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# A SINGULAR INTEGRAL

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ABSTRACT. In this paper we show that if $K(x) = \Omega(x)/|x|^n$ is a Calderón-Zygmund kernel, where $\Omega \in L^q(S^{n-1})$ for some $1 < q \leq \infty$, and $b$ is a radial bounded function, then $b(x)K(x)$ is the kernel of a convolution operator which is bounded on $L^p(R^n)$ for $1 < p < \infty$ and $n \geq 2$.

This paper is related to boundedness properties of a variation of Calderón-Zygmund operators.

Let $H(x) = b(|x|)\Omega(x)/|x|^n$ be a kernel, where $\int_{S^{n-1}} \Omega(x)\, d\sigma(x') = 0$, $\Omega(\lambda x) = \Omega(x)$ for $x \in R^n$, $\lambda > 0$, and $b$ is radial. Define

$$f(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} H(y) f(x-y)\, dy = \text{P.V. } H * f(x). \tag{1}$$

If $b \equiv 1$ and $\Omega \in L^q(S^{n-1})$ for some $q > 1$, then $T$ is a Calderón-Zygmund operator which is bounded on $L^p(R^n)$ for $1 < p < \infty$ and $n \geq 1$; see [**4**].

In [**1**] R. Fefferman showed that if $\Omega$ satisfies a Lipschitz condition and $b$ is bounded, then $T$ is bounded on $L^p(R^n)$, $1 < p < \infty$, $n \geq 2$. This is not true when $n = 1$; for instance, if $H(x) = \sin(|x|)/x$, then its Fourier transform $\hat{H}(\xi)$ is unbounded, indicating that $T$ cannot be bounded on $L^2(R^1)$. It turns out that the smoothness condition on $\Omega$ can be relaxed a great deal.

THEOREM. *If $\Omega \in L^q(S^{n-1})$ for some $1 < q \leq \infty$, and $b(|x|) \in L^\infty$, then $T$ is bounded on $L^p(R^n)$ for $1 < p < \infty$ and $n \geq 2$.*

PROOF. First we show that the Fourier transform of $H$ is bounded; thereby $T$ is bounded on $L^2(R^n)$. Using polar coordinates and letting $x = \rho x'$, $\rho = |x|$, we get

$$\begin{aligned} \hat{H}(\xi) &= \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{b(|x|)}{|x|^n} \Omega(x) e^{-ix\cdot\xi}\, dx \\ &= \lim_{\varepsilon \to 0} \int_{\varepsilon<\rho<1/\varepsilon} \frac{b(\rho)}{\rho}\, d\rho \left( \int_{S^{n-1}} \Omega(x') e^{i\rho x'\cdot\xi}\, d\sigma(x') \right). \end{aligned} \tag{2}$$

By a change of variable $r = \rho|\xi|$, (2) becomes

$$\begin{aligned} \hat{H}(\xi) &= \lim_{\varepsilon \to 0} \int_{\varepsilon|\xi|}^{|\xi|/\varepsilon} \frac{b(r/|\xi|)}{r}\, dr \left( \int_{S^{n-1}} \Omega(x') e^{-irx'\cdot\xi'}\, d\sigma(x') \right) \\ &= \lim_{\varepsilon \to 0} \int_{\varepsilon|\xi|}^{|\xi|/\varepsilon} \frac{b(r/|\xi|)}{r} g(r, \xi')\, dr \end{aligned} \tag{3}$$

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where

$$g(r,\xi') = \int_{S^{n-1}} \Omega(x') e^{-irx'\cdot\xi'}\, d\sigma(x'), \qquad |\xi'| = 1.$$

Now if

$$\text{(4)} \qquad \operatorname*{Sup}_{|\xi'|=1} \|g(\cdot,\xi')\|_{L^p(1,\infty)} \le A$$

for some $1 < p < \infty$ and a constant $A$, then we claim that $\hat{H}(\xi)$ is bounded. To prove the claim, first we observe that

$$\begin{aligned} |g(r,\xi')| &= \left| \int_{S^{n-1}} \Omega(x') e^{-irx'\cdot\xi'}\, d\sigma(x') \right| \\ &= \left| \int_{S^{n-1}} \Omega(x') (e^{-irx'\cdot\xi'} - 1)\, d\sigma(x') \right| \\ &\le 2r\|\Omega\|_{L^1(S^{n-1})} = cr \end{aligned}$$

by the mean value theorem. Using this and Hölder's inequality, we see that

$$\begin{aligned} \int_0^\infty \left| \frac{b(r/|\xi|)}{r} g(r,\xi') \right| dr &\le \|b\|_\infty \int_0^\infty \frac{|g(r,\xi')|}{r}\, dr \\ &\le \|b\|_\infty \left( \int_0^1 \frac{cr}{r}\, dr + \|g(\cdot,\xi')\|_{L^p(1,\infty)} \cdot \left\| \frac{1}{r} \right\|_{L^{p'}(1,\infty)} \right) \\ &\le \|b\|_\infty (c + A\|1/r\|_{L^{p'}(1,\infty)}) \le c, \end{aligned}$$

where $1/p' = 1 - 1/p$. Hence, by Lebesgue's dominated convergence theorem, the limit in (3) exists and $|\hat{H}(\xi)| \le c$.

Now let us see when condition (4) may hold. Let $\rho$ be a rotation such that $\rho(\xi') = (1, 0, 0, \dots, 0)$. In the integral of $g(r,\xi')$ make the change of variable $x' \to \rho^{-1}x'$, and let $\Omega^\rho(x') = \Omega(\rho^{-1}x')$. Then

$$\begin{aligned} g(r,\xi') &= \int_{S^{n-1}} \Omega(x') e^{-irx'\cdot\xi'}\, d\sigma(x') = \int_{S^{n-1}} \Omega^\rho(x') e^{-ir(\rho^{-1}x')\cdot\xi'}\, d\sigma(x') \\ &= \int_{S^{n-1}} \Omega^\rho(x') e^{-irx'\cdot\rho\xi'} = \int_{S^{n-1}} \Omega^\rho(x') e^{-irx'_1}\, d\sigma(x'), \end{aligned}$$

where $x' = (x'_1, x'_2, \dots, x'_n) = (x'_1, x'')$. Taking $x'_1 = \cos\theta$, we can write the last integral as

$$g(r,\xi') = \int_0^\pi e^{-ir\cos\theta}\, d\theta \left( \int_{S_{\cos\theta}} \Omega^\rho(\cos\theta, x'')\, d\sigma_\theta(x') \right),$$

where $S_{\cos\theta} = \{|x'| = 1 \colon x' = (\cos\theta, x'')\}$ and $d\sigma_\theta(x')$ is its surface measure. By a further change of variable $s = \cos\theta$, the last integral is reduced to

$$g(r,\xi') = \int_{-1}^1 e^{-irs}(1-s^2)^{-1/2} \left( \int_{S_s} \Omega^\rho(s, x'')\, d\sigma_s(x') \right) ds.$$

Set $x'' = \sqrt{1-s^2}\,y'$. Then $y' \in S^{n-2}$ and

$$\text{(5)} \qquad g(r,\xi') = \int_{-1}^1 e^{-irs}(1-s^2)^{(n-3)/2} \left( \int_{S^{n-2}} \Omega^\rho(s, \sqrt{1-s^2}\,y')\, d\sigma_{n-2}(y') \right),$$

where $d\sigma_{n-2}(y')$ is the surface measure of $S^{n-2}$.

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From (5) we observe that $g(r,\xi') = (h(s)w(s))\hat{}(r)$, where

$$h(s) = \int_{S^{n-2}} \Omega^\rho(s, \sqrt{1-s^2}y')\, d\sigma_{n-2}(y') \quad \text{and} \quad w(s) = (1-s^2)^{(n-3)/2}\chi_{(-1,1)}(s).$$

Without loss of generality, we may assume that $\Omega \in L^q(S^{n-1})$ for some $1 < q < \infty$. Let $q'$ be the conjugate of $q$; $1/q' + 1/q = 1$. By Hölder's inequality,

$$\begin{aligned} |h(s)| &= \left| \int_{S^{n-2}} \Omega^\rho(s, \sqrt{1-s^2}y')\, d\sigma_{n-2}(y') \right| \\ &\le \omega_{n-2}^{1/q'} \left( \int_{S^{n-2}} |\Omega^\rho(s, \sqrt{1-s^2}y')|^q\, d\sigma_{n-2}(y') \right)^{1/q} \\ &= c \left( \int |\Omega^\rho(s, \sqrt{1-s^2}y')|^q\, d\sigma_{n-2}(y') \right)^{1/q}. \end{aligned}$$

$\omega_{n-2}$ is the area of $S^{n-2}$. Hence,

$$\begin{aligned} \int_{-1}^1 |h(s)|^q w(s)\, ds &\le c \int_{-1}^1 (1-s^2)^{(n-3)/2}\, ds \int_{S^{n-2}} |\Omega^\rho(s, \sqrt{1-s^2}y')|^q\, d\sigma_{n-2}(y') \\ &= c\|\Omega\|^q_{L^q(S^{n-1})}. \end{aligned}$$

Therefore, $|h(s)|^q w(s) \in L^1(-1,1)$.

Let $1 < p' < q$ and $r = q/p'$. Using Hölder's inequality again,

$$\begin{aligned} \int_{-1}^1 |h(s)w(s)|^{p'}\, ds &= \int_{-1}^1 |h(s)|^{p'} w(s)^{1/r} w(s)^{p'-1/r}\, ds \\ &\le \left( \int_{-1}^1 |h(s)|^q w(s)\, ds \right)^{1/r} \left( \int_{-1}^1 w(s)^{r'(p'-1/r)}\, ds \right)^{1/r'} \\ &\le c\|\Omega\|^{q/r}_{L^q(S^{n-1})} \left( \int_{-1}^1 (1-s^2)^{[(n-3)/2]r'(p'-1/r)}\, ds \right)^{1/r'} \\ &= c\|\Omega\|^{p'}_{L^q(S^{n-1})} \left( \int_{-\pi/2}^{\pi/2} |\sin\theta|^{(n-3)r'(p'-1/r)+1}\, d\theta \right)^{1/r'}. \end{aligned}$$

This last integral is convergent if $(n-3)r'(p'-1/r)+1 > -1$. This inequality, obviously, holds for all $p'$ when $n \ge 3$. In the case $n = 2$, it holds whenever $1 < p' < 2/(1+1/q)$. Therefore, if $\Omega \in L^q(S^{n-1})$, $1 < q \le \infty$, then there always exists some $p'$, $1 < p' < 2$, such that $h(s)w(s) \in L^{p'}(-1,1)$. By the Hausdorff-Young inequality

$$\begin{aligned} \|g(\cdot,\xi')\|_{L^p(1,\infty)} &\le \|g(\cdot,\xi')\|_{L^p(-\infty,\infty)} = \|(h(s)w(s))\hat{}\|_{L^p(-\infty,\infty)} \\ &\le \|h(s)w(s)\|_{L^{p'}(-\infty,\infty)} \\ &= \|h(s)w(s)\|_{L^{p'}(-1,1)} \le c\|\Omega\|^{p'}_{L^q(S^{n-1})} = A. \end{aligned}$$

This proves that $\hat{H}(\xi)$ is bounded, and thus $T$ is bounded on $L^2(R^n)$, $n \ge 2$.

Using the technique of complex interpolation and an argument similar to the one in [**1**], the $L^p$ boundedness can be established. We mention it briefly.

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Define

$$m_z(\xi) = \text{P.V.} \int_{R^n} \frac{H(x)}{|x|^z} e^{-ix\cdot\xi}\, dx \cdot |\xi|^{-z}$$

and

$$(T_z f)\hat{\ }(\xi) = m_z(\xi)\hat{f}(\xi),$$

$|\text{Re}(z)| < \eta$ for some $\eta > 0$. $T_z$ is an analytic family of operators in the sense of Stein. Also $T_0 = T$. Each $T_z$ is a principal-valued convolution operator, $T_z f =$ P.V. $H_z * f$, where $H_z = \check{m_z}$. As proved in [**1**], if $0 < \text{Re}(z) \le \eta$ and $\eta$ small enough, then

$$\sup_{|y|>0} \int_{|x|>2|y|} |H_z(x+y) - H_z(x)|\, dx \le c_z, \tag{6}$$

where the constant $c_z$ depends only on the real part of $z$. Also a similar argument as in the $L^2$ case will show that $|m_z(\xi)| \le c$ for $|\text{Re}(z)| \le \eta$, when $\eta$ is small enough. Therefore $\|T_{-\eta+iy}\|_2 \le c$ and $\|T_{\eta+iy}\|_r \le c_\eta$ for $-\infty < y < \infty$ and $1 < r < \infty$. By complex interpolation, if $1/p = (1-1/2)/2 + (1/2)/r$ then $\|T\|_p = \|T_0\|_p \le c$. Hence, $T$ is bounded on $L^p(R^n)$, $1 < p < \infty$, $n \ge 2$.

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