$, $D^*(\lambda-\lambda z)=\dfrac{\beta}{\beta+\lambda-\lambda z}$, $R_1^*(\lambda-\lambda z)=\dfrac{\gamma_1}{\gamma_1+\lambda-\lambda z}$, and $R_2^*(\lambda-\lambda z)=\dfrac{\gamma_2}{\gamma_2+\lambda-\lambda z}$.

Thus we obtain

$$W(z)=\frac{\left[(\lambda z-\lambda-\alpha)zQ+\alpha QzD^*(\lambda-\lambda z)R_1^*(\lambda-\lambda z)R_2^*(\lambda-\lambda z)\right]\left[\dfrac{1}{\mu+\lambda-\lambda z+\alpha}\right]}{\left[z-S^*(\lambda-\lambda z+\alpha)\right]-\left(\dfrac{1}{\mu+\lambda-\lambda X(z)+\alpha}\right)\alpha zD^*(\lambda-\lambda z)R_1^*(\lambda-\lambda z)R_2^*(\lambda-\lambda z)}, \quad (6.1)$$

$$F^D(z)=\left[\alpha Q+\alpha W(z)\right]\left(\frac{1}{\beta+\lambda-\lambda z}\right), \quad (6.2)$$

$$F^{R_1}(z)=\left(\alpha Q+\alpha W(z)\right)D^*(\lambda-\lambda z)\left(\frac{1}{\gamma_1+\lambda-\lambda z}\right), \quad (6.3)$$

(6.4)

where $W(z)$ is given by (6.1).

$$Q=\frac{1-\left(\dfrac{1}{\mu+\alpha}\right)\left\{\alpha+\lambda\left[1+\alpha\left(\dfrac{1}{\beta}+\dfrac{1}{\gamma_1}+\dfrac{1}{\gamma_2}\right)\right]\right\}}{\left(\dfrac{\mu}{\mu+\alpha}\right)\left[1+\alpha\left(\dfrac{1}{\beta}+\dfrac{1}{\gamma_1}+\dfrac{1}{\gamma_2}\right)\right]}, \quad (6.5)$$

where the stability condition is

$$\left(\frac{1}{\mu+\alpha}\right)\left\{\alpha+\lambda\left[1+\alpha\left(\frac{1}{\beta}+\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\right]\right\}<1. \quad (6.6)$$

$$W(1)=\frac{\lambda}{\mu}=\rho, \quad (6.7)$$

$$F^D(1)=\frac{\left(\dfrac{\alpha}{\beta}\right)}{\left[1+\alpha\left(\dfrac{1}{\beta}+\dfrac{1}{\gamma_1}+\dfrac{1}{\gamma_2}\right)\right]}, \quad (6.8)$$

$$F^{R_1}(1) = \frac{\left(\frac{\alpha}{\gamma_1}\right)}{\left[1+\alpha\left(\frac{1}{\beta}+\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\right]}, \tag{6.9}$$

$$F^{R_2}(1) = \frac{\left(\frac{\alpha}{\gamma_2}\right)}{\left[1+\alpha\left(\frac{1}{\beta}+\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\right]}, \tag{6.10}$$

$$L = \rho + \frac{\lambda^2}{(1-\rho)(\mu+\alpha)^2} + \frac{\alpha\lambda^2}{(1-\rho)}\left\{\frac{1}{\beta^2}+\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}+\frac{1}{\beta}\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\right\}, \tag{6.11}$$

where $\rho = \dfrac{\lambda}{\mu}$.

Further, if we let $\kappa = \alpha\left(\dfrac{1}{\beta}+\dfrac{1}{\gamma_1}+\dfrac{1}{\gamma_2}\right)$, the above results from (6.5) to (6.10) are simplified as follows:

$$Q = \frac{(1-\rho)(\mu+\alpha)}{\mu(1+\kappa)} = \frac{\mu-\lambda(1+\kappa)}{\mu(1+\kappa)} = \frac{1}{1+\kappa} - \frac{\lambda}{\mu}, \tag{6.12}$$

Stability condition is

$$\frac{\alpha+\lambda(1+\kappa)}{\mu+\alpha} < 1. \tag{6.13}$$

$$W(1) = \frac{\lambda}{\mu} = \rho \ , \tag{6.14}$$

$$F^{D}(1) = \frac{\alpha}{\beta(1+\kappa)} \ , \tag{6.15}$$

$$F^{R_1}(1) = \frac{\alpha}{\gamma_1(1+\kappa)}, \tag{6.16}$$

$$F^{R_2}(1) = \frac{\alpha}{\gamma_2(1+\kappa)} \ . \tag{6.17}$$

Case 2: No Second Phase Repairs $\left(M^X / G(RB)/G(D)/G(R_1)/1\right)$

The results corresponding to this case can be obtained by substituting $R_2^*\left(\lambda-\lambda X(z)\right)=1$ in the main results of section 4 and by putting, $E(R_2^2)=0$ in (5.1).

Case 3: No Delay in Repairs to Start $\left(M^X / G(RB)/G(R_1,R_2)/1\right)$

The results corresponding to this case can be obtained by substituting $D^*(\lambda - \lambda X(z)) = 1$ in the main results of section 4 and by putting, $E(D^2) = 0$ in (5.1).

Case 4: No Breakdowns $(M^X/G/1)$

In this case, we let $\alpha = 0$, $S^*(\alpha) = 1$, $\lim_{\alpha \to 0}\left(\frac{1 - S^*(\alpha)}{\alpha}\right) = E(S)$ and

$\lim_{\alpha \to 0}\left\{\frac{-\alpha E(S e^{-s\alpha}) + (1 - S^*(\alpha))}{\alpha^2}\right\} = \frac{E(S^2)}{2}$ in the main results of sections 4 and 5. Thus we obtain

$F^D(z) = 0$, $F^{R_1}(z) = 0$, $F^{R_2}(z) = 0$,

$$W(z) = \frac{(1-\rho_0)(1-z)S^*(\lambda - \lambda X(z))}{[S^*(\lambda - \lambda X(z)) - z]}, \tag{6.18}$$

$$\rho = \lim_{\alpha \to 0}\left(\frac{1 - S^*(\alpha)}{\alpha S^*(\alpha)}\right)\lambda E(X) = \lambda E(X)E(S),$$

$$L = \rho + \frac{\lambda E(X)\rho_0 E(S_R)}{(1-\rho)} + \frac{\rho E(X_R)}{(1-\rho)}, \tag{6.19}$$

where $E(S_R) = \frac{E(S^2)}{2E(S)}$ is the mean residual service time.

The results in (6.18), (6.19) agree with the results of Gaver [6] for the ordinary $M^X/G/1$ queuing system without server breakdowns.

7. NUMERICAL EXAMPLES

λ	μ	α	γ_2	$W(1) = \rho$
1	4	0	1	0.25

β	γ_1	κ	S.C.	L_q	L	Q	$F^D(1)$	$F^{R_1}(1)$	$F^{R_2}(1)$
1	**2**	0.00	0.25	0.08	0.33	0.75	0.00	0.00	0.00
5	**10**	0.00	0.25	0.08	0.33	0.75	0.00	0.00	0.00
6	**1**	0.00	0.25	0.08	0.33	0.75	0.00	0.00	0.00
0.2	**2**	0.00	0.25	0.08	0.33	0.75	0.00	0.00	0.00
100	**7**	0.00	0.25	0.08	0.33	0.75	0.00	0.00	0.00
10	**25**	0.00	0.25	0.08	0.33	0.75	0.00	0.00	0.00

Table 1: When the failure rate $\alpha = 0$, the above table gives the results of an ordinary M/M/1 queue without breakdowns. Obviously the delay parameter β and the completion of the two phases of repairs parameters γ_1 and γ_2 have no effect.

We base the following numerical examples on the results of particular case 1 in order to check the validity of our results and to see the effect of various parameters involved in our model (namely the failure rate α, the delay parameter β and the completion of the two phases of repairs parameters γ_1 and γ_2) on the utilization factor ρ and on probabilities of various states of the system, namely probabilities of the idle state, the working state and the failure state waiting for repairs to start, under repairs of phase 1 and under repairs of phase 2. We assume the fixed values of the arrival rate $\lambda = 1$

and the service rate $\mu = 4$ and arbitrarily choose values of the other various parameters such that the stability condition (6.6) of the particular case 1 is not violated. We obtain the following numerical values which depict results as expected.

λ	μ	α	β	γ_1	$W(1)=\rho$
1	4	0.4	0.2	0.7	0.25

γ_2	κ	s.c.	L_q	L	Q	$F^D(1)$	$F^{R_1}(1)$	$F^{R_2}(1)$
1	2.97	0.99	21.50	21.75	0.00	0.50	0.14	0.10
1.1	2.94	0.99	21.17	21.42	0.00	0.51	0.15	0.09
2	2.77	0.95	19.77	20.02	0.02	0.53	0.15	0.05
20	2.59	0.91	18.43	18.68	0.03	0.56	0.16	0.01
50	2.58	0.90	18.35	18.60	0.03	0.56	0.16	0.00
200	2.57	0.90	18.31	18.56	0.03	0.56	0.16	0.00

Table 2: For fixed values of α , β and γ_1, we note that as γ_2 increases, the probability of the idle state Q increases and the average queue length L decreases.

λ	μ	α	γ_1	γ_2	$W(1)=\rho$
1	4	1	3	2	0.25

β	κ	s.c.	L_q	L	Q	$F^D(1)$	$F^{R_1}(1)$	$F^{R_2}(1)$
0.5	2.83	0.97	8.09	8.34	0.01	0.52	0.09	0.13
0.6	2.50	0.90	6.09	6.34	0.04	0.48	0.10	0.14
1	1.83	0.77	2.98	3.23	0.10	0.35	0.12	0.18
2	1.33	0.67	1.42	1.67	0.18	0.21	0.14	0.21
50	0.85	0.57	0.56	0.81	0.29	0.01	0.18	0.27
200	0.84	0.57	0.54	0.79	0.29	0.00	0.18	0.27

Table 3: For fixed values of α , γ_1 and γ_2, we note that as β increases, the probability of the idle state Q increases and the average queue length decreases.

λ	μ	β	γ_1	γ_2	$W(1)=\rho$
1	4	5	5	5	0.25

α	κ	s.c.	L_q	L	Q	$F^D(1)$	$F^{R_1}(1)$	$F^{R_2}(1)$
0.1	0.06	0.28	0.11	0.36	0.69	0.02	0.02	0.02
0.5	0.30	0.40	0.20	0.45	0.52	0.08	0.08	0.08
1	0.60	0.52	0.32	0.57	0.38	0.13	0.13	0.13
2	1.20	0.70	0.57	0.82	0.20	0.18	0.18	0.18
3.5	2.10	0.88	0.96	1.21	0.07	0.23	0.23	0.23
4.9	2.94	0.99	1.32	1.57	0.00	0.25	0.25	0.25

Table 4: For fixed values of β , γ_1 and γ_2, we note that as α increases, the probability of the idle state Q decreases and the average queue length L increases.

λ	α	β	γ_1	γ_2
1	1	2	3	3

μ	κ	S.C.	L_q	L	Q	$F^D(1)$	$F^{R_1}(1)$	$F^{R_2}(1)$	$W(1)=\rho$
2.3	1.17	0.96	1.59	2.02	0.03	0.23	0.15	0.15	0.43
3	1.17	0.79	1.30	1.64	0.13	0.23	0.15	0.15	0.33
4	1.17	0.63	1.13	1.38	0.21	0.23	0.15	0.15	0.25
6	1.17	0.45	0.99	1.16	0.29	0.23	0.15	0.15	0.17
10	1.17	0.29	0.90	1.00	0.36	0.23	0.15	0.15	0.10
15	1.17	0.20	0.87	0.93	0.39	0.23	0.15	0.15	0.07

Table 5: For fixed values of α , β , γ_1 and γ_2, we note that as μ increases, the probability of the idle state Q increases, the utilization factor ρ decreases and the average queue length L decreases.

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REFERENCES

[1] AVI-ITZHAK, B. and NAOR, P.(1963): Some Queueing Problems with the Service Station Subject to Breakdowns. **Operations Research**, 11, 303–320.

[2] DORDA, M. (2013): On Two Modifications of E2/E2/1/*m* Queueing System with a Server Subject to Breakdowns. **Applied Mathematical Sciences**, 7-11, 539–550.

[3] GAVER, D.P. (1962): A Waiting Line with Interrupted Service Including Priorities. **Journal of Royal Statistical Society**, *Ser.- B*, 24, 73–90.

[4] KHALAF, R. F., MADAN, K. C. and LUKAS, A. C. (2011): An $M^{[X]}/G/1$ Queue with Bernoulli Schedule General Vacation Times, Random Breakdowns, General Delay Times and General Repair Times. **Applied Mathematical Sciences**, 5, 35–51.

[5] LI, W., SHI D. and CHAO, X. (1997): Reliability Analysis of M/G/1 Queueing System with Server Breakdowns and Vacations. **Journal of Applied Probability**, 34, 546–555.

[6] Madan, K.C., A Queueing System with Random Failures and Delayed Repairs, *Journal of Indian Statistical Association*, 32 (1994), 39–48.
[7] Madan, K.C., An M/G/1 Queue with Time-Homogeneous Breakdowns and Deterministic Repair Times, *Soochow Journal of Mathematics*, 29 (2003), 103–110.
[8] Sengupta, B., A Queue with Service Interruptions in an Alternating Random Environment, *Operations Research,* 38 (1990), 308–318.
[9] Takine, T. and Sengupta, B., A Single Server Queue with Service Interruptions, *Queueing Systems*, 26 (1997), 285–300.
[10] Thiruvengadan, K., Queueing with Breakdowns, *Operations Research,* 11 (1963), 62–71.
[11] Towsley, D. and Tripathy, S. K., A Single Server Priority Queue with Server Failures and Queue Flushing, *Operations Research Letters*, 10 (1991), 353–362.
[13] Choudhury, G. and Tadj, L., An M/G/1 Queue with Two Phases of Service Subject to the Server Breakdown and Delayed Repair, *Applied Mathematical Modeling*, 33 (2009), 2699–2709.
[14] Wang, J., An M/G/1 Queue with Second Optional Service and Server Breakdowns, Computers and Mathematics with Applications, 47 (2004), 1713–1723.