# A Singular Perturbation Problem for the p-Laplace Operator 

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ABSTRACT. In this paper we initiate the study of the nonlinear one phase singular perturbation problem

$$
\operatorname{div}\left(\left|\nabla u^{\varepsilon}\right|^{p-2} \nabla u^{\varepsilon}\right)=\beta_{\varepsilon}\left(u^{\varepsilon}\right), \quad(1<p<\infty)
$$

in a domain $\Omega$ of $\mathbb{R}^{N}$. We prove uniform Lipschitz regularity of uniformly bounded solutions. Once this is done we can pass to the limit to obtain a solution to the stationary case of a combustion problem with a nonlinearity of power type. (The case $p=2$ has been considered earlier by several authors.)

## 1. INTRODUCTION

Our objective in this paper is to study the singular perturbation problem

$$
\Delta_{p} u^{\varepsilon}=\beta_{\varepsilon}\left(u^{\varepsilon}\right), \quad u^{\varepsilon} \geq 0
$$

in a domain $\Omega$ of $\mathbb{R}^{N}$. Here, for $1<p<\infty, \Delta_{p}$ denotes the $p$-Laplace operator, i.e., $\Delta_{p} u=\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$. We recall that a solution to $\left(P_{\varepsilon}\right)$ is a function $u^{\varepsilon} \in W^{1, p}(\Omega) \cap L^{\infty}(\Omega)$ such that

$$
\begin{equation*}
\int_{\Omega}\left|\nabla u^{\varepsilon}\right|^{p-2} \nabla u^{\varepsilon} \cdot \nabla \varphi d x=-\int_{\Omega} \varphi \beta_{\varepsilon}\left(u^{\varepsilon}\right) d x \tag{1.1}
\end{equation*}
$$

for all $\varphi \in C_{0}^{\infty}(\Omega)$. We require $\beta_{\varepsilon}$ to be $\operatorname{Lip}(\mathbb{R})$ and to satisfy

$$
\begin{equation*}
0 \leq \beta_{\varepsilon} \leq \frac{A}{\varepsilon} \chi_{(0, \varepsilon)} \quad \text { and } \quad \int_{0}^{\varepsilon} \beta_{\varepsilon}(s) d s=M \tag{1.2}
\end{equation*}
$$

for positive constants $A$ and $M$. In particular, these conditions are fulfilled when the functions $\beta_{\varepsilon}$ are constructed from a single nonnegative Lipschitz function $\beta$
supported in [0, 1] by setting

$$
\begin{equation*}
\beta_{\varepsilon}(s)=\frac{1}{\varepsilon} \beta\left(\frac{s}{\varepsilon}\right) . \tag{1.3}
\end{equation*}
$$

Although our analysis applies to a general type of operators, as the ones considered in [17] of the form $\operatorname{div}(\mathcal{A}(x, u, \nabla u))$, for simplicity and clarity of the arguments we focus on the specific form of the $p$-Laplacian $\Delta_{p}$.

The motivation of the study in this paper comes from the applications to the one-phase case of the combustion problem, appearing in the description of laminar flames as an asymptotic limit for high activation energy, that corresponds to the limit as $\varepsilon \rightarrow 0$ in $\left(P_{\varepsilon}\right)$. For the case $p=2$ there is an extensive study of the problem and more or less a complete resolution of it; see [2], [15] for the elliptic case and [8], [3], [6], [7], [4], [9] for the parabolic one. However, the nonlinear case, addressed here, has never been considered earlier. This might partly depend on the lack of an established theory for the $p$-Laplace operator, and partly on the fact that some of the earlier techniques fail in the absence of linearity.

We show that, in a sense, the limit of $\left(P_{\varepsilon}\right)$ as $\varepsilon \rightarrow 0$ is a free boundary problem

$$
\begin{cases}\Delta_{p} u=0 & \text { in }\{u>0\}  \tag{P}\\ |\nabla u|=c & \text { on } \partial\{u>0\} \cap \Omega\end{cases}
$$

with $c=((p /(p-1)) M)^{1 / p}$. Namely, the main result of this paper (Theorem 4.3 in Section 4) asserts that the uniform limits $u$ of $u^{\varepsilon}$ have the asymptotic development

$$
u(x)=\left(\frac{p}{p-1} M\right)^{1 / p}\left\langle x-x_{0}, \eta\right\rangle^{+}+o\left(\left|x-x_{0}\right|\right)
$$

near $x_{0} \in \partial\{u>0\}$, provided $\partial\{u>0\}$ admits a measure theoretic normal and $u$ is not degenerate at $x_{0}$ (see Definitions 4.1-4.2).

The free boundary problem $(P)$ for the $p$-Laplacian was studied earlier under certain geometric (convexity) assumptions, by different techniques; see e.g. [1] by Acker and Meyer and a series of papers [11]-[13] by Henrot and Shahgholian.

To prove the main theorem (Theorem 4.3) we need a uniform bound (Theorem 2.1) for the gradient of the solutions, in order to have some stability of the problem as one passes to the limit. This type of uniform bounds on the gradient usually constitutes the basics of the analysis to follow, and it is by no means an obvious generalization of earlier results. Indeed, it needs to be pointed out that one of the main difficulties in the consideration of operators that do not admit linearization, as it is for the $p$-Laplacian, appears in the deduction of the uniform gradient bound, which has its own independent interest. In this part of our analysis we apply techniques that have been recently developed for related free boundary problems, see [14] and [5].

## 2. The Uniform Gradient Bound for Solutions

In this section we prove that the solutions $u^{\varepsilon}$ of the singular perturbation problem $\left(P_{\varepsilon}\right)$ are locally uniformly Lipschitz. Our main theorem in this section is the following.

Theorem 2.1. Let $u^{\varepsilon}$ be a nonnegative solution of $\left(P_{\varepsilon}\right)$ in a domain $\Omega$ of $\mathbb{R}^{N}$ with $\beta_{\varepsilon}$ satisfying (1.2) and such that $\left\|u^{\varepsilon}\right\|_{L^{\infty}(\Omega)} \leq L$. Then for every compact $K \Subset \Omega$ there is a constant $C=C(N, p, A, L, \operatorname{dist}(K, \partial \Omega))$ independent of $\varepsilon$ such that

$$
\left\|\nabla u^{\varepsilon}\right\|_{L^{\infty}(K)} \leq C .
$$

It is also noteworthy that as far as the proof of Theorem 2.1 goes, one can relax the conditions on $\beta_{\varepsilon}$. An important observation is that the same technique to follow shows that, in the case of two-phase problems (see [6]-[7]), one may deduce gradient bound for the non-negative part of the solution if one already knows that the negative part of the solution is Lipschitz. In [3], L. Caffarelli applied this idea in combination with the monotonicity formulas to deduce gradient bound for the solution of the two-phase singular perturbation problem for the Laplacian; see also [4]. In the absence of the monotonicity formula we are not able to prove a similar result as that in [3]. It is apparent that some new technique is to be developed to handle the sign change in the case of the $p$-Laplacian or any other nonlinear case. This remains an open and tantalizing problem.

The proof of Theorem 2.1 will be based on the following lemma.
Lemma 2.2. Let $v$ be a bounded nonnegative solution of

$$
0 \leq \Delta_{p} v \leq A \chi_{\{0<v<1\}}
$$

in the unit ball $B_{1}$ of $\mathbb{R}^{N}$, with $v(0) \leq 1$. Then there is a constant $C=C(N, p, A)$ such that

$$
\|v\|_{L^{\infty}\left(B_{1 / 4}\right)} \leq C
$$

Remark 2.3. We explicitly observe that $v \in C^{1, \alpha}(\Omega)$ for some $\alpha>0$, thanks to the results in [17]. In the case when $\Delta_{p} v=\beta(v)$ with $|\beta(s)| \leq$ $c_{0}|s|^{p}+c_{1}|s|^{p-1}$ for some constants $c_{0}$ and $c_{1}$, the conclusion of the Lemma 2.2 follows directly from Serrin's Harnack inequality for nonhomogeneous quasilinear operators, see [16]. Our proof, however, uses only Harnack inequality for homogeneous operators and is based on compactness rather than energy methods, which allows to generalize it to a broad range of operators.

Proof. Indeed, assume the contrary. Then there exists a sequence of functions $\left\{v_{k}\right\}, k=1,2, \ldots$, satisfying the assumptions of the lemma and such that

$$
\max _{\bar{B}_{1 / 4}} v_{k}(x)>\frac{4}{3} k
$$

Consider the sets

$$
\Omega_{k}=\left\{x \in B_{1} \mid v_{k}(x)>1\right\} \quad \text { and } \Gamma_{k}=\partial \Omega_{k} \cap B_{1} .
$$

Note that $v_{k}$ is $p$-harmonic in $\Omega_{k}$. Let now $\delta_{k}(x)=\operatorname{dist}\left(x, B_{1} \backslash \Omega_{k}\right)$ and define

$$
\mathcal{O}_{k}=\left\{x \in B_{1} \left\lvert\, \delta_{k}(x) \leq \frac{1}{3}(1-|x|)\right.\right\} \supset B_{1} \backslash \Omega_{k} .
$$

Observe that $\bar{B}_{1 / 4} \subset \mathcal{O}_{k}$. In particular

$$
\mathrm{m}_{k}:=\sup _{\mathcal{O}_{k}}(1-|x|) v_{k}(x) \geq \frac{3}{4} \max _{\bar{B}_{1 / 4}} v_{k}(x)>k .
$$

Since $v_{k}(x)$ is bounded (for fixed $k$ ), we will have $(1-|x|) v_{k}(x) \rightarrow 0$ as $|x| \rightarrow 1$, and therefore $\mathrm{m}_{k}$ will be attained at some point $x_{k} \in \mathcal{O}_{k}$ :

$$
\begin{equation*}
\left(1-\left|x_{k}\right|\right) v_{k}\left(x_{k}\right)=\max _{\mathcal{O}_{k}}(1-|x|) v_{k}(x) . \tag{2.1}
\end{equation*}
$$

Clearly,

$$
v_{k}\left(x_{k}\right)=\frac{\mathrm{m}_{k}}{1-\left|x_{k}\right|} \geq \mathrm{m}_{k}>k .
$$

Since $x_{k} \in \mathcal{O}_{k}$, by the definition we will have

$$
\begin{equation*}
\delta_{k}:=\delta_{k}\left(x_{k}\right) \leq \frac{1}{3}\left(1-\left|x_{k}\right|\right) . \tag{2.2}
\end{equation*}
$$

Let now $y_{k} \in \Gamma_{k}$ be a point where $\delta_{k}=\operatorname{dist}\left(x_{k}, \Gamma_{k}\right)$ is realized, so that

$$
\begin{equation*}
\left|y_{k}-x_{k}\right|=\delta_{k} . \tag{2.3}
\end{equation*}
$$

Then we will have two inclusions, $B_{2 \delta_{k}}\left(y_{k}\right) \subset B_{1}$ and $B_{\delta_{k} / 2}\left(y_{k}\right) \subset \mathcal{O}_{k}$, both consequences of (2.2)-(2.3). In particular, for $z \in B_{\delta_{k} / 2}\left(y_{k}\right)$ the following inequality holds

$$
(1-|z|) \geq\left(1-\left|x_{k}\right|\right)-\left|x_{k}-z\right| \geq\left(1-\left|x_{k}\right|\right)-\frac{3}{2} \delta_{k} \geq \frac{1}{2}\left(1-\left|x_{k}\right|\right) .
$$

This, in conjunction with (2.1), implies that

$$
\max _{\overline{B_{\delta_{k} / 2}\left(y_{k}\right)}} v_{k} \leq 2 v_{k}\left(x_{k}\right)
$$

Next, since $B_{\delta_{k}}\left(x_{k}\right) \subset \Omega_{k}, v_{k}$ satisfies $\Delta_{p} v_{k}=0$ in $B_{\delta_{k}}\left(x_{k}\right)$. By the Harnack inequality for $p$-harmonic functions there is a constant $c=c(N, p)>0$ such that

$$
\min _{\dot{B_{3} \delta_{k} / 4}\left(x_{k}\right)} v_{k} \geq c v_{k}\left(x_{k}\right) .
$$

In particular,

$$
\max _{\overline{\bar{B}_{\delta_{k} / 4}\left(y_{k}\right)}} v_{k} \geq c v_{k}\left(x_{k}\right)
$$

Further, define

$$
w_{k}(x)=\frac{v_{k}\left(y_{k}+\delta_{k} x\right)}{v_{k}\left(x_{k}\right)} \quad \text { for } x \in B_{2} .
$$

Summarizing the properties of $v_{k}$ above, we see that $w_{k}$ satisfies the following system

$$
\left\{\begin{array}{l}
0 \leq \Delta_{p} w_{k} \leq \frac{A\left(\delta_{k}\right)^{p}}{k^{p-1}} \quad \text { in } B_{2} \\
\max _{\bar{B}_{1 / 2}} w_{k} \leq 2, \quad \max _{\bar{B}_{1 / 4}} w_{k} \geq c>0 \\
w_{k} \geq 0, \quad w_{k}(0) \leq \frac{1}{k}
\end{array}\right.
$$

Therefore, from a priori estimates, we can conclude that a subsequence of $\left\{w_{k}\right\}$ will converge in $C^{1, \alpha}$ norm on every compact subset of $B_{1 / 2}$ to a function $w_{0}$ that satisfies

$$
\left\{\begin{array}{l}
\Delta_{p} w_{0}=0 \quad \text { in } B_{1 / 2} \\
\max _{\bar{B}_{1 / 4}} w_{0} \geq c>0 \\
w_{0} \geq 0, \quad w_{0}(0)=0
\end{array}\right.
$$

This, however, contradicts the strong maximum principle for $p$-harmonic functions. The lemma is proved.

Proof of Theorem 2.1. We start with the observation that it is enough to prove the theorem in the case when $\Omega=B_{1}, K=\bar{B}_{1 / 8}$, and under the assumptions $u^{\varepsilon}(0)=\varepsilon$ and $\left\|u^{\varepsilon}\right\|_{L^{\infty}\left(B_{1}\right)}=1$. Denote

$$
\Omega^{\varepsilon}=\left\{x \in B_{1} \mid u^{\varepsilon}>\varepsilon\right\} \quad \text { and } \quad \Gamma^{\varepsilon}=\partial \Omega^{\varepsilon} \cap B_{1} .
$$

Step 1. Prove that there is a constant $C=C(N, p, A)$ such that

$$
\begin{equation*}
\left|\nabla u^{\varepsilon}(x)\right| \leq C, \quad \text { for } x \in \bar{B}_{1 / 2} \backslash \Omega^{\varepsilon} . \tag{2.4}
\end{equation*}
$$

Indeed, take a point $x_{0} \in \bar{B}_{1 / 2}$ with $\mathcal{u}^{\varepsilon}\left(x_{0}\right) \leq \varepsilon$ and consider a function

$$
v^{\varepsilon}(x)=\frac{u^{\varepsilon}\left(x_{0}+\varepsilon x\right)}{\varepsilon} .
$$

Direct computation shows that $v^{\varepsilon}$ satisfies

$$
0 \leq \Delta_{p} v^{\varepsilon} \leq A X_{\left\{0<v^{\varepsilon}<1\right\}} .
$$

All the assumptions of Lemma 2.2 are thus fulfilled for $v=v^{\varepsilon}$ and we can conclude

$$
\begin{equation*}
\max _{\bar{B}_{1 / 4}} v^{\varepsilon} \leq C(N, p, A) . \tag{2.5}
\end{equation*}
$$

From interior gradient estimates we obtain

$$
\left|\nabla u^{\varepsilon}\left(x_{0}\right)\right|=\left|\nabla v^{\varepsilon}(0)\right| \leq C_{1}(N, p, A) \max _{\bar{B}_{1 / 4}} v^{\varepsilon} \leq C_{2}(N, p, A)
$$

for all $x_{0} \in \bar{B}_{1 / 2} \backslash \Omega^{\varepsilon}$. Hence (2.4) is proved.
Step 2. Prove that

$$
\begin{equation*}
\left|u^{\varepsilon}(x)\right| \leq \varepsilon+C(N, p, A) \operatorname{dist}\left(x, B_{1} \backslash \Omega^{\varepsilon}\right) \quad \text { for } x \in \bar{B}_{1 / 4} \cap \Omega^{\varepsilon} . \tag{2.6}
\end{equation*}
$$

Indeed, for $x_{0} \in \bar{B}_{1 / 4} \cap \Omega^{\varepsilon}$ denote

$$
\mathrm{m}_{0}=u^{\varepsilon}\left(x_{0}\right)-\varepsilon \quad \text { and } \quad \delta_{0}=\operatorname{dist}\left(x_{0}, B_{1} \backslash \Omega^{\varepsilon}\right)
$$

Notice that, since $0 \in \Gamma^{\varepsilon}, \delta_{0} \leq \frac{1}{4}$. We want to prove that

$$
\begin{equation*}
\mathrm{m}_{0} \leq C(N, p, A) \delta_{0} . \tag{2.7}
\end{equation*}
$$

Since $B_{\delta_{0}}\left(x_{0}\right)$ is contained in $\Omega^{\varepsilon}, u^{\varepsilon}-\varepsilon$ will be nonnegative and $p$-harmonic there. We can thus apply the Harnack inequality to conclude

$$
\min _{B_{\delta_{0} / 2}\left(x_{0}\right)}\left(u^{\varepsilon}(x)-\varepsilon\right) \geq c_{1} \mathrm{~m}_{0}
$$

for $c_{1}=c_{1}(N, p)>0$. Next, consider the $p$-capacitary potential $\varphi(x)$ of the ring $B_{1} \backslash \bar{B}_{1 / 2}$ which satisfies

$$
\Delta_{p} \varphi(x)=0 \quad \text { in } B_{1} \backslash \bar{B}_{1 / 2},\left.\varphi\right|_{\partial B_{1}}=0, \text { and }\left.\varphi\right|_{\partial B_{1 / 2}}=1 .
$$

The function $\varphi$ will be spherically symmetric with $|\nabla \varphi|=c_{0}=c_{0}(N, p)>0$ on $\partial B_{1}$. Define

$$
\psi(x)=c_{1} \mathrm{~m}_{0} \varphi\left(\frac{x-x_{0}}{\delta_{0}}\right) \quad \text { for } x \in \bar{B}_{\delta_{0}}\left(x_{0}\right) \backslash \bar{B}_{\delta_{0} / 2}\left(x_{0}\right) .
$$

From the comparison principle for $p$-harmonic functions we will have

$$
\begin{equation*}
\psi(x) \leq u^{\varepsilon}(x)-\varepsilon \quad \text { for } x \in \bar{B}_{\delta_{0}}\left(x_{0}\right) \backslash \bar{B}_{\delta_{0} / 2}\left(x_{0}\right) . \tag{2.8}
\end{equation*}
$$

Take $y_{0} \in \partial B_{\delta_{0}}\left(x_{0}\right) \cap \Gamma_{\varepsilon}$. Then $y_{0} \in \bar{B}_{1 / 2}$, and

$$
\begin{equation*}
\psi\left(y_{0}\right)=u^{\varepsilon}\left(y_{0}\right)-\varepsilon=0 . \tag{2.9}
\end{equation*}
$$

We infer from (2.8), (2.9), and (2.4) that $\left|\nabla \psi\left(y_{0}\right)\right| \leq\left|\nabla u^{\varepsilon}\left(y_{0}\right)\right| \leq c_{2}(N, p, A)$. Observe now that $\left|\nabla \psi\left(y_{0}\right)\right|=c_{1} \mathrm{~m}_{0} c_{0} / \delta_{0}$ and therefore we obtain

$$
\mathrm{m}_{0} \leq \frac{c_{2}}{c_{0} c_{1}} \delta_{0}
$$

Thus, inequalities (2.7) and (2.6) are proved.
Step 3. Prove that

$$
\begin{equation*}
\left|\nabla u^{\varepsilon}(x)\right| \leq C(n, p, A) \quad \text { for } x \in \bar{B}_{1 / 8} \cap \Omega^{\varepsilon} . \tag{2.10}
\end{equation*}
$$

Indeed, let $x_{0} \in \bar{B}_{1 / 8} \cap \Omega^{\varepsilon}, \delta_{0}=\operatorname{dist}\left(x_{0}, B_{1} \backslash \Omega^{\varepsilon}\right)$ and define

$$
w(x)=\frac{u^{\varepsilon}\left(x_{0}+\delta_{0} x\right)-\varepsilon}{\delta_{0}} \quad \text { for } x \in B_{1}
$$

Then, from the inclusion $B_{\delta_{0}}\left(x_{0}\right) \subset \Omega^{\varepsilon}$ and inequality (2.6) we will have

$$
0 \leq w \leq C(N, p, A) \quad \text { in } B_{1} .
$$

Since also $w$ is $p$-harmonic in $B_{1}$, from the interior gradient estimates we obtain

$$
\left|\nabla u^{\varepsilon}\left(x_{0}\right)\right|=|\nabla w(0)| \leq C_{1}(N, p, A)
$$

which proves (2.10).
Now the theorem follows from (2.4) and (2.10).

## 3. Passage to the Limit: $\varepsilon_{j} \searrow 0$

From now on we will assume that functions $\beta_{\varepsilon}$ in $\left(P_{\varepsilon}\right)$ satisfy (1.2)-(1.3).
This section embodies the main technical tools that one needs to establish the main theorem (Theorem 4.3).

Lemma 3.1. Let $\left\{u^{\varepsilon}\right\}$ be a uniformly bounded family of solutions to $\left(P_{\varepsilon}\right)$. Then for every sequence $\varepsilon_{j} \rightarrow 0$ there exists a subsequence $\varepsilon_{j}^{\prime} \rightarrow 0$ and $u \in \operatorname{Lip}(\Omega)$ such that:
(i) $u^{\varepsilon_{j}^{\prime}} \rightarrow u$ uniformly on compact subsets of $\Omega$;
(ii) $\Delta_{p} u=0$ in $\Omega \backslash \partial\{u>0\}$;
(iii) $\nabla u^{\varepsilon_{j}^{\prime}} \rightarrow \nabla u$ in $L_{\mathrm{loc}}^{p}(\Omega)$.

Proof. Part (i) follows by Theorem 2.1 and a standard compactness argument. Let now $E \Subset\{u>0\}$ be open. Then $u \geq c>0$ in $E$. By the uniform convergence, we will have $u^{\varepsilon_{j}^{\prime}}>c / 2$ in $E$ for small $\varepsilon_{j}^{\prime}$. Hence, if also $\varepsilon_{j}^{\prime}<c / 2$, $u^{\varepsilon_{j}^{\prime}}$ will be $p$-harmonic in $E$. This implies that $u$ is $p$-harmonic in $E$ and since $E$ was arbitrary, (ii) follows.

Finally, we prove (iii). Let $\psi$ be a nonnegative $C_{0}^{\infty}(\Omega)$ function, and $\delta>0$. Take $(u-\delta)^{+} \psi$ as a test function. Since $\Delta_{p} u=0$ in the positivity set of $u$, integrating by parts, we obtain

$$
\int_{\{u>\delta\}}|\nabla u|^{p} \psi=-\int_{\{u>\delta\}}|\nabla u|^{p-2} \nabla u \cdot \nabla \psi u+\delta \int_{\{u>\delta\}}|\nabla u|^{p-2} \nabla u \cdot \nabla \psi .
$$

Letting $\delta \rightarrow 0$ we find

$$
\begin{equation*}
\int_{\{u>0\}}|\nabla u|^{p} \psi=-\int_{\{u>0\}}|\nabla u|^{p-2} \nabla u \cdot \nabla \psi u . \tag{3.1}
\end{equation*}
$$

On the other hand, the observation $\beta_{\varepsilon}\left(\mathcal{u}^{\varepsilon}\right) \mathcal{u}^{\varepsilon} \geq 0$ yields

$$
\begin{equation*}
\int_{\Omega}\left|\nabla u^{\varepsilon}\right|^{p} \psi \leq-\int_{\Omega}\left|\nabla u^{\varepsilon}\right|^{p-2} \nabla u^{\varepsilon} \cdot \nabla \psi u^{\varepsilon} . \tag{3.2}
\end{equation*}
$$

Using the uniform convergence of $u^{\varepsilon}$ to $u$ and the weak convergence of $\left|\nabla u^{\varepsilon}\right|^{p-2} \nabla u^{\varepsilon}$ to $|\nabla u|^{p-2} \nabla u$ in $L_{\text {loc }}^{p /(p-1)}(\Omega)$, we infer from (3.1) and (3.2) that

$$
\begin{equation*}
\underset{j \rightarrow \infty}{\lim \sup } \int\left|\nabla u^{\varepsilon_{j}}\right|^{p} \psi \leq \int|\nabla u|^{p} \psi . \tag{3.3}
\end{equation*}
$$

Since $\nabla u^{\varepsilon_{j}}-\nabla u$ in $L_{\text {loc }}^{p}(\Omega)$, we have

$$
\begin{equation*}
\int|\nabla u|^{p} \psi \leq \underset{j \rightarrow \infty}{\liminf } \int\left|\nabla u^{\varepsilon_{j}}\right|^{p} \psi . \tag{3.4}
\end{equation*}
$$

It follows from (3.3), (3.4), and a simple compactness argument that $\nabla u^{\varepsilon_{j}} \rightarrow \nabla u$ in $L_{\text {loc }}^{p}(\Omega)$.

The conclusion of part (iii) is proved, and so is the lemma.
We now prove that limit solutions are solutions to the free boundary problem in a very weak sense.

Proposition 3.2. Let $\left\{\mathcal{u}^{\varepsilon_{j}}\right\}$ be a family of solutions to $\left(P_{\varepsilon_{j}}\right)$. Assume that $u^{\varepsilon_{j}} \rightarrow$ $u$ uniformly on compact subsets of $\Omega$ as $\varepsilon_{j} \rightarrow 0$. Then there exists a locally finite measure $\mu$ supported on the free boundary $\Omega \cap \partial\{u>0\}$ such that $\beta_{\varepsilon_{j}}\left(u^{\varepsilon_{j}}\right)$ $\mu$ in $\Omega$. In particular, $\Delta_{p} u=\mu$ in $\Omega$, i.e.,

$$
\begin{equation*}
\int_{\Omega}|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi d x=-\int_{\Omega} \varphi d \mu \tag{3.5}
\end{equation*}
$$

for all $\varphi \in C_{0}^{\infty}(\Omega)$.
Proof. By definition of weak solutions to $\left(P_{\varepsilon}\right)$, if $\varphi \in C_{0}^{\infty}(\Omega)$, one has

$$
\begin{equation*}
\int_{\Omega}\left|\nabla u^{\varepsilon}\right|^{p-2} \nabla u^{\varepsilon} \cdot \nabla \varphi=-\int_{\Omega} \beta_{\varepsilon}\left(u^{\varepsilon}\right) \varphi \tag{3.6}
\end{equation*}
$$

Since $u^{\varepsilon_{j}} \rightarrow u$ uniformly on compact subsets of $\Omega$, by Lemma 3.1 we know $\nabla u^{\varepsilon_{j}} \rightarrow \nabla u$ in $L_{\text {loc }}^{p}(\Omega)$, and so the left-hand side of (3.6) converges to the lefthand side of (3.5). Now let $F \subset \Omega$ be compact, and take $\varphi \in C_{0}^{\infty}(\Omega), \varphi \geq 0$, $\varphi \equiv 1$ in $F$. The sequence $\left\{\int_{\Omega} \beta_{\varepsilon_{j}}\left(u^{\varepsilon_{j}}\right) \varphi d x\right\}$ is convergent, and therefore it is bounded. Hence

$$
\int_{F} \beta_{\varepsilon_{j}}\left(u^{\varepsilon_{j}}\right) d x \leq \int_{\Omega} \beta_{\varepsilon_{j}}\left(u^{\varepsilon_{j}}\right) \varphi d x \leq C(\varphi) .
$$

This implies that there exists a locally finite measure $\mu$ such that, passing to a subsequence (still denoted by $\varepsilon_{j}$ ) if necessary, $\beta_{\varepsilon_{j}}\left(u^{\varepsilon_{j}}\right) \rightarrow \mu$ as measures in $\Omega$. Passing to the limit in (3.6), we get (3.5). Moreover, since $\Delta_{p} u=0$ in $\Omega \backslash \partial\{u>$ $0\}$ by Lemma 3.1, we conclude that $\mu$ is supported in $\Omega \cap \partial\{u>0\}$. The proof is thus complete.

Lemma 3.3. Let $\left\{u^{\varepsilon_{j}}\right\}$ be a family of solutions to $\left(P_{\varepsilon_{j}}\right)$ in $\Omega$ such that $u^{\varepsilon_{j}} \rightarrow u$ uniformly on compact subsets of $\Omega$ and $\varepsilon_{j} \rightarrow 0$ as $j \rightarrow \infty$. Let $x_{0}, x_{n} \in \Omega \cap \partial\{u>0\}$ be such that $x_{n} \rightarrow x_{0}$ as $n \rightarrow \infty$. Let $\lambda_{n} \rightarrow 0, u_{\lambda_{n}}(x)=\left(1 / \lambda_{n}\right) u\left(x_{n}+\lambda_{n} x\right)$, and $\left(u^{\varepsilon_{j}}\right)_{\lambda_{n}}(x)=\left(1 / \lambda_{n}\right) u^{\varepsilon_{j}}\left(x_{n}+\lambda_{n} x\right)$. Suppose that $u_{\lambda_{n}} \rightarrow U$ as $n \rightarrow \infty$ uniformly on compact sets of $\mathbb{R}^{N}$. Then, there exists $j(n) \rightarrow \infty$ such that for every $j_{n} \geq j(n)$ there holds that $\varepsilon_{j_{n}} / \lambda_{n} \rightarrow 0$, and
(i) $\left(u^{\varepsilon_{j n}}\right)_{\lambda_{n}} \rightarrow U$ uniformly on compact sets of $\mathbb{R}^{N}$;
(ii) $\nabla\left(u^{\varepsilon_{j n}}\right)_{\lambda_{n}} \rightarrow \nabla U$ in $L_{\mathrm{loc}}^{p}\left(\mathbb{R}^{N}\right)$;
(iii) $\nabla u_{\lambda_{n}} \rightarrow \nabla U$ in $L_{\mathrm{loc}}^{p}\left(\mathbb{R}^{N}\right)$.

Proof. The proof is along the lines of the one of Lemma 3.2 in [6]. We discuss here only the relevant modifications. For simplicity we assume $x_{n}=x_{0}$. Proceeding as in the cited reference, one can show that (i) holds. The functions $\left(u^{\varepsilon_{j n}}\right)_{\lambda_{n}}$ are solutions to

$$
\Delta_{p}\left(u^{\varepsilon_{j n}}\right)_{\lambda_{n}}=\beta_{\varepsilon_{j_{n}} / \lambda_{n}}\left(\left(u^{\varepsilon_{j_{n}}}\right)_{\lambda_{n}}\right)
$$

in $B_{k}$, where $k$ is a fixed positive number. By Lemma 3.1 there exists a subsequence, still denoted by $j_{n}$, such that $\nabla\left(u^{\varepsilon_{j n}}\right)_{\lambda_{n}} \rightarrow \nabla U$ in $L^{p}\left(B_{k}\right)$. Then also (ii) holds. In order to prove (iii), let $\delta>0$ and consider

$$
\left\|\nabla u_{\lambda_{n}}-\nabla U\right\| \leq\left\|\nabla u_{\lambda_{n}}-\nabla\left(u^{\varepsilon_{j}}\right)_{\lambda_{n}}\right\|+\left\|\nabla\left(u^{\varepsilon_{j}}\right)_{\lambda_{n}}-\nabla U\right\|=I+I I,
$$

where all the norms are in $L^{p}\left(B_{k}\right)$. By (ii) we already know that $I I<\delta$ if $j \geq j_{n}$ and $n$ is sufficiently large. Moreover, by virtue of Lemma 3.1 it holds

$$
\begin{aligned}
I^{p} & =\int_{B_{k}}\left|\nabla u-\nabla u^{\varepsilon_{j}}\right|^{p}\left(x_{0}+\lambda_{n} x\right) d x \\
& =\frac{1}{\lambda_{n}^{N}} \int_{B_{\lambda_{n k} k}\left(x_{0}\right)}\left|\nabla u-\nabla u^{\varepsilon_{j}}\right|^{p}(x) d x<\delta^{p}
\end{aligned}
$$

if $j$ and $n$ are sufficiently large. This proves (iii).
We now turn our attention to the special case when the limit function $u$ is onedimensional.

Proposition 3.4. Let $x_{0} \in \Omega$, and let $u^{\varepsilon_{k}}$ be solutions to

$$
\Delta_{p} u^{\varepsilon_{k}}=\beta_{\varepsilon_{k}}\left(u^{\varepsilon_{k}}\right)
$$

in $\Omega$. If $u^{\varepsilon_{k}}$ converge to $\alpha\left(x-x_{0}\right)_{1}^{+}$uniformly on compact subsets of $\Omega$, with $\alpha \in \mathbb{R}$, and $\varepsilon_{k} \rightarrow 0$ as $k \rightarrow \infty$, then

$$
0 \leq \alpha \leq\left(\frac{p}{p-1} M\right)^{1 / p} .
$$

Proof. Without loss of generality, we assume $x_{0}=0$. Since $u^{\varepsilon_{k}} \geq 0$, we readily have $\alpha \geq 0$. Next, let $\psi \in C_{0}^{\infty}(\Omega)$. Choosing $u_{\chi_{1}}^{\varepsilon_{k}} \psi$ as a test function in the weak formulation of $\Delta_{p} u^{\varepsilon_{k}}=\beta_{\varepsilon_{k}}\left(\mathcal{u}^{\varepsilon_{k}}\right)$ (see Remark 3.5 below) and integrating by parts, we obtain

$$
\begin{equation*}
-\frac{1}{p} \int_{\Omega}\left|\nabla u^{\varepsilon_{k}}\right|^{p} \psi_{x_{1}}+\int_{\Omega}\left|\nabla u^{\varepsilon_{k}}\right|^{p-2} u_{x_{1}}^{\varepsilon_{k}} \nabla u^{\varepsilon_{k}} \cdot \nabla \psi=\int_{\Omega} B_{\varepsilon_{k}}\left(u^{\varepsilon_{k}}\right) \psi_{x_{1}} . \tag{3.7}
\end{equation*}
$$

Here, $B_{\varepsilon_{k}}(s)=\int_{0}^{s} \beta_{\varepsilon_{k}}(\tau) d \tau$. Since $0 \leq B_{\varepsilon_{k}}(s) \leq M$, there exists $M(x) \in L^{\infty}(\Omega)$, $0 \leq M(x) \leq M$, such that on a subsequence (still denoted by $\left.\varepsilon_{k}\right) B_{\varepsilon_{k}}\left(u^{\varepsilon_{k}}\right) \rightarrow$ $M(x)$ *-weakly in $L^{\infty}(\Omega)$. If $y \in \Omega \cap\left\{x_{1}>0\right\}$, then $u^{\varepsilon_{k}} \geq \alpha y_{1} / 2$ in a neighborhood of $y$ for $k$ sufficiently large. Hence, if $u^{\varepsilon_{k}}(x) \geq \varepsilon_{k}$ we have, by (1.2),

$$
B_{\varepsilon_{k}}\left(u^{\varepsilon_{k}}\right)(x)=\int_{0}^{u^{\varepsilon_{k}(x) / \varepsilon_{k}}} \beta(s) d s=M .
$$

Moreover, using Proposition 3.2, it is immediate to recognize that $\nabla B_{\varepsilon_{k}}\left(\mathcal{u}^{\varepsilon_{k}}\right)=$ $\beta_{\varepsilon_{k}}\left(u^{\varepsilon_{k}}\right) \nabla u^{\varepsilon_{k}} \rightarrow 0$ in $L_{\text {loc }}^{1}\left(\Omega \cap\left\{x_{1}<0\right\}\right)$. Hence $M(x)=\bar{M} \in[0, M]$ in $\Omega \cap\left\{x_{1}<0\right\}$. Passing to the limit in (3.7) yields

$$
\frac{p-1}{p} \alpha^{p} \int_{\left\{x_{1}>0\right\}} \psi_{x_{1}} d x=M \int_{\left\{x_{1}>0\right\}} \psi_{x_{1}} d x+\bar{M} \int_{\left\{x_{1}<0\right\}} \psi_{x_{1}} d x,
$$

and integrating by parts we find

$$
\frac{p-1}{p} \alpha^{p} \int_{\left\{x_{1}=0\right\}} \psi d x^{\prime}=M \int_{\left\{x_{1}=0\right\}} \psi d x^{\prime}-\bar{M} \int_{\left\{x_{1}=0\right\}} \psi d x^{\prime} .
$$

The arbitrariness of $\psi \in C_{0}^{\infty}(\Omega)$ allows to conclude $((p-1) / p) \alpha^{p}=M-\bar{M} \leq M$, since $\bar{M} \geq 0$. Hence $\alpha^{p} \leq(p /(p-1)) M$, and the proof is complete.

Remark 3.5. We recall that the weak solution $u$ of the equation $\Delta_{p} u=f$ in $\Omega$ with bounded $f$ has a representative in $W_{\text {loc }}^{2,2}(\Omega)$ if $2 \leq p<\infty$, and in $W_{\text {loc }}^{2, p}(\Omega)$ for $1<p \leq 2$, see e.g. [17]. This, in conjunction with local $L^{\infty}$ bounds on $\left|\nabla u^{\varepsilon_{k}}\right|$ in $\Omega$, justifies the integration by parts in the proof of Proposition 3.4 above.

Proposition 3.6. Let $x_{0} \in \Omega$, and let $\mathcal{u}^{\varepsilon_{k}}$ be solutions to

$$
\Delta_{p} u^{\varepsilon_{k}}=\beta_{\varepsilon_{k}}\left(u^{\varepsilon_{k}}\right)
$$

in $\Omega$. If $u^{\varepsilon_{k}}$ converge to $\alpha\left(x-x_{0}\right)_{1}^{+}+\gamma\left(x-x_{0}\right)_{1}^{-}$uniformly on compact subsets of $\Omega$, with $\alpha, \gamma>0$ and $\varepsilon_{k} \rightarrow 0$ as $k \rightarrow \infty$, then

$$
\alpha=\gamma \leq\left(\frac{p}{p-1} M\right)^{1 / p} .
$$

Proof. Without loss of generality we assume $x_{0}=0$. As in Proposition 3.4, $u^{\varepsilon_{k}}$ satisfies (3.7), and it is immediate to recognize that $B_{\varepsilon_{k}}\left(u^{\varepsilon_{k}}\right) \rightarrow M$ in $L_{\text {loc }}^{1}(\Omega)$. Passing to the limit in (3.7), and integrating by parts in the resulting equation, we find that $\alpha=\gamma$.

Now we assume that $\alpha>((p /(p-1)) M)^{1 / p}$ and show that this leads to a contradiction.

Step 1. Let $\mathcal{R}_{2}:=\left\{x=\left(x_{1}, x^{\prime}\right) \in \mathbb{R}^{N}:\left|x_{1}\right|<2,\left|x^{\prime}\right|<2\right\}$. Without loss of generality, we may assume $\mathcal{R}_{2} \Subset \Omega$. First of all, we construct a family $\left\{v^{\varepsilon_{j}}\right\}$ of solutions to $\left(P_{\varepsilon_{j}}\right)$ in $\mathcal{R}_{2}$ with the property

$$
\begin{equation*}
v^{\varepsilon_{j}}\left(x_{1}, x^{\prime}\right)=v^{\varepsilon_{j}}\left(-x_{1}, x^{\prime}\right) \quad \text { in } \mathcal{R}_{2}, \tag{3.8}
\end{equation*}
$$

and such that $v^{\varepsilon_{j}} \rightarrow u$ uniformly on compact subsets of $\mathcal{R}_{2}$, where $u(x)=\alpha\left|x_{1}\right|$. To this end, we let $b_{\varepsilon_{j}}=\sup _{\mathcal{R}_{2}}\left|u^{\varepsilon_{j}}-u\right|$ and $v^{\varepsilon_{j}}$ be the minimal solution (i.e., minimum of all supersolutions) to ( $P_{\varepsilon_{j}}$ ) in $\mathcal{R}_{2}$ with boundary values $v^{\varepsilon_{j}}=u-b_{\varepsilon_{j}}$ on $\partial \mathcal{R}_{2}$. By virtue of Lemma 3.1, there exists $v \in \operatorname{Lip}_{\text {loc }}\left(\mathcal{R}_{2}\right)$ such that, on a subsequence, $v^{\varepsilon_{j}} \rightarrow v$ uniformly on compact subsets of $\mathcal{R}_{2}$. From the minimality of $v^{\varepsilon_{j}}$ it is immediate to recognize that $u \geq v$.

To prove the reverse inequality, we consider $w \in C^{2}(\mathbb{R})$, satisfying

$$
\left(\left|w^{\prime}\right|^{p-2} w^{\prime}\right)^{\prime}=\beta(w) \text { in } \mathbb{R}, \quad w(0)=1, \quad w^{\prime}(0)=\alpha
$$

and let

$$
w^{\varepsilon_{j}}\left(x_{1}\right)=\varepsilon_{j} w\left(\frac{x_{1}}{\varepsilon_{j}}-\frac{b_{\varepsilon_{j}}}{\gamma \varepsilon_{j}}+\bar{s}\right) .
$$

Here $\bar{s}<0$ is a constant, determined as in [6, Proposition 5.3], such that

$$
w(s)= \begin{cases}1+\alpha s, & s \geq 0 \\ \gamma(s-\bar{s}), & s \leq \bar{s}\end{cases}
$$

Here $\alpha$ and $\gamma$ are related by $\alpha^{p}-\gamma^{p}=(p /(p-1)) M$ and therefore $\gamma>0$ under the assumption $\alpha>((p /(p-1)) M)^{1 / p}$. Moreover, we have $w^{\prime}\left(x_{1}\right) \geq \gamma$ for all $x_{1} \in \mathbb{R}$. Observe also that $w^{\varepsilon_{j}} \leq u-b_{\varepsilon_{j}}=v^{\varepsilon_{j}}$ on $\partial \mathcal{R}_{2}$. From Lemma 3.7 below, it follows that $w^{\varepsilon_{j}} \leq v^{\varepsilon_{j}}$ in $\mathcal{R}_{2}$. Let us point out that such an implication is not immediate since, in general, there is no comparison principle for $\Delta_{p}-\beta$. As a consequence, $u \leq v$ in $\mathcal{R}_{2} \cap\left\{x_{1}>0\right\}$. Using (3.8), finally, we see that $u \leq v$ in $\mathcal{R}_{2}$, and the construction is complete.

Step 2. Let $\mathcal{R}^{+}=\left\{x\left|0<x_{1}<1,\left|x^{\prime}\right|<1\right\}\right.$. Then using the weak formulation of $\left(P_{\varepsilon_{j}}\right)$ in $\mathcal{R}^{+}$we have

$$
\begin{aligned}
E_{j} & =\iint_{\mathcal{R}^{+}} \frac{\partial}{\partial x_{1}}\left(\frac{1}{p}\left|\nabla v^{\varepsilon_{j}}\right|^{p}\right) d x=\iint_{\mathcal{R}^{+}}\left|\nabla v^{\varepsilon_{j}}\right|^{p-2} \nabla v^{\varepsilon_{j}} \cdot \nabla v_{x_{1}}^{\varepsilon_{j}} d x \\
& =\iint_{\mathcal{R}^{+}} \operatorname{div}\left(\left|\nabla v^{\varepsilon_{j}}\right|^{p-2} \nabla v^{\varepsilon_{j}} v_{x_{1}}^{\varepsilon_{j}}\right) d x-\iint_{\mathcal{R}^{+}} \beta_{\varepsilon_{j}}\left(v^{\varepsilon_{j}}\right) v_{x_{1}}^{\varepsilon_{j}} d x=F_{j}-G_{j} .
\end{aligned}
$$

Using the divergence theorem and that $v_{x_{1}}^{\varepsilon_{j}}\left(0, x^{\prime}\right)=0$ (from symmetry in the $x_{1}$ variable) we find that

$$
F_{j}=\int_{\partial \mathcal{R}^{+} \cap\left\{x_{1}=1\right\}}\left|\nabla v^{\varepsilon_{j}}\right|^{p-2}\left(v_{x_{1}}^{\varepsilon_{j}}\right)^{2} d x^{\prime}+\int_{\partial \mathcal{R}^{+} \cap\left\{\left|x^{\prime}\right|=1\right\}}\left|\nabla v^{\varepsilon_{j}}\right|^{p-2} v_{n}^{\varepsilon_{j}} v_{x_{1}}^{\varepsilon_{j}} d S
$$

where $v_{n}^{\varepsilon_{j}}$ is the exterior normal derivative of $v^{\varepsilon_{j}}$ on $\partial \mathcal{R}^{+} \cap\left\{\left|x^{\prime}\right|=1\right\}$. From the convergence $v^{\varepsilon_{j}} \rightarrow u=\alpha x_{1}^{+}+\alpha x_{1}^{-}$in $\mathcal{R}_{2}$ and Lemma 3.1 it follows (at least for a subsequence) that

$$
\nabla v_{x_{1}}^{\varepsilon_{j}} \rightarrow \alpha \mathbf{e}_{1} \quad \text { pointwise a.e. in } \mathcal{R}_{2}^{+}=\mathcal{R}_{2} \cap\left\{x_{1}>0\right\}
$$

Since $\left|\nabla v^{\varepsilon_{j}}\right|$ are uniformly bounded, from the dominated convergence theorem we deduce that

$$
\begin{equation*}
\lim _{j \rightarrow \infty} F_{j}=\int_{\partial \mathcal{R}^{+} \cap\left\{x_{1}=1\right\}} \alpha^{p} d x^{\prime} \tag{3.9}
\end{equation*}
$$

On the other hand

$$
\begin{aligned}
E_{j}+G_{j} & =\iint_{\mathcal{R}^{+}} \frac{\partial}{\partial} x_{1}\left(\frac{1}{p}\left|\nabla v^{\varepsilon_{j}}\right|^{p}+B_{\varepsilon_{j}}\left(v^{\varepsilon_{j}}\right)\right) d x \\
& \leq \int_{\partial \mathcal{R}^{+} \cap\left\{x_{1}=1\right\}}\left(\frac{1}{p}\left|\nabla v^{\varepsilon_{j}}\right|^{p}+B_{\varepsilon_{j}}\left(v^{\varepsilon_{j}}\right)\right) d x^{\prime} .
\end{aligned}
$$

Using again that $v^{\varepsilon_{j}} \rightarrow u=\alpha x_{1}^{+}+\alpha x_{1}^{-}$uniformly on compact subsets of $\mathcal{R}_{2}$, we have $\left|\nabla v^{\varepsilon_{j}}\right| \rightarrow \alpha$ uniformly and $B_{\varepsilon_{j}}\left(v^{\varepsilon_{j}}\right)=M$ on $\partial \mathcal{R}^{+} \cap\left\{x_{1}=1\right\}$, and therefore

$$
\begin{equation*}
\underset{j \rightarrow \infty}{\limsup }\left(E_{j}+G_{j}\right) \leq \int_{\mathcal{R}^{+} \cap\left\{x_{1}=1\right\}}\left(\frac{1}{p} \alpha^{p}+M\right) d x^{\prime} \tag{3.10}
\end{equation*}
$$

Combining (3.9) and (3.10) we obtain

$$
\alpha^{p} \leq \frac{1}{p} \alpha^{p}+M
$$

or equivalently

$$
\alpha \leq\left(\frac{p}{p-1} M\right)^{1 / p}
$$

which is a contradiction. The proof is thus complete.
Lemma 3.7. Let $w^{\varepsilon}\left(x_{1}\right)$ be a strictly increasing solution of

$$
\left(\left|w^{\varepsilon^{\prime}}\right|^{p-2} w^{\varepsilon^{\prime}}\right)^{\prime}=\beta_{\varepsilon}\left(w^{\varepsilon}\right)
$$

on $\mathbb{R}$, and $v^{\varepsilon}(x)$ be a solution of $\Delta_{p} v^{\varepsilon}=\beta_{\varepsilon}\left(v^{\varepsilon}\right)$ in $\mathcal{R}=\left\{x=\left(x_{1}, x^{\prime}\right) \mid a<\right.$ $\left.x_{1}<b,\left|x^{\prime}\right|<r\right\}$, continuous up to $\partial \mathcal{R}$. Then the following comparison principle holds: if $v^{\varepsilon}(x) \geq w^{\varepsilon}\left(x_{1}\right)$ for all $x \in \partial \mathcal{R}$, then $v^{\varepsilon}(x) \geq w^{\varepsilon}\left(x_{1}\right)$ for all $x \in \mathcal{R}$.

Proof. Without loss of generality we assume that $w^{\varepsilon}(0)=0$. Since $w^{\varepsilon}$ is strictly increasing, we can find $\tau>\max \{|a|,|b|\}$ such that

$$
w^{\varepsilon}\left(x_{1}-\tau\right)<v^{\varepsilon}(x) \quad \text { on } \overline{\mathcal{R}}
$$

For $\eta>0$ sufficiently small define $w^{\varepsilon, \eta}\left(x_{1}\right):=w^{\varepsilon}\left(\varphi_{\eta}\left(x_{1}-c_{\eta}\right)\right)$, where $\varphi_{\eta}(s)=$ $s+\eta s^{2}$ and $c_{\eta}>0$ is the smallest constant such that $\varphi_{\eta}\left(s-c_{\eta}\right) \leq s$ on $[-2 \tau, 2 \tau]$. By the construction we readily have $w^{\varepsilon, \eta} \leq w^{\varepsilon}$ on $[-2 \tau, 2 \tau]$ and the straightforward computation shows that

$$
\begin{aligned}
& \Delta_{p} w^{\varepsilon, \eta}=\left(\varphi_{\eta}^{\prime}\right)^{p}\left(\Delta_{p} w^{\varepsilon}\right)\left(\varphi_{\eta}\right)+\left(\left(w^{\varepsilon}\right)^{\prime}\left(\varphi_{\eta}\right)\right)^{p-1} \Delta_{p} \varphi_{\eta} \\
&>\left(\varphi_{\eta}^{\prime}\right)^{p} \beta_{\varepsilon}\left(w^{\varepsilon, \eta}\right) \geq \beta_{\varepsilon}\left(w^{\varepsilon, \eta}\right)
\end{aligned}
$$

on $[-2 \tau, 2 \tau]$. The first inequality follows from the observation that $\Delta_{p} \varphi_{\eta}>0$ on $[-2 \tau, 2 \tau$ ] (for small $\eta>0$ ) and the second inequality follows from the fact that $x_{1} \leq c_{\eta}$ implies $\beta_{\varepsilon}\left(w^{\varepsilon, \eta}\right)=0$ and that for $x_{1} \geq c_{\eta}$ we have $\varphi_{\eta}^{\prime} \geq 1$.

Summarizing the construction above, we see that, for small $\eta>0$, the function $w^{\varepsilon, \eta}$ is strictly increasing on the interval $[-2 \tau, 2 \tau]$, and satisfies $\Delta_{p} w^{\varepsilon, \eta}>$ $\beta_{\varepsilon}\left(w^{\varepsilon, \eta}\right)$ and $w^{\varepsilon, \eta} \leq w^{\varepsilon}$. Moreover, as $\eta \rightarrow 0, w^{\varepsilon, \eta}$ converges uniformly to $w^{\varepsilon}$.

Let now $\tau^{*} \geq 0$ be the smallest constant with the property

$$
w^{\varepsilon, \eta}\left(x_{1}-\tau^{*}\right) \leq v^{\varepsilon}(x) \quad \text { on } \overline{\mathcal{R}} .
$$

Evidently, $\boldsymbol{T}_{*}<\boldsymbol{\tau}$, and in fact, we claim that $\boldsymbol{\tau}^{*}=0$. Indeed, the minimality of $\tau^{*}$ implies that there is a point $x^{*} \in \overline{\mathcal{R}}$ such that $w^{\varepsilon, \eta}\left(x_{1}^{*}-\tau^{*}\right)=v^{\varepsilon}\left(x^{*}\right)$. If $\tau^{*}>0$, we have $w^{\varepsilon, \eta}\left(\cdot-\tau^{*}\right)<w^{\varepsilon, \eta} \leq w^{\varepsilon} \leq v^{\varepsilon}$ on $\partial \mathcal{R}$, and hence $x^{*}$ is an interior point of $\mathcal{R}$. At this point we observe that the gradient of $w^{\varepsilon, \eta}\left(x_{1}-\tau^{*}\right)$ is non-degenerate. We can thus apply the strong comparison principle for the $p$-Laplacian to obtain a contradiction, since at $x^{*}$ we have $\Delta_{p} w^{\varepsilon, \eta}\left(x_{1}^{*}-\tau^{*}\right)>$ $\Delta_{p} v^{\varepsilon}\left(x^{*}\right)$. This shows that $\tau^{*}=0$, and in particular that $w^{\varepsilon, \eta} \leq v^{\varepsilon}$ on $\overline{\mathcal{R}}$. Letting $\eta \rightarrow 0$, we conclude the proof of the lemma.

## 4. Asymptotic Behavior of Limit Solutions

In this section we prove the asymptotic development of solutions to $\left(P_{\varepsilon}\right)$. We begin with the relevant definitions.

Definition 4.1. A unit vector $\eta \in \mathbb{R}^{N}$ is said to be the inward unit normal in the measure theoretic sense to the free boundary $\partial\{u>0\}$ at a point $x_{0} \in \partial\{u>0\}$ if

$$
\begin{equation*}
\lim _{r \rightarrow 0} \frac{1}{r^{N}} \int_{B_{r}\left(x_{0}\right)}\left|\chi_{\{u>0\}}-\chi_{\left\{x \mid\left\langle x-x_{0}, \eta\right\rangle>0\right\}}\right| d x=0 \tag{4.1}
\end{equation*}
$$

Definition 4.2. Let $v$ be a continuous function in a domain $\Omega \subset \mathbb{R}^{N}$. We say that $v$ is non-degenerate at a point $x_{0} \in \Omega \cap\{v=0\}$ if there exist $c, r_{0}>0$ such that

$$
\frac{1}{r^{N}} \int_{B_{r}\left(x_{0}\right)} v d x \geq c r \quad \text { for any } r \in\left(0, r_{0}\right)
$$

The main result in this section is the following.
Theorem 4.3. Let $u^{\varepsilon_{j}}$ be solutions to $\left(P_{\varepsilon_{j}}\right)$ in a domain $\Omega \subset \mathbb{R}^{N}$ such that $u^{\varepsilon_{j}} \rightarrow u$ uniformly on compact subsets of $\Omega$ and $\varepsilon_{j} \rightarrow 0$. Let $x_{0} \in \Omega \cap \partial\{u>0\}$ be such that $\partial\{u>0\}$ has an inward unit normal $\eta$ in the measure theoretic sense at $x_{0}$, and suppose that $u$ is non-degenerate at $x_{0}$. Under these assumptions, we have

$$
u(x)=\left(\frac{p}{p-1} M\right)^{1 / p}\left\langle x-x_{0}, \eta\right\rangle^{+}+o\left(\left|x-x_{0}\right|\right)
$$

The proof of Theorem 4.3 relies heavily on the following result.
Theorem 4.4. Let $u^{\varepsilon_{j}}$ be a solution to $\left(P_{\varepsilon_{j}}\right)$ in a domain $\Omega \subset \mathbb{R}^{N}$ such that $u^{\varepsilon_{j}} \rightarrow u$ uniformly on compact subsets of $\Omega$ as $\varepsilon_{j} \rightarrow 0$. Then

$$
\limsup _{x \rightarrow x_{0}}|\nabla u(x)| \leq\left(\frac{p}{p-1} M\right)^{1 / p}
$$

Proof. Let $\alpha=\lim \sup _{x \rightarrow x_{0}}|\nabla u(x)|$. Since $u \in \operatorname{Lip}(\Omega)$, clearly $\alpha<\infty$. If $\alpha=0$, there is nothing to prove, so we may assume $\alpha>0$. There exists a sequence $x_{n} \rightarrow x_{0}$ such that $\left|\nabla u\left(x_{n}\right)\right| \rightarrow \alpha$ and $u\left(x_{n}\right)>0$. Let $z_{n} \in \Omega \cap \partial\{u>0\}$ be such that $d_{n}=\left|z_{n}-x_{n}\right|=\operatorname{dist}\left(x_{n}, \partial\{u>0\}\right)$. Define $u_{d_{n}}(x)=\left(1 / d_{n}\right) u\left(z_{n}+\right.$ $\left.d_{n} x\right)$. Since $u \in \operatorname{Lip}(\Omega)$ and $u_{d_{n}}(0)=0$ for every $n,\left\{u_{d_{n}}\right\}$ is uniformly bounded on compact subsets of $\mathbb{R}^{N}$, and therefore for a subsequence (still denoted by $d_{n}$ ) $u_{d_{n}} \rightarrow u_{0}$ uniformly on compact subsets of $\mathbb{R}^{N}$, where $u_{0} \in \operatorname{Lip}\left(\mathbb{R}^{N}\right)$.

Now, set $\bar{x}_{n}=\left(x_{n}-z_{n}\right) / d_{n} \in \partial B_{1}$. We may choose the subsequence $d_{n}$ so that $\bar{x}_{n} \rightarrow \bar{x} \in \partial B_{1}$. Then $u_{0}$ is $p$-harmonic and nonnegative in $B_{1}(\bar{x})$. Consider now the sequence

$$
v_{n}=\frac{\nabla u_{d_{n}}\left(\bar{x}_{n}\right)}{\left|\nabla u_{d_{n}}\left(\bar{x}_{n}\right)\right|}=\frac{\nabla u\left(x_{n}\right)}{\left|\nabla u\left(x_{n}\right)\right|} .
$$

Passing to a subsequence, we assume, without loss of generality, that $v_{n} \rightarrow \mathbf{e}_{1}$. At this point we observe that $B_{2 / 3}(\bar{x}) \subset B_{1}\left(\bar{x}_{n}\right)$ for $n$ sufficiently large, and therefore $u_{d_{n}}$ is $p$-harmonic in $B_{2 / 3}(\bar{x})$. By interior gradient estimates, $u_{d_{n}} \rightarrow u_{0}$ in $C^{1, \sigma}\left(B_{1 / 2}(\bar{x})\right)$ for some $\sigma>0$. This suffices to show that $\nabla u_{d_{n}} \rightarrow \nabla u_{0}$ uniformly in $B_{1 / 3}(\bar{x})$ and thus, as a consequence, $\left|\nabla u\left(x_{n}\right)\right| \rightarrow \partial_{x_{1}} u_{0}(\bar{x})$. In particular, $\partial_{x_{1}} u_{0}(\bar{x})=\alpha$.

Next, it is easy to show that $\left|\nabla u_{0}\right| \leq \alpha$ in $\mathbb{R}^{N}$. Indeed, let $R>1, \delta>0$. Then there exists $\tau_{0}>0$ such that $|\nabla u(x)| \leq \alpha+\delta$ for any $x \in B_{\tau_{0} R}\left(x_{0}\right)$. Observe now that $\left|x_{n}-x_{0}\right|<\tau_{0} R / 2$ and $d_{n}<\tau_{0} / 2$ imply $B_{d_{n} R}\left(x_{n}\right) \subset B_{\tau_{0} R}\left(x_{0}\right)$, and therefore $\left|\nabla u_{d_{n}}(x)\right| \leq \alpha+\delta$ in $B_{R}$ for $n$ large enough. In particular, $\nabla u_{d_{n}} \rightarrow$ $\nabla u_{0} *$-weakly in $L^{\infty}\left(B_{R}\right)$ and thus $\left|\nabla u_{0}\right| \leq \alpha+\delta$ in $B_{R}$. Since $\delta$ and $R$ are arbitrary, we conclude

$$
\left|\nabla u_{0}\right| \leq \alpha \quad \text { in } \mathbb{R}^{N}
$$

Let $w=\partial_{x_{1}} u_{0}$, which is a weak solution to the equation

$$
\partial_{x_{j}}\left(\theta_{i j}\left(\nabla u_{0}\right) \partial_{x_{i}} w\right)=0 \quad \text { in } B_{1}(\bar{x})
$$

with $\theta_{i j}(\xi)=(p-2)|\xi|^{p-4} \xi_{i} \xi_{j}+\delta_{i j}|\xi|^{p-2} \xi_{i j}$. We also know that $w \leq \alpha$ in $B_{1}(\bar{x}), w(\bar{x})=\alpha$. By the maximum principle (recall that $\partial_{x_{1}} u_{0}(\bar{x})=\alpha>0$ ), $w \equiv \alpha$ in $B_{1}(\bar{x})$ and so $u_{0}(x)=\alpha\left(x_{1}-y_{1}\right)$ in $B_{1}(\bar{x})$ for some $y \in \mathbb{R}^{N}$. It is not difficult to recognize that

$$
u_{0}(x)=\alpha\left(x_{1}-y_{1}\right) \quad \text { in }\left\{x_{1} \geq y_{1}\right\} .
$$

We now apply Lemma A. 1 from the Appendix to $u_{0}$ in $\left\{x_{1}-y_{1}<0\right\}$ and obtain

$$
u_{0}(x)=\gamma(x-y)_{1}^{-}+o(|x-y|) \quad \text { in }\left\{x_{1}-y_{1}<0\right\}
$$

for some $\gamma \geq 0$. Define, for $\lambda>0,\left(u_{0}\right)_{\lambda}(x)=(1 / \lambda) u_{0}(\lambda x+y)$. There exist a sequence $\lambda_{n} \rightarrow 0$ and a function $u_{00} \in \operatorname{Lip}\left(\mathbb{R}^{N}\right)$ such that $\left(u_{0}\right)_{\lambda_{n}} \rightarrow u_{00}$ uniformly on compact sets of $\mathbb{R}^{N}$. We have $u_{00}(x)=\alpha x_{1}^{+}+\gamma x_{1}^{-}$in $\mathbb{R}^{N}$. By Lemma 3.3, there exists a sequence $\varepsilon_{j}^{0} \rightarrow 0$ such that $\mathcal{u}^{\varepsilon_{j}^{0}}$ is a solution to $\left(P_{\varepsilon_{j}^{0}}\right)$ and $u^{\varepsilon_{j}^{0}} \rightarrow u_{0}$ uniformly on compact subsets of $B_{1}$. Applying Lemma 3.3 again (the second "blowup") we find a sequence $\varepsilon_{j}^{00} \rightarrow 0$ and solutions $u^{\varepsilon_{j}^{00}}$ to $\left(P_{\varepsilon_{j}^{00}}\right)$ converging uniformly on compact subsets of $B_{1}$ to $u_{00}(x)=\alpha x_{1}^{+}+\gamma x_{1}^{-}$. Finally, we may apply Proposition 3.4 or Proposition 3.6, depending on whether $\gamma=0$ or $\gamma>0$, to conclude $\alpha \leq((p /(p-1)) M)^{1 / p}$.

We are now ready to prove Theorem 4.3.
Proof of Theorem 4.3. Without loss of generality we may assume $x_{0}=0$ and $\eta=\mathbf{e}_{1}$. Define, for $\lambda>0$,

$$
u_{\lambda}(x)=\frac{1}{\lambda} u(\lambda x),
$$

and let $\rho>0$ be such that $B_{\rho} \Subset \Omega$. Since $u_{\lambda} \in \operatorname{Lip}\left(B_{\rho / \lambda}\right)$ uniformly in $\lambda$, and $u_{\lambda}(0)=0$, there exist a subsequence $\lambda_{j} \rightarrow 0$ and a function $U \in \operatorname{Lip}\left(\mathbb{R}^{N}\right)$ such that $u_{\lambda_{j}} \rightarrow U$ uniformly on compact subsets of $\mathbb{R}^{N}$. From Lemmas 3.1 and 3.3 it follows that $u_{\lambda}$ is $p$-harmonic in its positivity set $\left\{u_{\lambda}>0\right\}$. Next, rescaling (4.1) we see that, for every fixed $k>0$

$$
\left|\left\{u_{\lambda}>0\right\} \cap\left\{x_{1}<0\right\} \cap B_{k}\right| \rightarrow 0 \quad \text { as } \lambda \rightarrow 0 .
$$

Hence, $U$ is nonnegative in $\left\{x_{1}>0\right\}$, $p$-harmonic in $\{U>0\}$, and vanishes on $\left\{x_{1} \leq 0\right\}$. By Lemma A.1, there exists $\alpha \geq 0$ such that

$$
U(x)=\alpha x_{1}^{+}+o(|x|) \quad \text { in }\left\{x_{1}>0\right\} .
$$

By virtue of Lemma 3.3, we can find a sequence $\varepsilon_{j}^{\prime} \rightarrow 0$ and solutions $u^{\varepsilon_{j}^{\prime}}$ to $\left(P_{\varepsilon_{j}^{\prime}}\right)$ such that $u^{\varepsilon_{j}^{\prime}} \rightarrow U$ uniformly on compact sets of $\mathbb{R}^{N}$ as $j \rightarrow \infty$.

On the other hand, if we define $U_{\lambda}(x)=(1 / \lambda) U(\lambda x), U_{\lambda} \rightarrow \alpha x_{1}^{+}$uniformly on compact subsets of $\mathbb{R}^{N}$ as $\lambda \rightarrow 0$. Applying Lemma 3.3 the second time, we find now a sequence $\sigma_{j} \rightarrow 0$ and solutions $u^{\sigma_{j}}$ to $\left(P_{\sigma_{j}}\right)$ such that $u^{\sigma_{j}} \rightarrow \alpha x_{1}^{+}$ uniformly on compact subsets of $\mathbb{R}^{N}$, and

$$
\begin{equation*}
\nabla u^{\sigma_{j}} \rightarrow \alpha \chi_{\left\{x_{1}>0\right\}} \mathbf{e}_{1} \quad \text { in } L_{\mathrm{loc}}^{p}\left(\mathbb{R}^{N}\right), \tag{4.2}
\end{equation*}
$$

by virtue of Lemma 3.1.
Further, we proceed as in the proof of Proposition 3.4. Let $\psi \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$ and choose $u_{x_{1}}^{\sigma_{j}} \psi$ as a test function in the weak formulation of $\Delta_{p} u^{\sigma_{j}}=\beta_{\sigma_{j}}\left(u^{\sigma_{j}}\right)$. Then we find that $B_{\sigma_{j}}\left(u^{\sigma_{j}}\right) \rightarrow M_{X}\left\{x_{1}>0\right\}+\bar{M} X_{\left\{x_{1}<0\right\}} *$-weakly in $L^{\infty}(\Omega)$, and $((p-1) / p) \alpha^{p}=M-\bar{M}$. We now claim that either $\bar{M}=0$ or $\bar{M}=M$. To this end, let $K \Subset\left\{x_{1}<0\right\}$. Then for any $\varepsilon>0$ there exists $0<\delta<1$ such that

$$
\begin{aligned}
\left|K \cap\left\{\varepsilon<B_{\sigma_{j}}\left(u^{\sigma_{j}}\right)<M-\varepsilon\right\}\right| & \leq\left|K \cap\left\{\delta<\frac{u^{\sigma_{j}}}{\sigma_{j}}<1-\delta\right\}\right| \\
& \leq\left|K \cap\left\{\beta_{\sigma_{j}}\left(u^{\sigma_{j}}\right) \geq \frac{a}{\sigma_{j}}\right\}\right| \rightarrow 0
\end{aligned}
$$

as $j \rightarrow \infty$, where $a=\inf _{[\delta, 1-\delta]} \beta>0$, since $\beta_{\sigma_{j}}\left(u^{\sigma_{j}}\right) \rightarrow 0$ in $L^{1}(K)$ by virtue of Proposition 3.2. At this point we observe that, thanks to a simple compactness argument, the latter fact also implies that the convergence of $B_{\sigma_{j}}\left(u^{\sigma_{j}}\right)$ to $\bar{M}$ is actually in $L_{\text {loc }}^{1}\left(\left\{x_{1}<0\right\}\right)$. We may thus conclude

$$
|K \cap\{\varepsilon<\bar{M}<M-\varepsilon\}|=0
$$

for every $\varepsilon>0$ and $K \Subset\left\{x_{1}<0\right\}$. Hence, the claim is proved.
Let us see now that $\alpha>0$. By virtue of the non-degeneracy assumption on $u$ at 0 , for every $r>0$ and $j$ sufficiently large,

$$
\frac{1}{r^{N}} \int_{B_{r}} u_{\lambda_{j}} \geq c r
$$

and passing to the limit as $j \rightarrow \infty$,

$$
\frac{1}{r^{N}} \int_{B_{r}} U \geq c r .
$$

Clearly, this forces $\alpha>0$, and as a consequence, $\bar{M}=0$ and $\alpha=$ $((p /(p-1)) M)^{1 / p}$. We have thus shown that

$$
U(x)= \begin{cases}\left(\frac{p}{p-1} M\right)^{1 / p} x_{1}+o(|x|), & x_{1}>0 ;  \tag{4.3}\\ 0, & x_{1} \leq 0 .\end{cases}
$$

Next, it follows from Theorem 4.4 that

$$
|\nabla U| \leq\left(\frac{p}{p-1} M\right)^{1 / p} \quad \text { in } \mathbb{R}^{N} .
$$

At this point it suffices to observe that $U \equiv 0$ on $\left\{x_{1}=0\right\}$ to conclude that $U \leq((p /(p-1)) M)^{1 / p} x_{1}$ in $\left\{x_{1}>0\right\}$. Applying Hopf's boundary principle we see that necessarily

$$
U(x)=\left(\frac{p}{p-1} M\right)^{1 / p} x_{1} \quad \text { in }\left\{x_{1}>0\right\} .
$$

As an immediate consequence we finally obtain

$$
u(x)=\left(\frac{p}{p-1} M\right)^{1 / p} x_{1}^{+}+o(|x|),
$$

and the proof is complete.

## Appendix A.

Here we prove an analogue of Corollary A. 1 in [6] on asymptotic development of $p$-harmonic functions near flat boundary points, by a slight modification of the proof of Lemma A. 1 in [6].

Lemma A.1. Let $U \in \operatorname{Lip}\left(\bar{B}_{1}^{+}\right)$and assume that $U$ is nonnegative in $B_{1}^{+}, p-$ harmonic in $\{U>0\}$ and vanishes on $\partial B_{1}^{+} \cap\left\{x_{1}=0\right\}$. Then in $B_{1}^{+}, U$ has the asymptotic development

$$
U(x)=\alpha x_{1}+o(|x|)
$$

with $\alpha \geq 0$.
Proof. Let $\ell_{k}:=\inf \left\{l \mid U(x) \leq l x_{1}\right.$ in $\left.B_{2-k}^{+}\right\}$. Since $\ell_{k}$ is a nonincreasing sequence of nonnegative finite numbers, there exists $\alpha=\lim _{k \rightarrow \infty} \ell_{k}$. Then

$$
U(x) \leq \alpha x_{1}+o(|x|) \quad \text { in } B_{1}^{+} .
$$

If $\alpha=0$, the conclusion of the lemma will follow. Assume therefore that $\alpha>0$. Then there exists a sequence $x^{k} \in B_{1}^{+}$with $r_{k}=\left|x^{k}\right| \rightarrow 0$ such that

$$
U\left(x^{k}\right) \leq \alpha x_{1}^{k}-\delta_{0} r_{k},
$$

for some $\delta_{0}>0$. Note that $x^{k}$ will belong to the cone $C=\left\{|x| \leq\left(\alpha / \delta_{0}\right) x_{1}\right\}$, and thus we can assume that $v_{k}:=x^{k} / r_{k}$ converges to some $v_{0} \in \partial B_{1}^{+} \cap C$. Next, let

$$
U^{k}(x):=\frac{U\left(r_{k} x\right)}{r_{k}} .
$$

Since $\left\{U^{k}\right\}$ are uniformly Lipschitz, we may assume that $U_{k}$ converges uniformly on $\bar{B}_{1}^{+}$to a nonnegative function $V$. Then from the construction we will have $V(x) \leq \alpha x_{1}$ in $B_{1}^{+}$, and in addition

$$
V(x) \leq \alpha x_{1}-\frac{\delta_{0}}{2} \quad \text { and } \quad U^{k}(x) \leq \ell_{k} x_{1}-\frac{\delta_{0}}{2} \quad \text { on } \partial B_{1}^{+} \cap B_{\varepsilon}\left(v_{0}\right)
$$

for a sufficiently small $\varepsilon>0$ and large $k$.
Let now $w$ be a $p$-harmonic function in $B_{1}^{+}$with smooth boundary values, such that

$$
\begin{array}{ll}
w=x_{1} & \text { on } \partial B_{1}^{+} \backslash B_{\varepsilon / 2}\left(v_{0}\right) \\
w=x_{1}-\frac{\delta_{0}}{4 \alpha} & \text { on } \partial B_{1}^{+} \cap B_{\varepsilon / 4}\left(v_{0}\right), \\
x_{1}-\frac{\delta_{0}}{4 \alpha} \leq w \leq x_{1} & \text { on } \partial B_{1}^{+} \cap B_{\varepsilon / 2}\left(v_{0}\right) .
\end{array}
$$

Then $w$ vanishes on the flat boundary $\left\{x_{1}=0\right\} \cap B_{1}$, it is $C^{1, \sigma}$ up to $\left\{x_{1}=\right.$ $0\} \cap B_{1}$, and by the Hopf boundary principle

$$
w(x) \leq(1-\mu) x_{1} \quad \text { in } B_{\gamma}^{+}
$$

for some small $\mu, \gamma$.
Now, from the comparison principle we will have $U^{k} \leq \ell_{k} w$ in $B_{1}^{+}$for large $k$, and consequently $U^{k} \leq \ell_{k}(1-\mu) x_{1}$ in $B_{\gamma}^{+}$. This implies $\alpha \leq(1-\mu) \alpha$, which contradicts the assumption that $\alpha>0$. The lemma is proved.

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