A special class of real concave functions

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Abstract

In this paper, one studies the concavity of some functions that can be written as powers of linear combinations of powers of some concave functions.

The necessary conditions (without any differentiability hypotheses on the functions) and the sufficient conditions (with differentiability hypotheses on the functions) for the optimal solutions of the optimization problems have, among other hypotheses, that of concavity (or generalized concavity) of the functions that define the problem.

In this paper, one studies the concavity of a function by showing that its values are the optimum values of a convex optimization problem. The studied functions are those that can be written as powers of linear combinations of powers of some concave functions.

Let $r > 1$ be a real number. Obviously, the function $\varphi :]0, +\infty[\to \mathbb{R}$ defined, for any $t \in]0, +\infty[$, by $\varphi(t) = t^{1/r}$ is concave and increasing. From this, we deduce that, if $D \subseteq \mathbb{R}^n$ is a nonempty convex set and $f : D \to]0, +\infty[$ is a concave function, then the function $\alpha : D \to \mathbb{R}$ defined, for any $x \in D$, by

$$\alpha(x) = (f(x))^{1/r}$$

is concave. Now, from the properties of the concave functions, we obtain that, if $a_1, \ldots, a_p$ are real positive numbers, $D \subseteq \mathbb{R}^n$ is a nonempty convex set and $f_1, \ldots, f_p : D \to]0, +\infty[$ are concave functions, then the function $\beta : D \to \mathbb{R}$ defined, for any $x \in D$, by

$$\beta(x) = a_1(f_1(x))^{1/r} + \ldots + a_p(f_p(x))^{1/r}$$

*2000 Subject Classification: 90C25, 90C64

Key words and phrases: Concave function, optimization problem

is concave.

Below, we want to answer to the following question:

If $a_1, \ldots, a_p$ are p positive real numbers, $r > 1$ is a real number, $D \subseteq \mathbb{R}^n$ is a nonempty convex set and $f_1, \ldots, f_p : D \to]0, +\infty[$ are concave functions, is the function $\gamma : D \to \mathbb{R}$ defined, for any $x \in D$, by:

$$\gamma(x) = \left(\sum_{i=1}^{p} a_i (f_i(x))^{1/r} \right)^r$$

concave?

The answer is not immediate, because, as shown in the following example, the square of a positive concave function may not be concave.

Example. Let $f :]0, +\infty[\to]0, +\infty[$ be the function defined, for each $x \in \mathbb{R}$, $x > 0$, by $f(x) = x^{2/3}$. Evidently, the function f is concave on $]0, +\infty[$. Let's consider now the function $\gamma :]0, +\infty[\to \mathbb{R}$ defined, for each $x > 0$, by $\gamma(x) = f^2(x)$. Since $\gamma''(x) = (4/9)x^{-2/3} > 0$, for all $x > 0$, we deduce that the square of a concave function might not be a concave function (might be even a convex function, as in this example).

LEMMA 1. *Let $q \in]0, +\infty[$, $a = (a_1, \ldots, a_p) \in]0, +\infty[^p$ and $b = (b_1, \ldots, b_p) \in]0, +\infty[^p$. Then the optimization problem*

$$\min a_1 z_1 + \ldots + a_p z_p$$

subject to

$$b_1(z_1)^{-q} + \ldots + b_p(z_p)^{-q} \leq 1 \tag{1}$$

$$z_1 > 0, \quad i = 1, \ldots, p$$

has an unique optimal solution $z^0 = (z_1^0, \ldots, z_p^0) \in \mathbb{R}^p$ given by:

$$z_i^0 = \left(\frac{b_i}{a_i} \right)^{\frac{1}{q+1}} \left(\sum_{i=1}^{p} \left(\frac{a_i}{b_i} \right)^{\frac{q}{q+1}} \right)^{\frac{1}{q}}, \quad i = 1, \ldots, p.$$

Proof. Let $\psi :]0, +\infty[^p \to \mathbb{R}$ be defined, for each $z = (z_1, \ldots, z_p) \in]0, +\infty[^p$, by:

$$\psi(z) = b_1(z_1)^{-q} + \ldots + b_p(z_p)^{-q} - 1.$$

Evidently, the function ψ is convex and differentiable on $]0,+\infty[^p$.

Let us suppose that problem (1) has an optimal solution $z^0=(z_1^0,\ldots,z_p^0)$. Since the function ψ satisfies Slater's constraint qualification $[\psi((b_1(p+1))^{1/q},\ldots,(b_p(p+1))^{1/q})<0]$, in view of Karush-Kuhn-Tucker necessary optimality theorem [see [4], pp.109-110], there exists a nonnegative number ν such that:

$$a_i - q\nu b_i (z_i^0)^{-q-1} = 0, \quad i \in \{1,\ldots,p\}, \tag{2}$$

$$\nu\left(\sum_{i=1}^{p}(b_i z_i^0)^{-q}-1\right)=0. \tag{3}$$

If $\nu=0$, then from (2) it follows that $a_i=0$, $i\in\{1,\ldots,p\}$ which contradicts $a\in]0,+\infty[^p$. Hence $\nu>0$. Then, from (3) we deduce that:

$$\sum_{i=1}^{p} b_i (z_i^0)^{-q}=1. \tag{4}$$

On the other hand, from (2) it follows that:

$$z_i^0=\left(\frac{q\nu b_i}{a_i}\right)^{\frac{1}{q+1}}, \quad i=1,\ldots,p. \tag{5}$$

By substitution in (4), this implies that:

$$1=\sum_{i=1}^{p}\left(\frac{a_i}{q\nu b_i}\right)^{\frac{q}{q+1}}=\left(\frac{1}{q\nu}\right)^{\frac{q}{q+1}}\sum_{i=1}^{p}\left(\frac{a_i}{b_i}\right)^{\frac{q}{q+1}}.$$

From this it follows that:

$$\nu=\frac{1}{q}\left(\sum_{i=1}^{p}\left(\frac{a_i}{b_i}\right)^{\frac{q}{q+1}}\right)^{\frac{q+1}{q}}. \tag{6}$$

Now, by substitution in (5), it implies that:

$$z_i^0=\left(\frac{b_i}{a_i}\right)^{\frac{1}{q+1}}\left(\sum_{i=1}^{p}\left(\frac{a_i}{b_i}\right)^{\frac{q}{q+1}}\right)^{\frac{1}{q}}, \quad i=1,\ldots,p. \tag{7}$$

Therefore, if problem (1) has an optimal solution $z^0=(z_1^0,\ldots,z_p^0)$, then it is unique and it is given by (7).

On the other hand, problem (1) is convex and for $z^0=(z_1^0,\ldots,z_p^0)$ given by (7), there exists a nonnegative number ν given by (6) such that the Karush-Kuhn-Tucker conditions

(2)-(3) are hold. Then, in view of Karush-Kuhn-Tucker sufficient optimality theorem [see [4], pp. 93-95], the point z^0 is an optimal solution for problem (1).

Using Lemma 1, we can state the following theorem:

THEOREM 1. *Let* $r \in \mathbb{R}$, $r > 1$ *and* $(a_1, \ldots, a_p) \in]0, +\infty[^p$. *Let* $D \subseteq \mathbb{R}^n$ *be a nonempty convex set, let* $f_1, \ldots, f_p : D \to]0, +\infty[$ *be concave functions and* $g : D \to \mathbb{R}$ *be defined for each* $x \in D$ *by:*

$$g(x) = \left(\sum_{i=1}^{p} a_i (f_i(x))^{1/r} \right)^r .$$

Then g *is a concave function.*

Proof. Let $q = 1/(r-1)$. Then $q > 0$. Consider the function $h : D \to \mathbb{R}$ defined for each $x \in D$ by

$$h(x) = \min \left\{ \sum_{i=1}^{p} a_i z_i f_i(z) : \ (z_1, \ldots, z_p) \in Z \right\}, \tag{8}$$

where

$$Z = \left\{ (z_1, \ldots, z_p) \in]0, +\infty[^p : \ \sum_{i=1}^{p} a_i (z_i)^{-q} \leq 1 \right\}.$$

From Lemma 1, since $f_1, \ldots, f_p$ are strictly positive on D, it follows that the minimum in (8) exists and is finite for each $x \in D$. If, for each $z = (z_1, \ldots, z_p) \in Z$, we define the function:

$$h_z : D \to \mathbb{R} \quad \text{by} \quad h_z(x) = \sum_{i=1}^{p} a_i z_i f_i(x),$$

then for each $x \in D$, $h(x)$ may also be written as

$$h(x) = \min\{h_z(x) : \ (z_1, \ldots, z_p) \in Z\}. \tag{9}$$

Evidently, for each $(z_1, \ldots, z_p) \in Z$, the function h_z is concave. From this and (9), we deduce that the function h is also concave.

To complete the proof, we will show that, for each $x \in D$ we have $h(x) = g(x)$. Toward this end, fix $x \in D$ and let $z(x) = (z_1(x), \ldots, z_p(x)) \in Z$ denote an optimal solution to problem (8). From Karush-Kuhn-Tucker theorem, it follows that there exists a nonnegative number $\nu(x)$ such that

$$f_i(x) - q\nu(x)(z_i(x))^{-q-1} = 0, \quad i \in \{1, \ldots, p\} \tag{10}$$

$$(11) \qquad \left(\sum_{i=1}^{p} \frac{a_i}{(z_i(x))^{q+1}} - 1\right) \nu(x) = 0.$$

If $\nu(x) = 0$, then from (10) it follows that $f_i(x) = 0$ for each $i \in \{1, \dots, p\}$, which contradicts the strict positivity of $f_1, \dots, f_p$ on D. Hence $\nu(x) > 0$. Then, from (11) we deduce that:

$$(12) \qquad \sum_{i=1}^{p} \frac{a_i}{(z_i(x))^{q}} = 1.$$

On the other hand, from (10) it follows that:

$$z_i(x) = \left(\frac{q\nu(x)}{f_i(x)}\right)^{\frac{1}{q+1}}, \quad i \in \{1, \dots, p\}.$$

By substitution in (12), it implies that:

$$1 = \sum_{i=1}^{p} a_i \left(\frac{f_i(x)}{q\nu(x)}\right)^{\frac{q}{q+1}} = \left(\frac{1}{q\nu(x)}\right)^{\frac{q}{q+1}} \sum_{i=1}^{p} a_i (f_i(x))^{\frac{q}{q+1}}.$$

Therefore

$$(13) \qquad \nu(x) = \frac{1}{q}\left(\sum_{i=1}^{p} a_i (f_i(x))^{\frac{q}{q+1}}\right)^{\frac{q+1}{q}}.$$

Now, from, (12) and (13) we have that

$$\sum_{i=1}^{p} a_i z_i(x) f_i(x) = \sum_{i=1}^{p} a_i q\nu(x)(z_i(x))^{-q} = q\nu(x) \sum_{i=1}^{p} \frac{a_i}{(z_i(x))^{q}} = q\nu(x) =$$

$$= \left(\sum_{i=1}^{p} a_i (f_i(x))^{\frac{q}{q+1}}\right)^{\frac{q+1}{q}}$$

hence

$$(14) \qquad \sum_{i=1}^{p} a_i z_i(x) f_i(x) = \left(\sum_{i=1}^{p} a_i (f_i(x))^{\frac{q}{q+1}}\right)^{\frac{q+1}{q}}.$$

Since $(z_1(x), \dots, z_p(x)) \in Z$, is an optimal solution of problem (8), we have that the left-hand-side of equality (14) coincides with $h(x)$. The right-hand-side of equality (14) is equal with $g(x)$, because $(q+1)/q = r$. The proof is complete.

The following remarks show us that, in theorem 1, the exponent r cannot be changed.

Remark 1. Let $r \in \mathbb{R}$, $r > 1$, $(r_1, \dots, r_p) \in \mathbb{R}^p$, with $r_i > 1$, for each $i \in \{1, \dots, p\}$ and $(a_1, \dots, a_p) \in]0, +\infty[^p$. Let $D \subseteq \mathbb{R}^n$ be a convex nonempty set, $f_1, \dots, f_p : D \to]0, +\infty[$

be concave functions and $g : D \to \mathbb{R}$ be defined, for each $x \in D$, by

$$g(x) = \left(\sum_{i=1}^{p} a_i (f_i(x))^{\frac{1}{r_i}} \right)^r .$$

Is g a concave function? Usually, the answer is no. Let $r = 2$, $p = 2$, $r_1 = 4/3$, $r_2 = 2$, $a_1 = a_2 = 1$ and $f_1, f_2 :]0, +\infty[\to \mathbb{R}$ be the concave functions defined, for each $x \in]0, +\infty[$, by $f_1(x) = f_2(x) = x$. The function $g :]0, +\infty[\to]0, +\infty[$ defined, for each $x \in]0, +\infty[$, by

$$g(x) = (x^{3/4} + x^{1/2})^2 = x^{3/2} + x + 2x^{5/4}$$

is not concave, because $g''(x) = (3/4)x^{-1/2} + (5/8)^{-3/4} > 0$, for each $x \in]0, +\infty[$. (The function g is strictly convex).

Remark 2. Let $(r_1, \ldots, r_p) \in \mathbb{R}^p$, with $r_i > 1$, for each $i \in \{1, \ldots, p\}$ and $(a_1, \ldots, a_p) \in]0, +\infty[^p$. Let $D \subseteq \mathbb{R}^n$ be a nonempty convex set $f_1, \ldots, f_p : D \to]0, +\infty[$ be concave functions and $h : D \to \mathbb{R}$ be defined, for each $x \in D$, by

$$h(x) = \left(\sum_{i=1}^{p} a_i (f_i(x))^{1/r_i} \right)^{\sum\limits_{i=1}^{p} r_i} .$$

Is h a concave function? The answer is usually no. Let $p = 2$, $r_1 = 2$, $r_2 = 3$, $a_1 = a_2 = 1$ and $f_1, f_2 :]0, +\infty[\to \mathbb{R}$ be the concave functions defined, for each $x \in]0, +\infty[$, by $f_1(x) = f_2(x) = x$. The function $h :]0, +\infty[\to \mathbb{R}$ defined, for each $x \in]0, +\infty[$, by

$$h(x) = (x^{1/2} + x^{1/3})^5$$

is not concave, because $h''(x) > 0$, for each $x \in]0, +\infty[$. In fact, the function h is strictly convex.

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