Letters to the Editor

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A Solution of the Combined Gravitational and Mesic Field Equations in General Relativity

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Tolman[1)] has pointed out the paucity of solutions of the non-linear field-equations of General Relativity:

$$\text{(a)}\quad -8\pi T_1^{\,1}=e^{-\lambda}(\nu'/r+1/r^2)-1/r^2+\Lambda$$

$$\text{(b)}\quad -8\pi T_2^{\,2}=-8\pi T_3^{\,3}=e^{-\lambda}\Big\{\nu''/2-\lambda'\nu'/4+(\nu')^2/4+\frac{\nu'-\lambda'}{2r}\Big\}+\Lambda \qquad (1)$$

$$\text{(c)}\quad -8\pi T_4^{\,4}=-e^{-\lambda}(\lambda'/r-1/r^2)-1/r^2+\Lambda.$$

The same reasons contribute to the very great difficulties in finding out the exact solutions of the combined gravitational and electromagnetic fields,—the simplest of which is the Reissner-Nordström solution. Further solutions have been attempted[2),3),4)] following Tolman's method[1)] of assuming suitable values of e^{ν} or e^{λ} and then fixing p and ρ. As a further exercise of this type, we may attempt solutions of the combined gravitational and mesic fields.

In spin 0 case the mesic tensor is[5)]

$$m^{\mu\nu}=\bar\phi^{\mu}\bar\phi^{\nu}+\tfrac{1}{2}g^{\mu\nu}(\mu^2\bar\phi^2-\bar\phi_{\lambda}\bar\phi^{\lambda})$$
$$\bar\phi_{\mu}=\partial\bar\phi/\partial x^{\mu}. \qquad (2)$$

Further, in this case, we may establish a scalar potential $\bar\phi=1/r\cdot e^{-kr}$.

Taking the interval as $ds^2=e^{\lambda}dr^2-\cdots+e^{\nu}dt^2$ and considering spherical symmetry and $\partial\bar\phi/\partial t=0$, we find

$$m^{\mu\nu}=\bar\phi^{\mu}\bar\phi^{\nu}+\tfrac{1}{2}g^{\mu\nu}[\mu^2\bar\phi^2-\bar\phi_{\lambda}\bar\phi^{\lambda}]$$

where $\quad \bar\phi_{\lambda}=\partial\bar\phi/\partial r,$

for $\bar\phi$ depends on r only. We now introduce $\bar\phi^{\mu}=\exp\Big\{-\frac{1}{2}(\lambda+\nu)\Big\}\frac{\partial\phi}{\partial r}$

[cf. the analogous case of an electromagnetic field (6)], which, as we shall see, admits of a certain solution of the field equations.

Thus

$$m^{11}=g^{44}g^{11}X^2+\tfrac{1}{2}g^{11}[\mu^2\phi^2\cdots]$$
$$m^{22}=m^{33}=m^{44}=\tfrac{1}{2}[\mu^2\bar\phi^2-(X^2)\bar\phi^{\lambda}] \qquad (3)$$

putting $\quad X=\partial\bar\phi/\partial r$
$$=[(e^{-kr}/r)(-k)-e^{-\kappa r}/r^2].$$

Thus we have for the space-time around a particle of mesic charge (within $r>0$ to $r\sim10^{-13}$) the field equations

$$-F_1^{\,1}=-(T_1^{\,1}+m_1^{\,1})=g^{44}X^2+\tfrac{1}{2}[\mu^2\phi^2-\bar\phi_{\lambda}\bar\phi^{\lambda}]+e^{-\lambda}(\cdots$$
$$-F_2^{\,2}=-(T_2^{\,2}+m_2^{\,2})$$

$$=\frac{1}{2}(\mu^2\bar{\phi}^2-\bar{\phi}_\lambda\bar{\phi}^\lambda)+e^{-\lambda}(\nu''/2\cdots$$

$$-F_4^4=-(T_4^4+m_4^4)$$

$$=\frac{1}{2}(\mu^2\bar{\phi}^2-\bar{\phi}_\lambda\bar{\phi}^\lambda)-e^{-\lambda}(\lambda'/r\cdots. \quad (4)$$

Transforming these according to Tolman[1] we have finally $e^\nu=$Const

$$\text{(a)}\quad (e^{-\lambda}-1)/r^2 = \int 2/r\cdot e^\nu X^2 dr$$

$$\text{(b)}\quad p+m_2^2=e^{-\lambda}/r^2-1/r^2+\Lambda \qquad (5)$$

$$\text{(c)}\quad \rho+m_4^4=e^{-\lambda}(\lambda'/r-1/r^2) +1/r^2-\Lambda.$$

(5a) immediately yields the value of $e^{-\lambda}$. This together with the help of (5b) fixes the value of p. Thus, by following this method, we cannot find a solution of the mesic field occurring in strictly empty space. Imagining, therefore, a nucleon in non-empty space ($p, \rho\neq 0$), we find a possible approximate solution of (3):
(considering that $1/r$ is negligible with respect to $1/r^2$ ($r\gtreqless 10^{-13}$))

$$e^\nu=\text{const}$$

$$e^\lambda=1/(1+e^\nu e^{kr}+(r^2)).$$

We can have positive values of ρ within a certain region. The negative value of pressure may correspond to the short range attractive force.

This solution is valid within a certain region $r=r_1$, the radius of the nucleon up to $r\sim 10^{-13}$.

1) R. C. Tolman, Phys. Rev. **55** (1939), 364.
2) R. L. Brahmachary, Nuovo Cimento **4** (1956), 1216.
3) R. L. Brahmachary, Nuovo Cimento **5** (1957), 1250.
4) R. L. Brahmachary, Nuovo Cimento **6** (1957), 1502.
5) Schweber, Bethe, and Hoffmann, *Fields and Mesons*, p. 100.
6) A. S. Eddington, *The Mathematical Theory of Relativity*, p. 185 (1924).