Biol. Cybernetics 38, 159–169 (1980)

Biological Cybernetics

A Spatial Perturbation Technique for the Investigation of Discrete Internal Representations of Visual Patterns

David H. Foster

Department of Communication and Neuroscience, University of Keele, Keele, Staffordshire, England

Abstract. A technique is proposed for the investigation of discrete internal representations assumed to be formed by the visual system in response to pattern stimuli. The proposed technique involves applying a local 1-parameter group of spatial transformations to a pattern to generate a "continuum" of patterns. The visual discriminability of pairs of perturbed patterns corresponding to small fixed increments in the transformation parameter is determined at various points in the parameter range. By means of a simple rule, the characteristics of this discrimination performance are then related to the probability density functions assumed to underlie the hypothesized discrete internal representations. Two experimental applications of the proposed technique are described. The first is concerned with a discrete internal representation involving the specification of the collinearity or non-collinearity of the points in a pattern; the second is concerned with a discrete internal representation involving the specification of the acuteness or obtuseness of the angle between two lines in a pattern.

1. Introduction

The purpose of this study is to set out an experimental technique for the investigation of discrete internal pattern representations. These representations are assumed to be produced by the visual system in response to the presentation of spatially structured stimuli and are considered to provide a basis for the visual discrimination and identification of those stimuli. A formal description of discrete internal representations has been outlined in Foster (1980). Internal representations are there conceived of as comprising finite combinations of "components" which designate certain simple pattern properties and which are drawn from a fixed and finite repertoire. Included within such descriptions are those hypothesized representations which entail the specification of local pattern features in the stimulus, such as oriented lines, curves, and vertices, and the spatial relations obtaining between these local features such as "left of", "above", and "joined to" (Barlow et al., 1972; Sutherland, 1973; Reed, 1974; Palmer, 1978; Foster and Mason, 1979; Kahn and Foster, 1980). Independent of the precise nature of the structure of these hypothesized representations, the general assumption may be made (Foster, 1980) that the process by which the internal representations are constructed from their repertoire of components is probabilistic. Different stimulus patterns then determine different probabilities of inclusion in the representation for the components available.

The characteristic property of discrete representations is that, because of their "discontinuous" structure, they cannot smoothly follow an arbitrary smooth variation in the shape of a stimulus pattern. A formally analogous situation has been the subject of a number of studies in auditory perception. It has been shown (Liberman et al., 1957, 1961) that if a speech-like stimulus is varied in small steps along an acoustic continuum, the auditory discriminability of adjacent stimuli is better when they fall on different sides of a phoneme boundary than when they fall within the same phoneme category. There are, however, a number of problems that have to be surmounted in order to employ an analogous approach in the analysis of discrete internal representations of visual stimuli (Foster, 1980). First, an appropriate procedure for smoothly varying the shape of a stimulus must be contrived. There are no natural "dimensions" for visual stimuli corresponding, for example, to voice-onset time for speech-like stimuli. A smoothly parameterized set of "free-form" pattern stimuli has been constructed by Shepard and Cermak (1973), but their procedure is limited to those patterns which are closed curves. Second, given that internal representations may be constructed from some fixed and finite reper-

0340-1200/80/0038/0159/$02.20

toire of components, some of these components might describe quantities in the stimulus that are "continuous" variables (Foster, 1980). Possible contributions to discrimination performance by such quantities must be distinguished from the contributions by components that are discrete. Third, there is the methodological problem of ensuring that the contribution of some hypothesized discrete component is sufficiently large that it is not masked by the contributions of the putative continuous components.

A possible approach to the solution of these problems is suggested here. The essence of the proposed technique for the investigation of discrete internal representations involves applying to a pattern a family of spatial transformations that has the property of being a *local* 1-*parameter group*. This local 1-parameter group generates a spatial "continuum" of patterns. The visual discriminability of pairs of perturbed patterns corresponding to small fixed increments in the transformation parameter is measured at various points in the range of the parameter. Provided certain conditions are satisfied, a simple rule may then be invoked to relate the characteristics of this discrimination performance to the variations in the probability density functions governing the construction of the internal representations. Although the investigatory technique is developed within the framework of discrete internal representation schemes described in Foster (1980), it is in principle applicable to any scheme in which the internal representations satisfy the condition of being *locally finite*, as explained later.

The first part of this study gives an account of the principles of the proposed technique. The second part describes two experimental applications involving the analysis of two hypothesized pattern attributes that may be used in discrete internal representations. An informal account of part of this work has already appeared (Foster, 1979).

The approach to the analysis of the encoding of visual pattern stimuli adopted in this work may be contrasted with a previously described approach (Foster, 1975, 1977) designed to reveal the underlying structure of visual space and the structure of patterns mapped into that space when that structure satisfies certain formal axioms (Bourbaki, 1968). Rather than attempting to characterize the type of formal structure with which some internal representations might be associated (for example, a metric structure or a differential-geometric structure), the intention here is to eventually analyze and identify the components of the internal representation that give rise to that structure. Operationally, the present approach is concerned with the process of pattern discrimination, instead of with the more complex visual judgement of pattern equivalence defined with respect to some chosen formal structure (see for example, Foster, 1978; Foster and Mason, 1979; Kahn and Foster, 1980).

2. Discrete Internal Representations and Pattern Discrimination

Discrete internal representations may be considered in two ways: first, in a "constructional" sense, where the proposed nature of their structures is explicitly taken into account, as indicated above; and, second, in a "set-membership" sense, where, without reference to their construction, they are regarded as members of a set which is itself characterized by some general property like finiteness or countability. Since the experimental applications of the proposed investigatory technique discussed here relate to the identity and characteristics of possible components in internal representations, the "constructional" definition will be outlined first. The "set-membership" definition will be described in the next section, after the notion of a local 1-parameter group of local spatial transformations has been defined.

The following is a summary of the description of discrete internal representations developed in Foster (1980). A discrete internal representation is interpreted as comprising a finite set of finite subsets of *basis elements*, each subset in association with a finite number of *attributes* and *attribute values*. Typical basis elements might denote points, lines and regions in a stimulus pattern, and basis-element subsets would denote finite sets of these. Appropriate attributes might designate some *relative position* of one point with respect to another, or some *length* of a line, or some *area* of a region. Note the distinction that is made between attributes and the values that they may assume. The set A of all attributes may be partioned into two subsets, A^D and A^C. The subset A^D consists of *discrete* attributes, which can only assume values in a set equivalent to a countable set $\{1, 2, \dots, M\} \subset \mathbf{N}$, where $\mathbf{N}$ is the set of positive integers and M depends on the attribute. The subset A^C consists of *continuous* attributes, which can only assume values in some interval $\mathbf{I}$ in Euclidean space $\mathbf{R}^l$, where $\mathbf{R}$ is the set of real numbers and $\mathbf{I}$ and l also depend on the attribute. Typically[1], $M=2$, or 3, and $l=1$, 2, or 3. The values of both continuous and discrete attributes are nominal. Examples of discrete

1 For discrete attributes specifying the truth or falsity of a predicate, $M=2$. An upper limit on M is, in practice, about seven (Miller, 1956). Because of the finiteness of the visual system, both "discrete" and "continuous" variables have ranges which are finite; the cardinality of the range of a continuous variable is, however, assumed here to be much larger than that of a discrete variable. See Rosenfeld (1969) for discussion of the finite approximation of functions used in pattern recognition

attributes are, for three or more distinct points in the visual field, an attribute designating their *alignment*, that is, whether the points are collinear or non-collinear, and, for a point and a fixed non-intersecting non-vertical line their *vertical ordering*, that is, whether the point is above or below the line. Examples of continuous attributes are the attributes designating *length* for a line, and *area* for a region.

The finite set X of basis elements and the finite set A of attributes together constitute the fixed and finite repertoire of components from which the internal representation is considered to be constructed.

Symbolically, the internal representation of a pattern may be generally written thus:

$$\{(\mathbf{x}_i; (\alpha_{i1}, a_{i1}), (\alpha_{i2}, a_{i2}), \ldots, (\alpha_{i\kappa(i)}, a_{i\kappa(i)})) : i = 1, 2, \ldots, n\},$$

where, for $i = 1, 2, \ldots, n$,

(i) basis-element subsets

$$\mathbf{x}_i = \{x_{i1}, x_{i2}, \ldots, x_{i\xi(i)}\}, \quad x_{i\eta} \in X, \quad \eta = 1, 2, \ldots, \xi(i),$$

(ii) attributes

$$\alpha_{i\mu} \in \boldsymbol{\alpha}_i(\mathbf{x}_i), \quad \mu = 1, 2, \ldots, \kappa(i),$$

(iii) attribute values

$$a_{i\mu} \in \begin{Bmatrix} \{1, 2, \ldots, M\}, & \text{if } \alpha_{i\mu} \in A^D \\ \mathbf{I} \subset \mathbf{R}^l, & \text{if } \alpha_{i\mu} \in A^C \end{Bmatrix}, \quad \mu = 1, 2, \ldots, \kappa(i).$$

The set $\boldsymbol{\alpha}_i(\mathbf{x}_i)$ consists of those attributes appropriate to the subset $\mathbf{x}_i$, and the quantities M, $\mathbf{I}$, and l depend on i and μ.

The selection for an internal representation of basis-element subsets, attributes, and attribute values is assumed to be a probabilistic process. To introduce the notation used in the analysis of the probabilities governing this process, suppose that two arbitrary (and not necessarily distinct) basis-element subsets $\mathbf{x}_i$ and $\mathbf{x}_j$ are included in an internal representation and suppose that a discrete attribute $\delta_{i\mu}$ and a continuous attribute $\gamma_{j\nu}$ are associated with $\mathbf{x}_i$ and $\mathbf{x}_j$ respectively. Let $p_{i\mu}$ be the conditional discrete probability density function of $\delta_{i\mu}$ and let $f_{j\nu}$ be the conditional continuous probability density function of $\gamma_{j\nu}$. Thus, $p_{i\mu}(m)$ is the probability that value $m \in \{1, 2, \ldots, M\}$ is assigned to $\delta_{i\mu}$, and, for some subset U in $\mathbf{R}^l$, $\int_U f_{j\nu}(u)\, du$ is the probability that the value of $\gamma_{j\nu}$ lies in U.

Let B_1 and B_2 be two stationary patterns in the visual field. In Foster (1980), it was suggested that if B_1 and B_2 are sufficiently close to each other in shape, then the visual discriminability of B_1 and B_2 is determined by the differences in the density functions $p_{i\mu}$ and in the density functions $f_{j\nu}$ for the two patterns. (The parameters i, j, μ, ν vary over finite ranges.) Let $p_{i\mu}(v, m)$, $m \in \{1, 2, \ldots, M\}$, and $f_{j\nu}(v, u)$, $u \in \mathbf{R}^l$, denote the different density functions for the patterns B_v, $v = 1, 2$. The notation $p_{i\mu}(v, \cdot)$ and $f_{j\nu}(v, \cdot)$ is also used for these functions. The differences $|p_{i\mu}(1, m) - p_{i\mu}(2, m)|$, $m \in \{1, 2, \ldots, M\}$, and $|f_{j\nu}(1, u) - f_{j\nu}(2, u)|$, $u \in \mathbf{R}^l$, may be measured with the l_p and L_p norms. Let

$$e_{i\mu}(1,2) = \left(\sum_{m=1}^{M} |p_{i\mu}(1, m) - p_{i\mu}(2, m)|^q \right)^{1/q}, \tag{1}$$

$$e_{j\nu}(1,2) = \left(\int_{\mathbf{R}^l} |f_{j\nu}(1, u) - f_{j\nu}(2, u)|^{q'} du \right)^{1/q'}, \tag{2}$$

where M, l, and the integers q, q' $(1 \leqq q, q' < \infty)$ depend on i, μ, j, ν. The discriminability $d'(1,2)$ of B_1 and B_2 is then assumed to be given by

$$d'(1,2) = \sum_{\text{all } i\mu} d'_{i\mu}(1,2) + \sum_{\text{all } j\nu} d'_{j\nu}(1,2), \tag{3}$$

where

$$d'_{i\mu}(1,2) = y_{i\mu}(w_{i\mu} e_{i\mu}(1,2)), \tag{4}$$

$$d'_{j\nu}(1,2) = y_{j\nu}(w_{j\nu} e_{j\nu}(1,2)), \tag{5}$$

and $w_{i\mu}, w_{j\nu}$, $0 \leqq w_{i\mu}, w_{j\nu} \leqq 1$, are weighting coefficients depending on the selection probabilities of the $\delta_{i\mu}$, $\gamma_{j\nu}$ and basis-element subsets $\mathbf{x}_i$, $\mathbf{x}_j$; the functions $y_{i\mu}$, $y_{j\nu}$ are monotonic. All discriminabilities d' are discrimination indices as used in signal detection theory (Green and Swets, 1966; Swets, 1973). Thus, $d' = 0$ corresponds to the patterns not being discriminable, and the more positive d' is, the more the patterns are discriminable. Note that (3) implies that there is no "interaction" between attributes in their contributions to discrimination performance.

In the next section, we discuss how the shapes of B_1 and B_2 might be varied, and how their discriminability $d'(1,2)$ might depend on that variation.

3. Analysis by Pattern Perturbation

In Foster (1980), the following situation was considered. The patterns B_1 and B_2 were given a smooth parameterization by a variable t ranging in a parameter space T (a subspace of Euclidean space), thus:

$$B_1 = B_{t+\Delta t}, \quad B_2 = B_{t-\Delta t}, \quad t \in T,$$

where the increment $\Delta t > 0$, determining the "closeness" of B_1 and B_2, is fixed and small in a sense made precise below. It was conjectured that a smooth curve $t(s)$, $a \leqq s \leqq b$, might be constructed in T such that along that curve it should be possible to observe peaks in pattern discriminability $d'(1,2)$ at points where the discrete probability density functions have their greatest changes in value.

The principal theoretical problem relating to an experimental test of this conjecture concerns the specification of this smooth parameterization of the patterns.

It is a desirable condition that the pattern parameterization should be *transformationally uniform*. This condition is derived from the requirement that for any two values $t, t' \in T$, the change in pattern B_t caused by incrementing t to $t+\Delta t$ should in some sense be the same as the change in pattern $B_{t'}$ caused by incrementing t' to $t'+\Delta t$, where $t+\Delta t, t'+\Delta t \in T$. In order for the constancy of these changes to be expressed in a form independent of the particular choice of pattern, we use spatial transformations which correspond to these pattern changes. Given that a (stationary) pattern B in $\mathbf{R}^2$ (or $\mathbf{R}^3$) is a mapping of a non-empty subset V_B of $\mathbf{R}^2$ into $\mathbf{R}$ such that $B(z) \geqq 0$ is the luminance of the pattern at the point $z \in V_B$ (suppose white-light stimuli), we define the action on B of a local spatial transformation τ of V into $\mathbf{R}^2$, where V is a neighbourhood of V_B, by

$$(\tau(B))(z) = B(\tau^{-1}(z)) \quad \text{for} \quad z \in \tau(V_B) ,$$

which assigns to each point z in the domain of the transformed pattern $\tau(B)$ the luminance at its preimage. Note that the restriction of the mapping B to a subset of V_B defines a *subpattern* of B. To devise a smooth variation in the pattern B so that the above uniformity condition is satisfied, we require the local transformations to be members of a local 1-parameter group of local transformations, defined as follows. Let V be a subset of the plane $\mathbf{R}^2$ (or of $\mathbf{R}^3$) and let J be an interval $[a,b]$ in $\mathbf{R}$, where $a<0$, $b>0$. A *local 1-parameter group of local transformations* defined on V is a smooth (differentiable) mapping ψ of $J \times V$ into $\mathbf{R}^2$ such that

(i) for each $s \in J$, $\psi_s = \psi(s, \cdot)$ is a smooth mapping (with smooth inverse) of V onto $\psi_s(V)$;

(ii) if $s, s', s+s' \in J$, and if $z, \psi_{s'}(z) \in V$, then

$$\psi_{s+s'}(z) = \psi_s(\psi_{s'}(z)) ;$$

(iii) for all $z \in V$, $\psi_0(z) = z$.

Given that ψ_s, $s \in J$, is a local 1-parameter group of local transformations and that $\Delta s \in J$, the change from ψ_s to $\psi_{s+\Delta s}$ is automatically the same as the change from $\psi_{s'}$ to $\psi_{s'+\Delta s}$, namely $\psi_{\Delta s}$, providing that these quantities are defined.

The "set-membership" characterization of discrete internal representations referred to at the beginning of Sect. 2 (see also Foster, 1979) may now be made precise with the aid of the notion of local 1-parameter groups of local transformations. The set of internal representations is assumed to have the property of *local finiteness*, which is defined as follows. For each pattern B and for all local 1-parameter groups of local transformations ψ_s, $s \in J$, defined on a neighbourhood of B, it is possible to choose the intervals J sufficiently small but non-zero so that associated with the transformed patterns there are, in all, not more than a finite number of internal representations with non-zero assignment probabilities[2]. A version of this property will be made use of in the subsequent computation.

We now formulate a relationship between the probability density functions of discrete attributes and visual discriminability of spatially transformed patterns. Let ψ_s, $s \in J$, $J=[a,b]$, be a local 1-parameter group of local transformations defined on a neighbourhood of a pattern B. Suppose that for all $s \in J$ the transformed patterns $\psi_s(B)$ are such that the probabilities governing the selection of basis-element subsets and attributes in the internal representations are identical for all the $\psi_s(B)$, and that only the probability density functions $p_{i\mu}$ and $f_{j\nu}$ of the attributes $\delta_{i\mu}$ and $\gamma_{j\nu}$ may vary with s. (The parameters i, j, μ, ν vary over finite ranges, and, as before, i, j need not be distinct.) This is a plausible assumption for the applications discussed later (see also Foster, 1980). Let $p_{i\mu}(s, \cdot)$ and $f_{j\nu}(s, \cdot)$ be the probability density functions of $\delta_{i\mu}$ and $\gamma_{j\nu}$ respectively for pattern $\psi_s(B)$. The values of $\delta_{i\mu}$ are drawn from the set $\{1, 2, \ldots, M\}$, where M depends on i, μ, and the values of $\gamma_{j\nu}$ are drawn from $\mathbf{I} \subset \mathbf{R}^l$, where $\mathbf{I}$, l depend on j, ν. Thus, $p_{i\mu}(s,m)$ is the probability that for $\psi_s(B)$ the attribute $\delta_{i\mu}$ assumes the value m, and, for $U \subset \mathbf{R}^l$, $\int_U f_{j\nu}(s,u)\, du$ is the probability that $\gamma_{j\nu}$ assumes a value in U. Suppose that $p_{i\mu}(s, \cdot)$ and $f_{j\nu}(s, \cdot)$ vary smoothly and continuously with s, and are such that the following conditions are satisfied.

(C1) For all discrete attributes $\delta_{i\mu}$ with $i\mu \neq h\lambda$,

$$\frac{\partial p_{i\mu}}{\partial s}(s,m) = 0 , \quad a \leqq s \leqq b , \quad 1 \leqq m \leqq M .$$

That is, for all discrete attributes except for the $h\lambda^{\text{th}}$, the probability density functions $p_{i\mu}(s, \cdot)$ are constant with s.

(C2) For $\delta_{h\lambda}$, there is for each m, $1 \leqq m \leqq M$, a single interval $[c_m, d_m] \subset J$, where $c_m \leqq d_m$, on which $p_{h\lambda}(\cdot, m)$ achieves is maximum value, and such that on $[a, c_m]$, $p_{h\lambda}(\cdot, m)$ is monotonic increasing and on $[d_m, b]$ it is monotonic decreasing, with at most a single point of inflexion in each case. For all m, $1 \leqq m \leqq M$, let $d_m < c_{m+1}$. The purpose of the condition is to ensure that there are no "irregularities" in the smooth fall-off with s in the probabilities of assignment of each attribute value.

2 "Finite" here means small, i.e., of the order of M. So that this condition is not trivially satisfied, the intervals J should be required to be sufficiently large that in each case there are two or more internal representations with non-zero probabilities associated with the transformed patterns $\psi_s(B)$, $s \in J$

(C3) For $\delta_{h\lambda}$, there is a constant ε, $0<\varepsilon\ll 1$, such that

$$0 \leqq p_{h\lambda}(s,m), \left|\frac{\partial p_{h\lambda}}{\partial s}(s,m)\right| \cdot k^{-1} \leqq \varepsilon ,$$

for $a \leqq s \leqq d_{m-1}$ when $1<m\leqq M$, and $c_{m+1} \leqq s \leqq b$ when $1 \leqq m < M$, where

$$k = \min_{1 \leqq m' \leqq M} \max_{a \leqq s \leqq b} \left|\frac{\partial p_{h\lambda}}{\partial s}(s,m')\right| .$$

This condition requires that for adjacent attribute values $m-1$, m, $m+1$, $1<m<M$, the assignment probability of m is not significant in either magnitude or relative rate of change when the pattern is transformed to or beyond the point at which $m-1$ and $m+1$ have maximum assignment probabilities. For each transformed pattern $\psi_s(B)$, there are thus effictively at most two values of the discrete attribute (these values depending on s) that may be considered to be "in competition" with each other. (Note that under some conditions, discrete attribute values may be merged to form just two complementary values, corresponding to some property P and its negation *not-P*. For $M=2$, the condition may be amended in an obvious way.)

(C4) For each continuous attribute $f_{j\nu}(\cdot,\cdot)$ and each $\Delta s \in J$, there exists a constant $\Delta u \in \mathbf{R}^l$ such that for all s, $s' \in J$, for which $s'=s+\Delta s$,

$$f_{j\nu}(s,u) = f_{j\nu}(s', u+\Delta u) \quad \text{for all} \quad u \in \mathbf{R}^l .$$

Under this condition, the parameterization by s is uniform with respect to changes in the probability density functions associated with constant changes Δs in s. This condition, more restrictive than the transformational uniformity condition defined earlier, may be partly relaxed, as will subsequently be shown.

For the discrete attribute $\delta_{h\lambda}$, consider the interval $d_{m-1} < s \leqq d_m$ for some fixed m, $1<m\leqq M$. Let the transformed pattern $\psi_s(B)$ be given incremental transformations $\psi_{-\Delta s}$ and $\psi_{\Delta s}$, $\Delta s>0$, where Δs is fixed. Then, $\psi_{-\Delta s}(\psi_s(B)) = \psi_{s-\Delta s}(B)$ and $\psi_{\Delta s}(\psi_s(B)) = \psi_{s+\Delta s}(B)$, providing that s, $s-\Delta s$, and $s+\Delta s$ are in J. By (1)–(5), the visual discriminability $d'(s)$ of the pair of perturbed patterns $\psi_{s-\Delta s}(B)$, $\psi_{s+\Delta s}(B)$ is determined by the difference terms:

$$e_{h\lambda}(s) = \left(\sum_{m=1}^{M} |p_{h\lambda}(s+\Delta s, m) - p_{h\lambda}(s-\Delta s, m)|^q \right)^{1/q} , \qquad (6)$$

$$e_{j\nu}(s) = \left(\int_{\mathbf{R}^l} |f_{j\nu}(s+\Delta s, u) - f_{j\nu}(s-\Delta s, u)|^{q'} du \right)^{1/q'} , \qquad (7)$$

where j, ν extend over their full (finite) ranges. Note that $e_{i\mu}=0$ for $i\mu \neq h\lambda$ by (C1).

Making use of Taylor's formula and ignoring terms of the order ε and $(\Delta s)^3$ and higher, we obtain the result that for sufficiently small values of Δs,

$$e_{h\lambda}(s) = \left|\frac{\partial p_{h\lambda}}{\partial s}(s,m)\right| \cdot \Delta s \cdot 2^{(q+1)/q} ,$$

and

$$\frac{\partial e_{j\nu}(s)}{\partial s} = 0, \quad \text{for all} \quad j\nu .$$

Discriminability is then determined by (3)–(5), thus:

$$d'(s) = y_{h\lambda}(w_{h\lambda} e_{h\lambda}(s)) + \sum_{\text{all } j\nu} y_{j\nu}(w_{j\nu} e_{j\nu}(s)) .$$

Since the only variation in $d'(s)$ with s over the interval $d_{m-1} < s \leqq d_m$ derives from $e_{h\lambda}(s)$, and the choice of m was arbitrary, we have the following rule.

(R) *Over the interval $a \leqq s \leqq b$, discriminability $d'(s)$ is maximum when $|\partial p_{h\lambda}(s,m)/\partial s|$ is maximum, and minimum when $p_{h\lambda}(s,m)$ is maximum, $1 \leqq m \leqq M$.*

Note that the above relationship holds only when Δs is "sufficiently small". In practice, one might test whether this condition is satisfied by reducing the magnitude of Δs; if Δs is indeed sufficiently small, there should be no change in the positions of extrema of d'. Note also that the rule (R) does not depend on the particular choice of l_p and L_p metrics.

4. Experimental Applications

The analysis in the preceding section forms the basis of the proposed technique for the investigation of discrete internal representations. Two practical applications of this technique are demonstrated in the following experiments. The first is designed as a test of the approach. The putative discrete attribute under consideration is one designating the collinearity or noncollinearity of the points in a pattern. The second experiment is more exploratory. Concern there is with the possible existence of a discrete attribute designating the acuteness or obtuseness of the angle between two connected lines.

The stimulus displays used in the two experiments consisted of patterns comprising a number of subpatterns. The choice of this type of presentation is motivated by the hypothesis that there is a limited pattern-encoding capacity for the visual system (Estes and Taylor, 1964, 1966; Eriksen and Eriksen, 1974; Lupker and Massaro, 1979). Given that there is such a limit, the presentation of patterns or subpatterns in sufficient numbers to approach the limit should make an internal representation containing discrete attributes more economical than one containing continuous attributes. In an information-theoretic sense, less information is specified by a discrete attribute than

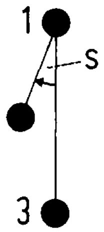

Fig. 1. Three-dot subpattern. The deformation of the subpattern is specified by the angle s. The centre-to-centre separations of dots 1 and 2 and dots 2 and 3 were constant and equal, and corresponded to 0.075° visual angle

by a continuous attribute. (The question of pattern-processing capacity is discussed further in Sect. 5.)

4.1. Experiment 1

Collinearity-Noncollinearity of the Points in a Pattern. Results from a number of different studies support the notion that the property of collinearity or noncollinearity of the parts of a pattern has a special role in visual recognition, discrimination, and other perceptual phenomena (see, for example, Wertheimer, 1923; Bouma and Andriessen, 1968; Andrews et al., 1973; Uttal, 1973; Prytulak, 1974; Caelli and Umansky, 1976; Prinzmetal and Banks, 1977). The experiment described below uses stimuli that might reasonably be expected to give internal representations containing an attribute designating collinearity or noncollinearity. The intention is to determine whether discrimination performance with such stimuli may be estimated on the basis of the rule (R) of Sect. 3.

Stimulus patterns consisted of various configurations of three-dot subpatterns, each similar to the one illustrated in Fig. 1. These subpatterns were subjected to local 1-parameter groups of local transformations ψ_s, $s \in J$, where ψ_s acts to deform the pattern through the angle s indicated in Fig. 1, the distances between dots 1 and 2 and between dots 2 and 3 being held constant. As before, we use $\psi_s(B)$ to denote the transformed subpattern. The interval $J = [-30°, 30°]$.

Let $\delta_{h\lambda}$ be the proposed discrete attribute, where the value 1 of $\delta_{h\lambda}$ corresponds to dot collinearity and the value 2 to dot noncollinearity. (Although it does not affect the subsequent analysis, the two forms of noncollinearity, associated with positive and negative values of s, may be assumed to give rise to distinct internal representations.) We suppose that the internal representations formed by the subpatterns are such that conditions (C1)–(C3) of Sect. 3 are satisfied, where $\partial p_{i\mu}(s, \cdot)/\partial s = 0$, $s \in J$, except for $i\mu = h\lambda$. It is not assumed that the uniformity condition (C4) is necessarily satisfied, but it is supposed that over the range J of s the weighting coefficients $w_{j\nu}$ (Sect. 3) for the continuous attributes are small in relation to $w_{h\lambda}$, the weighting coefficient for the discrete attribute $\delta_{h\lambda}$ (see comments below).

A plausible assumption is that the probability $p_{h\lambda}(s, 1)$ of the attribute $\delta_{h\lambda}$ assuming the value 1, expressing dot collinearity, should have its maximum value when $s = 0°$. It is not possible to specify *a priori*, however, the value of s where $|\partial p_{h\lambda}(s, 1)/\partial s|$ should attain its maximum, except that it should be close to $s = 0°$. Certainly, as s increases, the probability $p_{h\lambda}(s, 2)$ of $\delta_{h\lambda}$ assuming the value 2, expressing dot noncollinearity, should eventually reach a maximum. From (R), Sect. 3, there then follows the qualitative prediction that the performance in discriminating dot subpatterns differing by fixed increments $\pm \Delta s$ in the parameter s should be minimum when $s = 0°$, be maximum for values of s a short distance away, and for sufficiently large magnitudes of s, fall to a minimum again.

Details of the experimental procedure have been described in full elsewhere (Foster, 1979). Stimulus patterns were produced on a display CRT controlled by an on-line computer which also recorded subjects' responses. There were two versions of the experiment, with different presentation regimes for the stimuli, and different discrimination tasks for the subjects. In both versions of the experiment, there were four subpatterns presented in each display. In version (a), subjects performed a two-interval forced-choice "same-different" discrimination task. In each trial, the display presented in one interval contained two perturbed "different" subpatterns $\psi_{s-\Delta s}(B)$ and $\psi_{s+\Delta s}(B)$ [along with two identical dummy subpatterns, different from $\psi_{s-\Delta s}(B)$ and $\psi_{s+\Delta s}(B)$]; and the display presented in the other interval contained two unperturbed "same" subpatterns $\psi_s(B)$ (with the same two dummy subpatterns). Subjects had to indicate the interval containing the "different" subpatterns. In version (b) of the experiment, subjects again performed a two-alternative forced-choice discrimination task, but each trial entailed the presentation of a single display containing three identical subpatterns $\psi_{s-\Delta s}(B)$ and one "odd" subpattern $\psi_{s+\Delta s}(B)$, all oriented vertically. Subjects had to indicate the direction of perturbation of the odd subpattern relative to the other subpatterns. In both versions of the experiment, the magnitude (10°, 14°, or 18°) of the fixed increment Δs was chosen for each subject to yield mean response scores about midway between chance and perfect performance.

Figures 2a and b show respectively the results of these two versions of the dot-subpattern discrimination experiment. Discriminability d' of the perturbed patterns is plotted as a function of the value s of the transformation parameter about which the increments $\pm \Delta s$ were made. Results are averaged over four subjects. The variation in d' with s shown in a and in b is evidently consistent with the predictions given above: there is a minimum in d' at $s = 0°$, a maximum

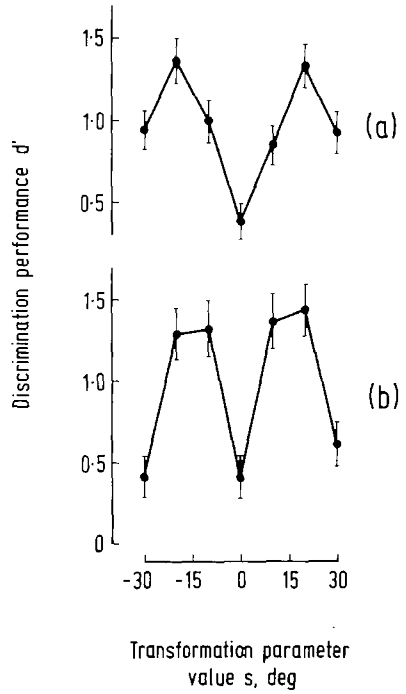

Fig. 2a and b. Performance in discriminating dot subpatterns. The discrimination index d' is shown as a function of value s of the transformation parameter. The vertical bars indicate ± 1 standard error of the mean. Data are pooled over four subjects. Performances in **a** and **b** refer to the different versions of the experiment

each side of this position, and a fall-off at larger magnitudes of s. In both versions of the experiment, the depression at $s=0°$ relative to $\pm 10°$ and $\pm 20°$ is highly significant ($P<0.001$); in version (a), the depression at $s=\pm 30°$ relative to $\pm 20°$ is significant ($P<0.05$), and, in version (b), highly significant ($P<0.001$).

Some alternative interpretations of these results have been examined elsewhere (Foster, 1979). In particular, the possibility that there is a significant continuous-attribute contribution to discrimination performance that varies with s, specifying for example the distances between the dots is a subpattern, was rejected on the basis of a control experiment which measured visual acuity for perturbations in subpattern *size*. These perturbations involved dot-displacements equal to the largest used in the main experiments, but discrimination performance for these subpatterns, for which there was now no change in "shape" of the figure, was found to be not significantly different from chance levels.

4.2. *Experiment 2*

Acuteness-Obtuseness of the Angle between Two Connected Lines in a Pattern. This experiment is designed to determine whether the internal representation of a pattern consisting of two connected lines may contain a discrete attribute designating the acuteness or obtuseness of the angle between the lines.

There were two types of subpattern. Only the one was supposed to give rise to the hypothesized discrete attribute; the other was designed to provide a control for the effect of continuous attributes designating angle size. The subpatterns were either asymmetric chevrons, as in Fig. 3a, or symmetric chevrons, as in Fig. 3b. For both, the action of the local 1-parameter group of local transformations ψ_s, $s \in J$, is specified by the angle s between the two arms of the figure. The interval $J=[30°, 150°]$ does not include 0°, but this may be achieved by a trivial translation in the scale for s. Again, we use $\psi_s(B)$ to denote the transformed subpattern.

Let $\delta_{h\lambda}$ be the hypothesized discrete attribute, where the value 1 of $\delta_{h\lambda}$ corresponds to angle acuteness, and the value 2 corresponds to angle obtuseness. We suppose, as before, that conditions (C1)–(C3) of Sect. 3 are satisfied, where $\partial p_{i\mu}(s, \cdot)/\partial s=0$, $s \in J$, except for $i\mu = h\lambda$. As a result of other studies (cited below) on visual acuity for changes in angle, we do not suppose that the weighting coefficients $w_{j\nu}$ for the proposed continuous attributes are necessarily small in relation to the weighting coefficient $w_{h\lambda}$ for $\delta_{h\lambda}$ (compare Experiment 1). The uniformity condition (C4) is not, however, assumed to be satisfied exactly. Instead, we suppose, on the basis of the following considerations, that:

(i) the contributions $e_{j\nu}(s)$ by the continuous attributes (7) to pattern discriminability vary slowly with s for both asymmetric and symmetric chevrons;

(ii) the weighting coefficient $w_{h\lambda}$ for $\delta_{h\lambda}$ is non-zero only for the asymmetric chevrons.

Support for assumption (i) comes from angle acuity experiments performed by Hakiel (1978) who showed that for acute angles the threshold for angle change increases smoothly and monotonically with angle. A plausible continuous angle attribute might refer to the distance between the two free endpoints of each arm of

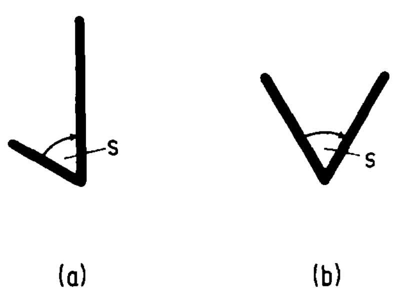

Fig. 3a and b. Chevron subpatterns: **a** asymmetric, and **b** symmetric. The deformation of each subpattern is specified by the angle s. The short arm of the asymmetric chevron corresponded to 0.22° visual angle, and each arm of the symmetric chevron corresponded to 0.33°

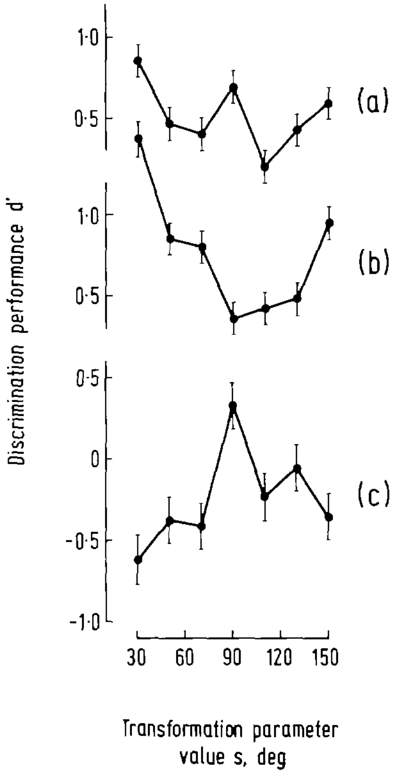

Fig. 4a–c. Performance in discriminating chevron subpatterns. The discrimination index d' is shown as a function of the value s of the transformation parameter. The vertical bars indicate ± 1 standard error of the mean. Data are pooled over four subjects. Performance in **a** refers to the asymmetric chevrons, in **b** to the symmetric chevrons, and in **c** to the difference of **a** and **b**

the subpatterns. Note that in the region about 90° the objectively determined value of that attribute depends similarly on s for each of the types of subpattern used here (Figs. 3a and b). With regard to assumption (ii), observations by Goldmeier (1936), Olson and Attneave (1970), and Rock (1973) have shown that the orthogonal nature of the angle between two lines and departures from orthogonality are most evident when the lines are close to "preferred or singular spatial directions". Attneave (1968) has also demonstrated that preferred spatial directions may be defined by the common alignments of the stimulus subpatterns in the field. As explained below, subpatterns were arranged in each display to favour the weighting defined in assumption (ii).

The experiment was divided into two parts; in each part, each display contained three identical subpatterns $\psi_{s-\Delta s}(B)$ and one "odd" subpattern $\psi_{s+\Delta s}(B)$. In part (a), the subpatterns were the asymmetric chevrons aligned so that the directions of the *long arms* of the four subpatterns were the same. (One arm of each subpattern was therefore codirectional with this "preferred" direction of the display.) In part (b), the subpatterns were the symmetric chevrons aligned so that the directions of the *bisectors* of the angles were the same. (Thus no arm of the subpatterns was codirectional with this "preferred" direction of the display.) The "preferred" directions were in each case chosen randomly from the range $22.5° + n\,45°$, $n = 1, 2, \ldots, 8$, so that discrimination performance was not confounded with the improvement in angle acuity observed with stimuli oriented vertically or horizontally (Rochlin, 1955; Onley and Volkmann, 1958; Weene and Held, 1966; Hakiel, 1978). For further details of the stimulus patterns, see Foster (1981).

It seems reasonable to hypothesize that the probability density function $p_{h\lambda}$ is such that $|\partial p_{h\lambda}(s, 1)/\partial s|$ is maximum at $s = 90°$. From the above considerations, rule (R) of Sect. 3 then implies that discrimination performance should have a maximum at $s = 90°$ for the asymmetric chevrons, and there should be no such maximum for the symmetric chevrons.

The methods of this experiment were similar to those of Experiment 1. Subjects performed a four-alternative forced-choice discrimination task; in each part of the experiment they had to indicate the location of the "odd" subpattern in each display. The magnitude of the increment Δs was fixed at 10° for all subjects.

Figures 4a and b show respectively the results for the asymmetric and symmetric chevron discrimination experiments. In each case, discriminability d', averaged over four subjects, is plotted as a function of the value s of the transformation parameter about which the increments $\pm \Delta s$ were made. Although both types of chevron give rise to similar performances at extreme values of s, there is a pronounced difference in the region about $s = 90°$. In accordance with the above hypothesis concerning the existence of a discrete attribute $\delta_{h\lambda}$ designating angle acuteness or obtuseness, there is a peak in d' at $s = 90°$ for the asymmetric chevron (Fig. 4a) and none for the symmetric chevron (Fig. 4b). For both types of subpattern, the value of d' outside the region about 90° increases monotonically with s as s moves towards the ends of its range. If the proposed contribution to discrimination performance by continuous attributes were the same for each of the two chevron types, then the contribution by the hypothesized attribute $\delta_{h\lambda}$ alone might be estimated by taking the differences of the values of d' in (a) and (b) [see (3)]. This differenced discrimination performance, averaged over four subjects, is shown in Fig. 4c. The maximum at $s = 90°$ relative to $s = 70°$ and 110° is highly significant ($P < 0.001$). Note that the differenced performance is negative for $s \neq 90°$, which may be interpreted as being due to the weighting coefficients w_{j_ν} for the continuous attributes being lower for the asymmetric chevrons than for the symmetric chevrons.

5. Summary and Discussion

The technique described in this study for the investigation of discrete internal pattern representations has been developed within the framework of the discrete representation schemes formalized in Foster (1980). Such schemes include those which hypothesize that patterns are encoded in terms of local pattern features like oriented lines, curves, and vertices (Gibson, 1969; Olson and Attneave, 1970; Beck, 1972; Beck and Ambler, 1973; Rumelhart and Siple, 1974; Pomerantz, 1978), and global pattern features such as symmetry, area, "compactness", and "jaggedness" (Sutherland, 1968; Mavrides and Brown, 1969; Aiken and Brown, 1971). More generally, discrete internal representations of the kind considered here include those putative representations based on forming relational structural descriptions, which were referred to in the Introduction. As shown elsewhere (Foster, 1979), the proposed technique may also be used for those discrete representation schemes which do not specify a detailed structure for the representations, but constrain them, as a set, to satisfy the local finiteness condition defined in Sect. 3.

The principal concern of the proposed investigatory technique is with the characterization of the discrete attributes in discrete internal representations. As explained in Sect. 3, the technique involves applying to a pattern B a local 1-parameter group of local transformations ψ_s, $s \in J$, and measuring the visual discriminability of the perturbed patterns $\psi_{s-\Delta s}(B)$ and $\psi_{s+\Delta s}(B)$ as a function of the transformation parameter s. The characteristics of this discrimination performance are then related to the variations in the probability density functions which are assumed to govern the assignment of values to the discrete attributes in the representation.

A fundamental condition for the applicability of the technique is that the contribution to discrimination performance of any continuous attributes present in the representation should be constant with the transformation parameter s [condition (C4) in Sect. 3]. The experimental applications considered in Sect. 4 dealt with this requirement in different ways. In the first experiment, concerned with a putative discrete attribute designating the collinearity-noncollinearity of dots in a dot pattern, visual discrimination was assumed to be determined solely by that attribute. This assumption was supported by an auxiliary experiment which showed that possible continuous attributes designating the distances between the dots were insufficiently sensitive to provide significant discrimination information. In the second experiment, concerned with a hypothesized discrete attribute designating the acuteness-obtuseness of the angle between two connected lines, it was not assumed that the contribution to discrimination performance by continuous angle attributes was zero. Instead, it was assumed that this contribution varied slowly with the parameter s. An estimate of the contribution was obtained with the symmetric chevron patterns, which, unlike the asymmetric chevron patterns, were not supposed to give rise to internal representations containing the hypothesized discrete attribute. Given a choice between these two approaches to the isolation of the contribution to discrimination performance by a discrete attribute, the first has the advantage that it is independent of the form which the variations in continuous-attribute contributions might take.

As already noted, the use of a local 1-parameter group of local transformations to generate a family of patterns parameterized by the variable s does not ensure the constancy of the contribution by continuous attributes to descrimination performance. Nevertheless, it does seem reasonable to conjecture that the transformational uniformity (Sect. 3) defined by the 1-parameter group secures at least some higher-order constancy in the variation with s of the discrimination performance determined by the continuous attributes. For example, the characteristics of discrimination performance for the symmetric chevrons in Experiment 2 (Sect. 4.2), assumed to depend only on continuous angle attributes, can be expressed simply as a combination of linear and quadratic trends with s (Lindman, 1974). In contrast, the characteristics for the asymmetric chevrons show, as might be expected, a significant quartic trend ($P < 0.05$). The assumption (i) introduced in Experiment 2 concerning the slow variation of the contribution to discrimination performance by continuous attributes might then be made specific in terms of the constancy of these higher-order trends.

In the analysis of the results of Experiment 1 (Sect. 4.1), one explanation that was not considered relates to the notion that the rapidly varying discrimination performance shown in Figs. 2a and b for the dot subpatterns is the result of a changeover from the use of one continuous attribute in the internal representation to another, the density function of each attribute depending monotonically on dot separation. Within the framework of a scheme of unstructured pointillistic internal representations summarized in Foster (1980), the characteristics of the observed performance might alternatively be explained by a changeover from the use of one smooth comparison operation to another. Both schemes are, however, essentially discrete in their operation and differ from the proposed discrete representation scheme only in the functional site of their discreteness. Practically, it may not be obvious how to demonstrate this difference (Foster and Mason, 1979).

The hypothesis that the visual system has a limited pattern-encoding capacity, referred to in the first part of Sect. 4, is given some support by the outcome of a modified version of Experiment 1. Two rather than four subpatterns were used in the display, and reliable discrimination performance (levels typically 80%) was obtained with the magnitude of the increments $\pm \Delta s$ in the transformation parameter s fixed at 2.5°, one quarter the value used in the displays with four subpatterns. Under these conditions, the discrimination task is more closely related to conventional vernier acuity and hyperacuity measurements (Ludvigh, 1953; Westheimer and McKee, 1977; Beck and Schwartz, 1979), for which there seems to be little evidence for discrimination determined by discrete attributes. The hypothesis that the number of subpatterns in the field determines the relative weighting of "coarse" discrete attributes and "fine" continuous attributes in the internal representation is analogous to a suggestion by Beck (Beck, 1972; Beck and Ambler, 1972, 1973) concerning the effect of subpattern number on "distributed" and "focal" attention in pattern discrimination. Although the investigatory technique proposed here considers internal representations that include continuous attributes, which may determine discrimination performance under conditions of "focal" attention or pattern scrutiny, the approach is, by its nature, suited to the identification and analysis of the discrete attributes of discrete internal representations.

Acknowledgements. I thank D.M. MacKay, S. Gravano, J.I. Kahn, R.J. Mason, and R.J. Watt for many useful discussions in the preparation of this work. I also thank R. Knapper for technical assistance.

References

Aiken, L.S., Brown, D.R.: A feature utilization analysis of the perception of pattern class structure. Percept. Psychophys. **9**, 279–283 (1971)

Andrews, D.P., Butcher, A.K., Buckley, B.R.: Acuities for spatial arrangement in line figures: human and ideal observers compared. Vision Res. **13**, 599–620 (1973)

Attneave, F.: Triangles as ambiguous figures. Am. J. Psychol. **81**, 447–453 (1968)

Barlow, H.B., Narasimhan, R., Rosenfeld, A.: Visual pattern analysis in machines and animals. Science **177**, 567–575 (1972)

Beck, J.: Similarity grouping and peripheral discriminability under uncertainty. Am. J. Psychol. **85**, 1–19 (1972)

Beck, J., Ambler, B.: Discriminability of differences in line slope and in line arrangement as a function of mask delay. Percept. Psychophys. **12**, 33–38 (1972)

Beck, J., Ambler, B.: The effects of concentrated and distributed attention on peripheral acuity. Percept. Psychophys. **14**, 225–230 (1973)

Beck, J., Schwartz, T.: Vernier acuity with dot test objects. Vision Res. **19**, 313–319 (1979)

Bouma, H., Andriessen, J.J.: Perceived orientation of isolated line segments. Vision Res. **8**, 493–507 (1968)

Bourbaki, N.: Theory of sets, Chap. IV. Paris: Hermann; Reading, MA, Menlo Park, CA, London, Don Mills (Ontario): Addison-Wesley 1968

Caelli, T.M., Umansky, J.: Interpolation in the visual system. Vision Res. **16**, 1055–1060 (1976)

Eriksen, B.A., Eriksen, C.W.: Effects of noise letters upon the identification of a target letter in a nonsearch task. Percept. Psychophys. **16**, 143–149 (1974)

Estes, W.K., Taylor, H.A.: A detection method and probabilistic models for assessing information processing from brief visual displays. Proc. Natl. Acad. Sci. **52**, 446–454 (1964)

Estes, W.K., Taylor, H.A.: Visual detection in relation to display size and redundancy of critical elements. Percept. Psychophys. **1**, 9–16 (1966)

Foster, D.H.: An approach to the analysis of the underlying structure of visual space using a generalized notion of visual pattern recognition. Biol. Cybernetics **17**, 77–79 (1975)

Foster, D.H.: Visual pattern recognition by assignment of invariant features and feature-relations. Opt. Acta **24**, 147–157 (1977)

Foster, D.H.: Visual comparison of random-dot patterns: evidence concerning a fixed visual association between features and feature-relations. Q. J. Exp. Psychol. **30**, 637–654 (1978)

Foster, D.H.: Discrete internal pattern representations and visual detection of small changes in pattern shape. Percept. Psychophys. **26**, 459–468 (1979)

Foster, D.H.: A description of discrete internal representation schemes for visual pattern discrimination. Biol. Cybernetics **38**, 151–157 (1980)

Foster, D.H.: Analysis of discrete internal representations of visual pattern stimuli. In: Processes of perceptual organization and representation. Beck, J. (ed.), (to appear) (1981)

Foster, D.H., Mason, R.J.: Transformation and relational-structure schemes for visual pattern recognition: two models tested experimentally with rotated random-dot patterns. Biol. Cybernetics **32**, 85–93 (1979)

Gibson, E.J.: Principles of perceptual learning and development. New York: Appleton-Century-Crofts 1969

Goldmeier, E.: Über Ähnlichkeit bei gesehenen Figuren. Psychologische Forschung **21**, 146–208 (1936); Transl. by Goldmeier: Similarity in visually perceived forms. Psychol. Issues **8**, No. 1, Monogr. **29**, 1–135 (1972)

Green, D.M., Swets, J.A.: Signal detection theory and psychophysics. New York, London, Sydney: Wiley 1966

Hakiel, S.R.: Variable and constant errors in perceived angle size. Unpublished doctoral dissertation, University of Keele, Keele, Staffordshire 1978

Kahn, J.I., Foster, D.H.: Visual comparison of transformed patterns as a function of their positional symmetry and separation in the field (in preparation) (1980)

Liberman, A.M., Harris, K.S., Hoffman, H.S., Griffith, B.C.: The discrimination of speech sounds within and across phoneme boundaries. J. Exp. Psychol. **54**, 358–368 (1957)

Liberman, A.M., Harris, K.S., Kinney, J.A., Lane, H.: The discrimination of relative onset-time of the components of certain speech and nonspeech patterns. J. Exp. Psychol. **61**, 379–388 (1961)

Lindman, H.R.: Analysis of variance in complex experimental designs. San Francisco: Freeman 1974

Ludvigh, E.: Direction sense of the eye. Am. J. Ophthalmol. **36**, 139–143 (1953)

Lupker, S.J., Massaro, D.W.: Selective perception without confounding contributions of decision and memory. Percept. Psychophys. **25**, 60–69 (1979)

Mavrides, C.M., Brown, D.R.: Discrimination and reproduction of patterns: feature measures and constraint redundancy as predictors. Percept. Psychophys. **6**, 276–280 (1969)

Miller, G.A.: The magical number seven, plus or minus two: some limits on our capacity for processing information. Psychol. Rev. **63**, 81–97 (1956)

Olson, R.K., Attneave, F.: What variables produce similarity grouping? Am. J. Psychol. **83**, 1–21 (1970)

Onley, J.W., Volkmann, J.: The visual perception of perpendicularity. Am. J. Psychol. **71**, 504–516 (1958)

Palmer, S.E.: Structural aspects of visual similarity. Mem. Cognition **6**, 91–97 (1978)

Pomerantz, J.R.: Are complex visual features derived from simple ones? In: Formal theories of visual perception. Leeuwenberg, E.L.J., Buffart, H.F.J.M. (eds.), pp. 217–229. Chichester, New York, Brisbane, Toronto: Wiley 1978

Prinzmetal, W., Banks, W.P.: Good continuation affects visual detection. Percept. Psychophys. **21**, 389–395 (1977)

Prytulak, L.S.: Good continuation revisited. J. Exp. Psychol. **102**, 773–777 (1974)

Reed, S.K.: Structural descriptions and the limitations of visual images. Mem. Cognition **2**, 329–336 (1974)

Rochlin, A.M.: The effect of tilt on the visual perception of parallelness. Am. J. Psychol. **68**, 223–236 (1955)

Rock, I.: Orientation and form. New York, London: Academic Press 1973

Rosenfeld, A.: Picture processing by computer. New York, London: Academic Press 1969

Rumelhart, D.E., Siple, P.: Process of recognizing tachistoscopically presented words. Psychol. Rev. **81**, 99–118 (1974)

Shepard, R.N., Cermak, G.W.: Perceptual-cognitive explorations of a toroidal set of free-form stimuli. Cog. Psychol. **4**, 351–377 (1973)

Sutherland, N.S.: Outlines of a theory of visual pattern recognition in animals and man. Proc. R. Soc. London, Ser. B **171**, 297–317 (1968)

Sutherland, N.S.: Object recognition. In: Handbook of perception, Vol. 3. Carterette, E.C., Friedman, M.P. (eds.), pp. 157–185. New York, London: Academic Press 1973

Swets, J.A.: The relative operating characteristic in psychology. Science **182**, 990–1000 (1973)

Uttal, W.R.: The effect of deviations from linearity on the detection of dotted line patterns. Vision Res. **13**, 2155–2163 (1973)

Weene, P., Held, R.: Changes in perceived size of angle as a function of orientation in the frontal plane. J. Exp. Psychol. **71**, 55–59 (1966)

Wertheimer, M.: Untersuchungen zur Lehre von der Gestalt. II. Psychol. Forsch. **4**, 301–350 (1923)

Westheimer, G., McKee, S.P.: Spatial configurations for visual hyperacuity. Vision Res. **17**, 941–947 (1977)

Received: March 6, 1980

Dr. D. H. Foster
Department of Communication and Neuroscience
University of Keele
Keele, Staffordshire ST5 5BG
England