FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 2 (2016), 529–541

A SPECIAL TYPE OF SEMI-SYMMETRIC NON-METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

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Abstract. The aim of the present paper is to study a Riemannian manifold admitting a type of semi-symmetric non-metric connection whose torsion tensor is pseudo symmetric.
Keywords: Semi-symmetric non-metric connection, Ricci-semisymmetric, locally symmetric

1. Introduction

In 1924, Friedmann and Schouten [11] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\widetilde{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection $\widetilde{\nabla}$ satisfies

$$T(X,Y) = u(Y)X - u(X)Y, \tag{1.1}$$

where u is a 1-form and ρ_1 is a vector field defined by

$$u(X) = g(X, \rho_1), \tag{1.2}$$

for all vector fields $X \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

In 1932, Hayden [12] introduced the idea of semi-symmetric metric connections on a Riemannian manifold (M,g). A semi-symmetric connection $\widetilde{\nabla}$ is said to be a semi-symmetric metric connection if

$$\widetilde{\nabla} g = 0. \tag{1.3}$$

A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of (M,g) was given by Yano [26]: $\tilde{\nabla}_X Y = \nabla_X Y + u(Y)X - g(X,Y)\rho_1$, where $u(X) = g(X,\rho_1)$.

Received January 12, 2016; accepted January 24, 2016
2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C25

The study of semi-symmetric metric connection was further developed by Amur and Pujar [2], Binh [5], De [8], Singh et al. [21], Ozgur et al ([14],[15]), Ozen, Uysal Demirbag [16], Zhao [28, 29], Velimirović et al [24, 25] and many others. After a long gap the study of a semi-symmetric connection $\bar{\nabla}$ satisfying

$$\bar{\nabla} g \neq 0. \tag{1.4}$$

was initiated by Prvanović [17] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [3].

A semi-symmetric connection $\bar{\nabla}$ is said to be a semi-symmetric non-metric connection if it satisfies the condition (1.4).

In 1992, Agashe and Chafle [1] studied a semi-symmetric non-metric connection $\bar{\nabla}$, whose torsion tensor $\bar{T}$ satisfies $\bar{T}(X,Y) = u(Y)X - u(X)Y$ and $(\bar{\nabla}_X g)(Y,Z) = -u(Y)g(X,Z) - u(Z)g(X,Y) \neq 0$. They proved that the projective curvature tensor of the manifold with respect to these two connections are equal to each other. In 1992, Barua and Mukhopadhyay [4] studied a type of semi-symmetric connection $\bar{\nabla}$ which satisfies

$$(\bar{\nabla}_X g)(Y,Z) = 2u(X)g(Y,Z) - u(Y)g(X,Z) - u(Z)g(X,Y).$$

Since $\bar{\nabla} g \neq 0$, this is another type of semi-symmetric non-metric connection. However, the authors preferred the name semi-symmetric semimetric connection.

In 1994, Liang [13] studied another type of semi-symmetric non-metric connection $\bar{\nabla}$ for which we have $(\bar{\nabla}_X g)(Y,Z) = 2u(X)g(Y,Z)$, where u is a non-zero 1-form and he called this a semi-symmetric recurrent metric connection.

The semi-symmetric non-metric connections was further developed by several authors such as De and Biswas [9], De and Kamilya [10], Liang [13], Singh et al. ([20], [22], [23]), Smaranda [18], Smaranda and Andonie [19] and many others.

We consider a type of linear connection given by

$$\bar{\nabla}_X Y = \nabla_X Y + a\omega(X)Y + b\omega(Y)X, \tag{1.5}$$

where a and b are two non-zero real numbers and ρ is a vector field defined by $\omega(X) = g(X, \rho)$, for all $X \in \chi(M)$, the set of all differentiable vector fields on M.

The torsion tensor $\bar{T}$ with respect to $\bar{\nabla}$ is

$$\bar{T}(X,Y) = (b-a)\omega(Y)X - (b-a)\omega(X)Y = \pi(Y)X - \pi(X)Y, \tag{1.6}$$

where $\pi(X) = (b-a)\omega(X)$.

Therefore, the connection $\bar{\nabla}$ is a semi-symmetric connection. Also

$$(\bar{\nabla}_X g)(Y,Z) = -2a\omega(X)g(Y,Z) - b\omega(Y)g(X,Z) - b\omega(Z)g(X,Y) \neq 0.$$

Hence the semi-symmetric connection $\bar{\nabla}$ defined by (1.5) is a semi-symmetric non-metric connection.

In 1987, Chaki [7] defined the notion of pseudo symmetric manifolds. A non-flat Riemannian manifold (M^n, g), $n \geq 2$ is said to be a pseudo symmetric manifold if its curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)U &= 2\omega(X)R(Y,Z)U + \omega(Y)R(X,Z)U \\ &\quad +\omega(Z)R(Y,X)U + \omega(U)R(Y,Z)X \\ &\quad +g(R(Y,Z)U,X)\rho,\end{aligned} \tag{1.7}$$

where ω is a non-zero 1-form and ρ is a vector field defined by

$$\omega(X) = g(X,\rho), \quad \text{for all } X,$$

and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The 1-form ω is called the associated 1-form of the manifold. If $\omega = 0$, then the manifold reduces to a symmetric manifold in the sense of Cartan [6]. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$.

A Riemannian manifold is said to be Ricci-semisymmetric with respect to the Levi-Civita connection ∇, if

$$(R(X,Y)\cdot S)(U,V) = 0.$$

A Riemannian manifold is said to be locally symmetric due to Cartan or Cartan symmetric if it satisfies $\nabla R = 0$.

The Weyl projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a n-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the projective curvature tensor vanishes. Here the Weyl projective curvature tensor $\mathbf{P}$ with respect to the Levi-Civita connection is defined by

$$\mathbf{P}(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{1.8}$$

for X, Y, $Z \in \chi(M)$. In fact, M is projectively flat if and only if it is of constant curvature [27]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

In this paper we study a special type of the semi-symmetric non-metric connection on Riemannian manifolds. The paper is organized as follows: After introduction in Section 2, we define a special type of semi-symmetric non-metric connection and we also construct an example of a special type semi-symmetric non-metric connection on Riemannian manifolds. In Section 3, we give some properties of a special type of semi-symmetric non-metric connection. Next Section deals with the relation of the curvature tensors between the Levi-Civita connection and the semi-symmetric non-metric connection on a Riemannian manifold whose torsion tensor is pseudo symmetric with respect to a special type semi-symmetric non-metric connection. Also we characterized a Riemannian manifold admitting a type of semisymmetric non-metric connection whose curvature tensor vanishes and the torsion tensor is pseudosymmetric. Weyl projective curvature tensor on Riemannian manifolds admitting a special type of the semi-symmetric non-metric connection have been studied in Section 5. Finally, we have classified the Ricci-semisymmetric Riemannian manifolds admitting a special type of the semi-symmetric non-metric connection.

2. Existence of a type of semi-symmetric non-metric connection

We consider a type of linear connection $\bar{\nabla}$ and the Levi-Civita connection ∇ of a Riemannian manifold M such that

$$\bar{\nabla}_X Y = \nabla_X Y + H(X, Y),$$

where H is a tensor of type $(1, 2)$ and $X, Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M. For $\bar{\nabla}$ to be a semi-symmetric non-metric connection in M, we have

$$H(X, Y) = \frac{1}{2}[\bar{T}(X, Y) - \grave{T}(X, Y) + \grave{T}(Y, X)] + a\omega(Y)X + b\omega(X)Y, \tag{2.1}$$

where $g(X, \rho) = \omega(X)$ and $\grave{T}$ is a tensor of type $(1, 2)$ such that

$$g(\bar{T}(Z, X), Y) = g(\grave{T}(X, Y), Z). \tag{2.2}$$

Combining (1.6) and (2.2), implies that

$$\grave{T}(X, Y) = \pi(X)Y - g(X, Y)\rho, \tag{2.3}$$

where $\pi(X) = (b - a)\omega(X)$. In view of (1.6), (2.1) and (2.3)yields

$$H(X, Y) = a\omega(X)Y + b\omega(Y)X.$$

Therefore, the semi-symmetric non-metric connection on a Riemannian manifold is given by

$$\bar{\nabla}_X Y = \nabla_X Y + a\omega(X)Y + b\omega(Y)X.$$

Conversely, we prove that a linear connection $\bar{\nabla}$ such that $\bar{\nabla}_X Y = \nabla_X Y + a\omega(X)Y + b\omega(Y)X$ is a semi-symmetric non-metric connection on a Riemannian manifold.

The torsion tensor $\bar{T}$ of the connection is given by

$$\bar{T}(X, Y) = (b - a)\omega(Y)X - (b - a)\omega(X)Y = \pi(Y)X - \pi(X)Y.$$

From the above equation, we obtain that the connection $\bar{\nabla}$ is a semi-symmetric connection. Also we have

$$(\bar{\nabla}_X g)(Y, Z) = -2a\omega(X)g(Y, Z) - b\omega(Y)g(X, Z) - b\omega(Z)g(X, Y) \neq 0.$$

Therefore, we are in a position to conclude that the connection $\bar{\nabla}$ is a semi-symmetric non-metric connection.

Now, we give an example of a special type semi-symmetric non-metric connection on Riemannian manifolds.

Example 2.1. *In local co-ordinate system let us denote the Riemannian - Christoffel symbols by* Γ_{ij}^h *and* $\{{}^{h}_{ij}\}$ *with respect to the semi-symmetric connection and the Levi-Civita connection respectively. Then we can express equation* (1.5) *as follows:*

$$\Gamma_{ij}^h = \{{}^{h}_{ij}\} + a\eta_j\delta_i^h + b\eta_i\delta_j^h. \tag{2.4}$$

Let us consider a Riemannian metric g *on* $\mathbb{R}^4$ *given by*

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (dx^3)^2 + (dx^4)^2, \tag{2.5}$$

$(i, j = 1, 2, 3, 4)$. *Then the only non-vanishing components of the Christoffel symbols with respect to the Levi-Civita connections are*

$$\{{}^{1}_{22}\} = -x^1, \{{}^{2}_{12}\} = \{{}^{2}_{21}\} = \frac{1}{x^1}.$$

Let us define η^i *by* $\eta^i = (0, -\frac{1}{(x^1)^2}, 0, 0)$. *If* Γ_{ij}^h *corresponds to the semi-symmetric connections, then from* (2.4)*, we have the non-zero components of* Γ_{ij}^h *as*

$$\Gamma_{22}^1 = \{{}^{1}_{22}\} + a\eta_2\delta_2^1 + b\eta_2\delta_2^1 = -x^1.$$

Similarly, we obtain

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{x^1},\ \Gamma_{32}^3 = \Gamma_{42}^4 = \Gamma_{12}^1 = -a,\ \Gamma_{23}^3 = \Gamma_{24}^4 = \Gamma_{21}^1 = -b.$$

Now we have

$$g_{22,1} = \frac{\partial g_{22}}{\partial x^1} - g_{2h}\Gamma_{21}^h - g_{2h}\Gamma_{21}^h = 0,$$

with respect to the semi-symmetric connection Γ*, where ”,” denotes the covariant derivative with respect to the semi-symmetric connection* Γ*. But*

$$g_{11,2} = g_{33,2} = g_{44,2} = 2a \neq 0,\ \ g_{12,1} = g_{32,3} = g_{42,4} = b \neq 0.$$

Thus, Γ *is not a metric connection. So,* Γ *is a semi-symmetric non-metric connection.*

3. Semi-symmetric non-metric connection

Definition 3.1. The 1-form ω is closed with respect to the Levi-Civita connection if

$$(\nabla_X \omega)(Y) - (\nabla_Y \omega)(X) = 0,$$

where ρ is a vector field defined by $\omega(X) = g(X, \rho)$, ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and $X, Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

The vector field ρ is irrotational if $g(Y, \nabla_X \rho) = g(X, \nabla_Y \rho)$ and the integral curves of the vector field ρ are geodesic if $\nabla_\rho \rho = 0$.

Equation (1.5) implies that

$$(\bar{\nabla}_X \omega)(Y) = (\nabla_X \omega)(Y) - (a+b)\omega(X)\omega(Y). \tag{3.1}$$

The above relation gives

$$(\bar{\nabla}_X \omega)(Y) - (\bar{\nabla}_Y \omega)(X) = (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X),$$

this means that 1-form ω is closed with respect to the Levi-Civita connection ∇ if and only if ω is closed with respect to the semi-symmetric non-metric connection $\bar{\nabla}$.

Putting $Y = \rho$ in (1.5), we get

$$\bar{\nabla}_X \rho = \nabla_X \rho + a\omega(X)\rho + b\omega(\rho)X. \tag{3.2}$$

The above equation yields

$$g(Y, \bar{\nabla}_X \rho) - g(X, \bar{\nabla}_Y \rho) = g(Y, \nabla_X \rho) - g(X, \nabla_Y \rho),$$

which implies that the vector field ρ is irrotational with respect to ∇ if and only if ρ is irrotational with respect to $\bar{\nabla}$.

Again putting $X = \rho$ in (3.2), we obtain

$$\bar{\nabla}_\rho \rho = \nabla_\rho \rho + (a+b)\omega(\rho)\rho. \tag{3.3}$$

If $a = -b$, then from the equation (3.3), it follows that

$$\bar{\nabla}_\rho \rho = \nabla_\rho \rho,$$

from this result we have the integral curves of the unit vector field ρ are geodesic with respect to ∇ if and only if the integral curves of the unit vector field ρ is geodesic with respect to $\bar{\nabla}$. From the above discussion we can state the following:

Theorem 3.1. *If a Riemannian manifold admits a special type of semi-symmetric non-metric connection, then*

(i) the 1*-form* ω *is closed with respect to the semi-symmetric non-metric connection if and only if the* 1*-form* ω *is also closed with respect to the Levi-Civita connection,*

(ii) the vector field ρ *is irrotational with respect to the semi-symmetric non-metric connection if and only if the vector field* ρ *is also irrotational with respect to the Levi-Civita connection and,*

(iii) the integral curves of the unit vector field ρ *are geodesic with respect to the semi-symmetric non-metric connection if and only if the integral curves of the unit vector field* ρ *are also geodesic with respect to the Levi-Civita connection provided the non-zero real numbers of the connection satisfy the relation* $a = -b$.

4. Expression of the curvature tensor of the semi-symmetric non-metric connection

In this section we obtain the expressions of the curvature tensor and Ricci tensor of M with respect to the semi-symmetric non-metric connection defined by (1.5).

Analogous to the definitions of the curvature tensor R of M with respect to the Levi-Civita connection ∇, we define the curvature tensor $\bar{R}$ of M with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ given by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z, \tag{4.1}$$

where $X, Y, Z \in \chi(M)$, the set of all differentiable vector fields on M. Using (1.5) in (4.1), we get

$$\begin{aligned}\bar{R}(X,Y)Z \;=\;& R(X,Y)Z - a(\nabla_Y\omega)(X)Z + a(\nabla_X\omega)(Y)Z - b(\nabla_Y\omega)(Z)X\\ &+b(\nabla_X\omega)(Z)Y + b^2\omega(Y)\omega(Z)X - b^2\omega(X)\omega(Z)Y.\end{aligned} \tag{4.2}$$

From (1.6) we obtain

$$(\bar{\nabla}_X C_1^1\bar{T})(Y) = (n-1)\pi(Y) = (n-1)(b-a)(\bar{\nabla}_X\omega)(Y), \tag{4.3}$$

where C_1^1 denotes the contraction.

Suppose the torsion tensor $\bar{T}$ with respect to the semi-symmetric non-metric connection is pseudo symmetyric, that is,

$$\begin{aligned}(\bar{\nabla}_X\bar{T})(Y,Z) \;=\;& \omega(X)\bar{T}(Y,Z) + \omega(Y)\bar{T}(X,Z) + \omega(Z)\bar{T}(Y,X)\\ &+g(\bar{T}(Y,Z),X)\rho,\end{aligned} \tag{4.4}$$

where $\omega(X) = g(X,\rho)$.

Contracting over Z in (4.4) and using (1.6), we obtain

$$(\bar{\nabla}_X C_1^1\bar{T})(Y) = 4(n-1)(b-a)\omega(X)\omega(Y) - (b-a)\omega(\rho)g(X,Y). \tag{4.5}$$

Combining (4.3) and (4.5), we have

$$(\bar{\nabla}_X\omega)(Y) = 4\omega(X)\omega(Y) - \frac{\omega(\rho)}{n-1}g(X,Y). \tag{4.6}$$

Therefore, from (3.1) and (4.6), it follows that

$$(\nabla_X\omega)(Y) = (a+b+4)\omega(X)\omega(Y) - \frac{\omega(\rho)}{n-1}g(X,Y). \tag{4.7}$$

In view of (4.7) the equation (4.2) takes the form

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z - b(a+4)\omega(Y)\omega(Z)X + b(a+4)\omega(X)\omega(Z)Y \\ &\quad + \frac{b\omega(\rho)}{n-1}g(Y,Z)X - \frac{b\omega(\rho)}{n-1}g(X,Z)Y.\end{aligned} \tag{4.8}$$

From (4.8), it follows that

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z,$$

and

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0. \tag{4.9}$$

We call (4.9) the *first Bianchi identity* with respect to the semi-symmetric non-metric connection $\bar{\nabla}$.

Taking the inner product of (4.8) with U, we obtain

$$\begin{aligned}{}'\bar{R}(X,Y,Z,U) &= {}'R(X,Y,Z,U) - b(a+4)\omega(Y)\omega(Z)g(X,U) \\ &\quad + b(a+4)\omega(X)\omega(Z)g(Y,U) + \frac{b\omega(\rho)}{n-1}g(Y,Z)g(X,U) \\ &\quad - \frac{b\omega(\rho)}{n-1}g(X,Z)g(Y,U),\end{aligned} \tag{4.10}$$

where ${}'\bar{R}(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U)$ and ${}'R(X,Y,Z,U) = g(R(X,Y)Z,U)$.

Let $\{e_1,...,e_n\}$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by putting $X = U = e_i$ in (4.10) and taking summation over i, $1 \le i \le n$, we have

$$\bar{S}(Y,Z) = S(Y,Z) + b\omega(\rho)g(Y,Z) - b(n-1)(a+4)\omega(Y)\omega(Z), \tag{4.11}$$

where $\bar{S}$ and S denote the Ricci tensor of M with respect to $\bar{\nabla}$ and ∇ respectively.

The above discussion helps us to state the following proposition:

Proposition 4.1. *For a Riemannian manifold M with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ whose torsion tensor is pseudo symmetric,* *(i) The curvature tensor $\bar{R}$ is given by*

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z - b(a+4)\omega(Y)\omega(Z)X + b(a+4)\omega(X)\omega(Z)Y \\ &\quad + \frac{b\omega(\rho)}{n-1}g(Y,Z)X - \frac{b\omega(\rho)}{n-1}g(X,Z)Y.\end{aligned}$$

(ii) The Ricci tensor $\bar{S}$ is given by

$$\bar{S}(Y,Z) = S(Y,Z) + b\omega(\rho)g(Y,Z) - b(n-1)(a+4)\omega(Y)\omega(Z),$$

(iii)

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z,$$

(iv)

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0,$$

(v) The Ricci tensor $\bar{S}$ is symmetric.

Let us suppose the curvature tensor $\bar{R}$ with respect to the semi-symmetric non-metric connection vanishes, that is,

$$'\bar{R} = 0.$$

Using the above relation in (4.10), we see that

$$\begin{aligned}'R(X,Y,Z,U) &= b(a+4)\omega(Y)\omega(Z)g(X,U) - b(a+4)\omega(X)\omega(Z)g(Y,U) \\ &\quad - \frac{b\omega(\rho)}{n-1}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].\end{aligned} \tag{4.12}$$

Putting $a = -4$ in (4.12), the above equation reduces to

$$'R(X,Y,Z,U) = -\frac{b\omega(\rho)}{n-1}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \tag{4.13}$$

This result shows that the manifold is of constant curvature.

Now, we are in a position to state the following:

Theorem 4.1. *A Riemannian manifold admitting a type of the semi-symmetric non-metric connection whose curvature tensor vanishes and the torsion tensor is pseudo symmetric is a manifold of constant curvature with respect to the Levi-Civita connection provided the value of the non-zero real number a of the connection is -4.*

5. Weyl projective curvature tensor on a Riemannian manifold admitting a special type of the semi-symmetric non-metric connection

The Weyl projective curvature tensor $\bar{P}$ with respect to the semi-symmetric non-metric connection is defined by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-1}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y]. \tag{5.1}$$

From (5.1), it follows that

$$'\bar{P}(X,Y,Z,U) =' \bar{R}(X,Y,Z,U) - \frac{1}{n-1}[\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U)], \tag{5.2}$$

where $'\bar{P}(X,Y,Z,U) = g(\bar{P}(X,Y)Z,U)$, for all X, Y, Z , $U \in \chi(M)$.

Using (4.10) and (4.11) in (5.2), it follows that

$$'\bar{P}(X,Y,Z,U) =' P(X,Y,Z,U), \tag{5.3}$$

where

$$'P(X,Y,Z,U) =' R(X,Y,Z,U) - \frac{1}{n-1}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U)]. \tag{5.4}$$

This leads us to state the following:

Theorem 5.1. *If a Riemannian manifold admits a type of the semi-symmetric non-metric connection whose torsion tensor is pseudo symmetric, then the Weyl projective curvature tensor with respect to the semi-symmetric non-metric connection is equal to the Weyl projective curvature tensor with respect to the Levi-Civita connection.*

6. Ricci-semisymmetric manifolds

A Riemannian manifold is said to Ricci-semisymmetric with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ if

$$(\bar{R}(X,Y) \cdot \bar{S})(U,V) = 0,$$

where $X, Y, U, V \in \chi(M)$. Then we have

$$(\bar{R}(X,Y) \cdot \bar{S})(U,V) = \bar{S}(\bar{R}(X,Y)U,V) + \bar{S}(U,\bar{R}(X,Y)V). \tag{6.1}$$

Using (4.11) in (6.1), we get

$$\begin{aligned}(\bar{R}(X,Y) \cdot \bar{S})(U,V) &= S(\bar{R}(X,Y)U,V) + S(\bar{R}(X,Y)V,U)\\ &\quad +b\omega(\rho)[g(\bar{R}(X,Y)U,V) + g(\bar{R}(X,Y)V,U)\\ &\quad -b(n-1)(a+4)[\omega(\bar{R}(X,Y)U)\omega(V)\\ &\quad +\omega(\bar{R}(X,Y)V)\omega(U)].\end{aligned} \tag{6.2}$$

By virtue of (4.8) and (6.2), we obtain

$$\begin{aligned}(\bar{R}(X,Y)\cdot\bar{S})(U,V) &= (R(X,Y)\cdot S)(U,V)+b\omega(\rho)[' R(X,Y,U,V)\\ &-\frac{1}{n-1}\{S(Y,U)g(X,V)-S(X,U)g(Y,V)\}]\\ &+b\omega(\rho)[' R(X,Y,V,U)-\frac{1}{n-1}\{S(Y,V)g(X,U)\\ &-S(X,V)g(Y,U)\}]-b(n-1)(a+4)\omega(R(X,Y)U)\omega(V)\\ &-b(a+4)\omega(Y)\omega(U)S(X,V)+b(a+4)\omega(X)\omega(U)S(Y,V)\\ &-b(n-1)(a+4)\omega(R(X,Y)V)\omega(U)-b(a+4)\omega(Y)\omega(V)S(X,U)\\ &+b(a+4)\omega(X)\omega(V)S(Y,U).\end{aligned} \tag{6.3}$$

Putting $a=-4$ in (6.3) and using (5.4), we have

$$\begin{aligned}(\bar{R}(X,Y)\cdot\bar{S})(U,V) &= (R(X,Y)\cdot S)(U,V)\\ &+b\omega(\rho)[' P(X,Y,U,V)+' P(X,Y,V,U)].\end{aligned} \tag{6.4}$$

Summing up we can state the following:

Theorem 6.1. *Ricci semi-symmetry of a Riemannian manifold with respect to the Levi-Civita connection and the semi-symmetric non-metric connection are equivalent, provided $a=-4$ and ρ is a null vector.*

7. Acknowledgments

The third author would like to thank Professor H. Z. Li for his encouragement and help.

The work was supported by National Natural Science Foundation of China (No.11371194) and by Graduate Student Innovation Engineering of Jiangsu Province (No.CXZZ130186)

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