A sparse h-adaptive boundary integral equation solution for the 2D Laplace equation using multi-wavelets

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Abstract

We present a sparse h-adaptive boundary integral equation solution for the 2D Laplace equation using the multi-wavelets of Alpert. we show that using the zero moment properties and the compact support properties of the multi-wavelet basis we can produce an auto-refining method. Furthermore using the same properties of the multi-wavelets on the matrices, they can be made sparse. Unfortunately the structure of the sparse matrices makes it very difficult to make use of fast iterative solvers. We show that the h-adaption proceeds in an identical fashion for both the untruncated and truncated system matrices (even with very severe truncation of modest sized system matrices).

1 Introduction

In this paper we give an h-adaptive method for the 2D Boundary Element Method (BEM) for Laplaces equation. The h-adaptivity arises from the use of the multi-wavelets of Alpert [1, 2]. Multi-wavelets are part of the more general subject of wavelets which is a relatively recent development in numerical analysis. Because of its vast potential wavelet analysis has been applied very rapidly to almost all areas of numerical approximation. See Chui [3] for a broad introduction to wavelets, the seminal text is still

Daubechies [6], though much of her work is more applicable to Signal Processing.

The wavelet bases are used to approximate functions or data and can be constructed to have very specific properties which suits the application at hand. Hence to describe wavelets in general is no easy task if all cases are to be included. We have opted instead to focus entirely upon the multi-wavelets of Alpert and even when we refer to multi-wavelets we are only considering a very small subset of all the possibilities.

Multi-wavelets have four very desirable properties; firstly they form an orthonormal basis. This enables the energy norm of the approximate solution to be rapidly calculated; secondly they are compactly supported, which means there is only a finite range in which thay are non-zero; thirdly the ordering of the basis elements can be viewed as a tree and it is this property which enables auto-refining to occur in a relatively straight-forward manner; finally they have a high zero moment property which provides a mechanism to drive the auto refining and it also enables the system matrices to be made sparse with little loss of accuracy by truncation of the small elements. It should be noted however that this sparsity cannot be fully exploited in a collocation scheme because of the complete loss of diagonal dominace shown in figure 2. Whereas a Galerkin method based on multi-wavelets retains its symmetry and diagonal dominace which enables the system matrix to be solved by a number of sparse iterative techniques that are much faster than LU decomposition.

It is this truncation of the matrix which gives the possibility of overcoming one of the disadvantages of the BEM, namely the denseness of the matrices that are produced. Many of the issues of sparsity have been considered by von Petersdorf and Schwab [10], Prössdorf and Schneider [7, 8] and Dahmen et al [5].

2   Multi-wavelets and auto-refining

A multi-wavelet is an element of a basis which can represent a general function. Wavelet bases differ from other more 'traditional' bases such as Fourier or spline in that the basis is constructed so that its elements can be ordered into a hierarchy. For approximations on a finite interval an additional set of basis functions are required—called scaling functions—which make the basis complete. In the case of multi-wavelets these are the Legendre polynomials. The orthogonalised Legendre polynomials up to quadratic order are given in Table 1 and the multi-wavelet basis elements in Table 2.

The wavelet basis functions are all related to a principal set of wavelets, $\phi^i(x)$, which are then scaled (squashed) and translated (shifted) to produce all the other members of the basis. For the multi-wavelets supported on

Order N	Legendre Polynomials $[-1,1)$
0	$\sqrt{\frac{1}{2}}$
1	$\sqrt{\frac{3}{2}}x$
2	$\frac{1}{2}\sqrt{\frac{5}{2}}(3x^2 - 1)$

Table 1: The Legendre polynomials up to order 2

Order N	Wavelets	
	$[-1,0)$	$[0,1)$
0	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
1	$\sqrt{\frac{3}{2}}(-1 - 2x)$ $\sqrt{\frac{1}{2}}(2 + 3x)$	$\sqrt{\frac{3}{2}}(1 - 2x)$ $\sqrt{\frac{1}{2}}(-2 + 3x)$
2	$-\frac{1}{3}\sqrt{\frac{1}{2}}(1 + 24x + 30x^2)$ $\frac{1}{2}\sqrt{\frac{3}{2}}(3 + 16x + 15x^2)$ $-\frac{1}{3}\sqrt{\frac{5}{2}}(4 + 15x + 12x^2)$	$\frac{1}{3}\sqrt{\frac{1}{2}}(1 - 24x + 30x^2)$ $\frac{1}{2}\sqrt{\frac{3}{2}}(3 - 16x + 15x^2)$ $\frac{1}{3}\sqrt{\frac{5}{2}}(4 - 15x + 12x^2)$

Table 2: The multi-wavelet bases up to order 2

$[-1, 1]$ the scaling and translation formula which achieves this is

$$\psi^i_{jk}(x) := 2^{\frac{j}{2}}\psi^i(2^j(x + 1) - 2k - 1) \tag{1}$$

where the j relates to the level of the basis function, in which the higher the j the smaller the interval of support, and the k relates to translation at the jth level. It is the localization property of wavelets which makes them so attractive to almost all areas of numerical analysis. Combining the localisation property with a high zero moment property enables large dense systems of linear equations to be truncated with little loss of accuracy, as will be seen later.

Thus an arbitrary function on $[-1, 1]$ can be represented by an Ith order multi-wavelet series by

$$f(x) = \sum_{i=0}^{I}\left\{\alpha^i\phi(x) + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}\beta^i_{j,k}(x)\psi^i_{j,k}(x)\right\} \tag{2}$$

where the ϕ^i's are the orthogonalized Legendre polynomials and the ψ^i's are

the multi-wavelets, the α's and β's are given by

$$\alpha^i = \int_{-1}^{1} f(x)\phi^i(x)dx \tag{3}$$

and

$$\beta_{j,k}^i = \int_{-1+\frac{2}{2^j}k}^{-1+\frac{2}{2^j}(k+1)} f(x)\psi_{j,k}^i(x)dx \tag{4}$$

The energy or $\mathbf{L}_2(-1,1)$ norm is

$$\|f\|_{\mathbf{L}_2(-1,-1)} = \left(\int_{-1}^{1} (f(x))^2 dx \right)^{\frac{1}{2}} \tag{5}$$

$$= \left(\sum_{i=0}^{N} \left\{ (\alpha^i)^2 + \sum_j \sum_k (\beta_{j,k}^i)^2 \right\} \right)^{\frac{1}{2}} \tag{6}$$

which comes immediately from the orthogonality of the basis. The norm of the approximation is a key ingredient in the test for auto-refining which we shall use later. The other property of multi-wavelets we exploit is their zero moment properties. An arbitrary function g has M zero moments over $[-1,1]$ if

$$\int_{-1}^{1} g(x)x^n dx = 0 \quad \forall n < M. \tag{7}$$

We now give an intuitive demonstration of why the truncation of the small multi-wavelet coefficients still gives an accurate but approximate representation of an arbitrary function. The function representation given by equation (2) requires the $\beta_{j,k}^i$ coefficients to be calculated by equation (4). Expanding the function f as a Taylor series about the midpoints, x_0, of the interval of support gives

$$\beta_{j,k}^i = \int_{-1+\frac{2}{2^j}k}^{-1+\frac{2}{2^j}(k+1)} \sum_n \frac{(x-x_0)^n}{n!} f^n(x_0)\psi_{j,k}^i(x)dx \tag{8}$$

The multi-wavelets of quadratic order given in Table 2 have two, three and four zero moments respectively. For the multi-wavelet with two zero moments we would have

$$\beta_{j,k}^0 = \frac{f^2(x_0)}{2!} \int_{-1+\frac{2}{2^j}k}^{-1+\frac{2}{2^j}(k+1)} (x-x_0)^2 \psi_{j,k}^i(x)dx + R_{n\geq 3} \tag{9}$$

If the function is 'almost linear' over an interval defined by a particular j and k then its second derivative $f^2(x_0)$ will be small, which in itself will be the dominant term. Truncation of this term will thus have little effect on the norm of the approximating function and the approximation will still be

'faithful' to the original function. Furthermore because the wavelet interval of support is halved between level j and level $j + 1$, even if a function could not be represented accurately by a linear function on level j it is more likely to be approximated by a linear function on level $j + 1$ since it is adding to the overall approximation over a smaller interval of support. The zero moment property is thus used very efficiently in wavelet approximations.

This gives us an overall strategy for auto-refining a multi-wavelet series. A predefined value for ε needs to be set. The scaling function coefficients and the multi-wavelet coefficients on level 0 are then calculated. The approximate norm squared $(\|f_{app}\|^2)$ is calculated. Also the contribution to the norm from the multi-wavelets is calculated (which in the first instance would have j and k both zero) with

$$\|\hat{\beta_{j,k}}\|^2 = \sum_{i=0}^{I} (\beta_{j,k}^i)^2 \qquad (10)$$

Then if

$$\frac{\|f_{app}\|^2 - \|\hat{\beta_{j,k}}\|^2}{\|f_{app}\|^2} < \varepsilon \qquad (11)$$

the multi-wavelet coefficients contribution to the norm is sufficiently small and no more refining is necessary. Otherwise the multi-wavelet coefficients of the multi-wavelets $\psi_{j+1,2k}^i$ and $\psi_{j+1,2k+1}^i$ must also be calculated. This process is repeated until no more multi-wavelet coefficients need to be calculated because the relative norm squared condition has been fully satisfied or it has reached a preset maximum number of levels.

3   Application to BIE's

The basic form for an interior boundary integral equations for 2D Laplace equation from Hall [11] is

$$\theta_B^e U(\mathbf{y}) = \int_B \frac{\mathbf{n}(\mathbf{x}).\mathbf{r}(\mathbf{x},\mathbf{y})}{r^2(\mathbf{x},\mathbf{y})} U(\mathbf{x})dS(\mathbf{x}) - \int_B \ln |\mathbf{r}(\mathbf{x},\mathbf{y})| \frac{\partial U(\mathbf{x})}{\partial n} dS(\mathbf{x}) \qquad (12)$$

where $\mathbf{n}(\mathbf{x})$ is the outward pointing normal to the boundary, $\mathbf{r}(\mathbf{x},\mathbf{y})$ is the distance between points on the boundary given by $\mathbf{r}(\mathbf{x},\mathbf{y}) = |\mathbf{x} - \mathbf{y}|$ and θ_B^e is the exterior boundary angle at the point $\mathbf{y}$.

The boundary is approximated by N quadratic elements. We have only considered Dirichelet boundary conditions, the known flux and unknown potential are initially represented by a multi-wavelet approximation using the scaling functions and multi-wavelets on level $j = 0$ *on each element*. On each boundary element there are thus six collocation points for a quadratic multi-wavelet approximation, none of which are at the element ends.

This is thus a non-conforming method. For a collocation formulation, after dividing the boundary into elements whose geometry is represented by quadratic parametric shape functions and the representation of the functions in terms of scaling functions and multi-wavelets the following integrals (and similar integrals involving the derivative kernel) arise

$$\int_{-1}^{1} \ln |\mathbf{x}_a(t) - \mathbf{y}_b^s| \phi^i(t) J_a(t) dt \tag{13}$$

and

$$\int_{-1+\frac{2}{2^j}k}^{-1+\frac{2}{2^j}(k+1)} \ln |\mathbf{x}_a(t) - \mathbf{y}_b^s| \psi_{jk}^i(t) J_a(t) dt \tag{14}$$

where $\mathbf{x}_a(t)$ is the quadratic shape function approximation of the boundary on the ath element, $\mathbf{y}_b^s$ is the sth collocation point on the bth element and $J_a(t)$ is the Jacobian of the ath element. The point we wish to raise here is that the integrals involving the multi-wavelets must not necessarily be evaluated over the whole range of the parameter t.

Once the solution to the initial system matrix has been obtained the solution is auto-refined by the method mentioned in section 2 and a new system matrix is constructed. Much of the matrix from the previous iteration can be used. This process is repeated until either no-more auto-refining is required or the relative error increases.

These matrices can be rendered sparse by truncating the small values in the system matrix. The condition for trunction which we apply to this is for non-singular entries in the system matrix ($K_{m,n}$) if

$$K_{m,n} < \alpha 2^{-j} \tag{15}$$

where α is a preset truncation parameter and the j relates to the level of the multi-wavelet appearing in the calculation of the integral. This was tested in two cases, both of which had ε set at 0.5. Table 3 gives the results for auto-refining with no truncation of the system matrix, whilst Table 4 gives the results for the system matrix with α in equation 15 set to 0.01. In both cases the approximate solution was refined in exactly the same way. The refinement took five steps in both cases, the final tree representation of the second element is shown in Figure 3b).

4   Results

In this section we first wish to demonstrate that auto-refining works and secondly that is not effected even by severe trunction of the matrices. We have chosen

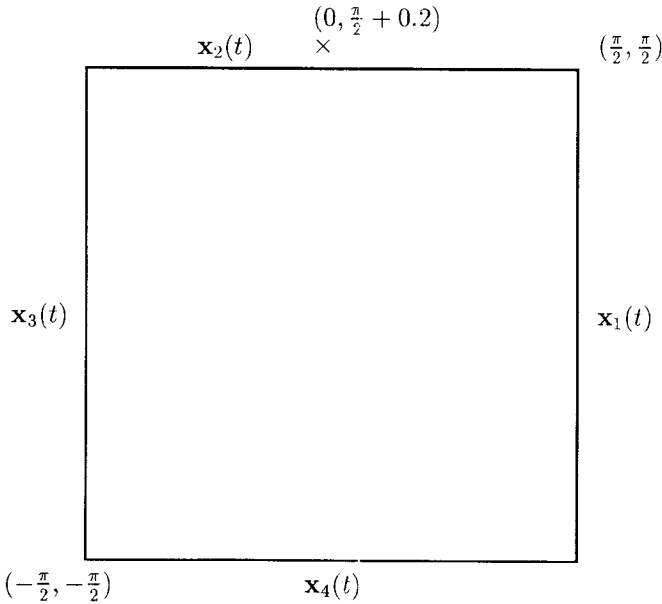$$U(x,y) = \frac{x+a}{(x-a)^2 + (y-b)^2} \tag{16}$$

Figure 1: The boundary

as the exact solution of the Laplace equation with $a = 0$ and $b = \frac{\pi}{2} + 0.2$. The boundary is a square with sides of length π and is shown in figure 1. The maximum error has been calculated by evaluating the exact solution and the approximate solution at 100 points on each boundary element including the end points.

The only visible difference between the two solutions is shown in figure 3 where at the end point of the second element the approximation 'wobbles' a bit. This feature however is mainly associated with the truncation scheme for the matrix not taking into account the corner points. The coefficients in the system matrix should be truncated will a little more care.

Step	Matrix size	Type of error	
		L_2	Max
0	24	5.0518	2.6781
1	30	5.8382	5.7055
2	42	4.7355	1.7566
3	60	1.3639	0.5812
4	78	0.6672	0.3620

Table 3: The errors for the untruncated matrix
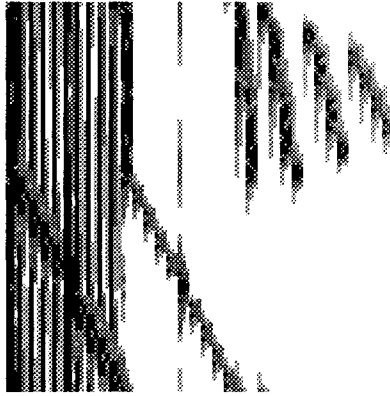
652 Boundary Elements

Figure 2: The truncated matrix

a) b)
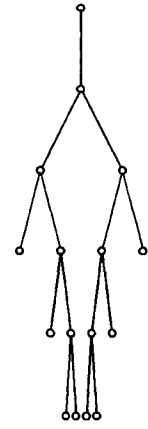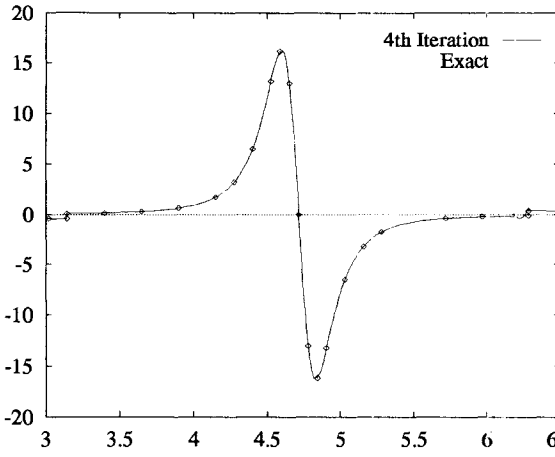
Figure 3: a) The solution and b) the tree representation of the flux on the 2nd element.

Iteration	n	Type of error		Sparsity
		L_2	Max	
0	24	5.0519	2.6775	25.0
1	30	5.8385	5.7055	33.56
2	42	4.7365	1.7591	43.65
3	60	1.3635	0.5787	52.47
4	78	0.6509	0.3617	58.56

Table 4: The errors when the truncation is set to $\alpha = 0.01$

5 Conclusions

We have shown that using the properties of multi-wavelets an auto-refining scheme can be applied to the solution of boundary integral equations. Because of the space limitations we have not been able to shown that this is fully scalable, though from further experiments we have performed the method works well for matrices less than 1000×1000. The sparsity in a collocation scheme cannot be put to a practical use as Figure 2 shows there is no symmetry and more importantly all daigonal dominance is lost. However using multi-wavelets in a Galerkin formulation of a BIE looks to be promising since then *all* of the system matrix from the previous step is retained when refining occurs. Furthermore there are a priori estimates which can be used to calculate which entries in the system matrix need to be calculated, further reducing the time required to calculate the matrix. An efficient implementation would then be able to use the solution from the previous step. There are other improvements that can be made, most notably in the type of wavelets we are using. Applying the lifting scheme of Sweldens [9] to the multi-wavelets it is possible to construct a biorthogonal basis where in the case of the quadratic multi-wavelets each basis function would have four zero moments instead of two, three, and four zero moments we currently have.

References

[1] B. Alpert, G. Bekylin, R. Coifman, and R. Rokhlin.
Wavelet-like bases for the fast solution of integral equations.
SIAM J. Sci. Comput., 14(1):159–184, January 1993.

[2] Bradley K. Alpert.
Wavelets and Other Bases for Fast Numerical Linear algebra, volume 2
of *Wavelet Analysis and its Applications*, chapter 12, pages 184–212.
Wiley, 1 edition, 1993.

[3] Charles K. Chui.
    An Introduction to wavelets.
    Academic Press, San Diego, 1992.

[4] W. Dahmen, S. Prössdorf, and R. Schneider.
    Wavelet approximation methods for pseudodifferential equations II:
    Matrix compression and fast solution.
    Advances in Computaitional Mathematics, 1:259–335, 1993.

[5] Ingrid Daubechies.
    Ten Lectures on Wavelets.
    SIAM: CBMS-NSF Regional Conference in Applied Mathematics, 1
    edition, 1992.

[6] S. Prössdorf and R. Schneider.
    A spline collocation method for multidimensional strongly elliptic
    pseudo-differential operators of order zero.
    Integral Equations and Operator Theory, 14:399–435, 1991.

[7] S. Prössdorf and R. Schneider.
    Spline approximation methods for multidimensional periodic pseudo-
    differential equations.
    Integral Equations and Operator Theory, 15:626–672, 1992.

[8] W. Sweldens.
    The lifting scheme: A new philosophy in biorthogonal wavelet con-
    structions.
    In A. F. Laine and M. Unser, editors, *Wavelet Applications in Signal
    and Image Processing III*, pages 68–79. Proc. SPIE 2569, 1995.

[9] Tobias von Petersdorff and Christoph Schwab.
    Wavelet approximations for first kind boundary integral equations on
    polygons.
    Numerische Mathematik, 74(4), 1996.

[10] W.S.Hall.
    The boundary element method.
    Kluwer Academic Publishers, 1994.