A SPECTRAL APPROACH FOR SCATTERING BY IMPEDANCE POLYGONS

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[Received 5 May 2005. Revise 1 January 2006. Accepted 4 July 2006]

Summary

We study a new spectral approach for scattering by two-dimensional polygonal objects with arbitrary surface impedance conditions. In this delicate exterior problem, the Wiener–Hopf method cannot be applied, while asymptotic methods can only be used if corners are widely spaced compared to wavelength. A new method based on the Sommerfeld–Maliuzhinets integral representation is presented to reduce the problem to simple spectral equations in the complex plane. For this, we use an expression of the spectral function, where we can isolate the contribution of any element of an arbitrary surface. Considering polygons with impedance boundary conditions, it then becomes possible to derive functional equations on spectral functions of Maliuzhinets type for finite or infinite objects. We apply this approach to an important class of three-part impedance polygons composed of a finite segment attached to two semi-infinite planes, and reduce this problem to non-singular Fredholm integral equations, suitable for approximation or numerical inversion. In the particular cases of a three-part impedance plane or symmetric impedance polygon, we show that the system of integral equations in the spectral domain can be simply uncoupled.

1. Introduction

This paper presents a new spectral approach for the problem of scattering by two-dimensional semi-infinite or finite polygonal objects with an imperfectly reflective surface, illuminated by a plane wave. Solutions for such generic objects play a crucial role in the calculation and the interpretation of the field scattered by complex objects.

Up to now, the study of scattering by objects with several singularities has remained a delicate task: the high-frequency techniques only yield asymptotic solutions for widely spaced singularities (**1** to **3**), while numerical methods with discretization of space are time-consuming because of singularities and oscillations of the field (even if some methods use some asymptotics (**4**, **5**)). Besides, the problem concerning a general impedance polygon in free space has not yet been studied by spectral methods (see for example (**1**, **6** to **10**)), even if some recent analyses have proposed representations valid in a convex polygon (**11**), outside a general wedge-shaped region (**10**), or for angular geometries with perfectly reflective surfaces (**12**, **13**). The presence of imperfectly reflective surfaces particularly complicates the problem. So we note that, in the particular case of diffraction by a three-part plane with three distinct impedances at real frequency, the well-known Wiener–Hopf method leads to coupled singular integral equations. Finally note some very recent promising

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Q. Jl Mech. Appl. Math, Vol. 59. No. 4 ©
Advance Access publication 27 October 2006. doi:10.1093/qjmam/hbl014

progress in numerical methods to address this problem using PML absorbing conditions (**14**) or a random walk method (**15**).

We present here a general method to handle the problem of n-part polygonal objects using the Sommerfeld–Maliuzhinets representation of the field. This representation has long been devoted to the sole analysis of isolated wedges (**16** to **24**). However, some of our recent developments (**10**) permit us to consider a new integral expression of the spectral function, in some domain of complex angles, where it becomes possible to take account of boundary conditions on a complex geometry.

This generalization of Maliuzhinets methods enables us to derive, for the first time, the functional equations for the spectral functions for scattering by a general impedance polygon with finite or infinite surface, and to reduce the problem concerning an infinite three-part impedance polygon to a system of two Fredholm integral equations of the second kind, with non-singular kernels allowing approximations.

The paper is organized as follows. In section 2, we develop the expression of the spectral function, related to the Sommerfeld–Maliuzhinets representation of the field, as an integral along a semi-infinite polygonal line. In section 3, we apply it to the basic problem of scattering by impedance polygons, and derive functional equations in spectral domain. In section 4, using the meromorphy of the functions involved in this problem, we then show how to decompose the field in simple elements. Next, we illustrate and develop the method for a three-part infinite polygonal scatterer. We then define two unknown spectral functions and write coupled functional equations for them in section 5. We reduce, in section 6, the system of functional equations to a set of non-singular Fredholm integral equations of the second kind. It is shown in section 7 that the scattering diagram can be derived in a simple manner from the solutions of these equations. In section 8, we give some important features of the integral equations. We show how to decouple the set of integral equations in some particular cases, then we study particular properties of kernels permitting approximations when the wave number k is large or small, and give some numerical results in this case.

2. Spectral function in the Sommerfeld–Maliuzhinets representation of the field: properties, expressions and particular relations with radiation of one face

Let us consider the case of diffraction in free space of a plane wave

$$u^i(\rho,\varphi) = e^{ik\rho\cos(\varphi-\varphi_\circ)} \tag{2.1}$$

by a scatterer enclosed in a wedge-shaped region, defined in cylindrical coordinates (ρ,φ) as the domain outside the free space angular sector with origin O, $-\Phi_r \leqslant \varphi \leqslant \Phi_l$ (Fig. 1). The plane wave comes from the free space and $-\Phi_r \leqslant \varphi_\circ \leqslant \Phi_l$. The characteristics of the scatterer are supposed to be independent of the z-coordinate. An implicit harmonic dependence on time $e^{i\omega t}$ is understood and henceforth suppressed. In (2.1) k denotes the wave number of the exterior medium with $|\arg(ik)| < \frac{1}{2}\pi$. Physically, $|\arg(ik)| < \frac{1}{2}\pi$ means that there are some losses in free space, and $|\arg(ik)| = \frac{1}{2}\pi$ is considered as a limit case. We assume that the total field in the free space region, $u = u_s + u^i$, satisfies the Helmholtz equation

$$(\Delta + k^2)u(\rho,\varphi) = 0, \tag{2.2}$$

and that $u(\rho,\varphi)$ is analytic with respect to ρ, φ and $\varphi_\circ$, except possibly at the origin, and that there exists a constant $s_\circ$ such that $\int_0^\infty |u(\rho,\varphi)e^{-s_\circ\rho}|d\rho < \infty$.

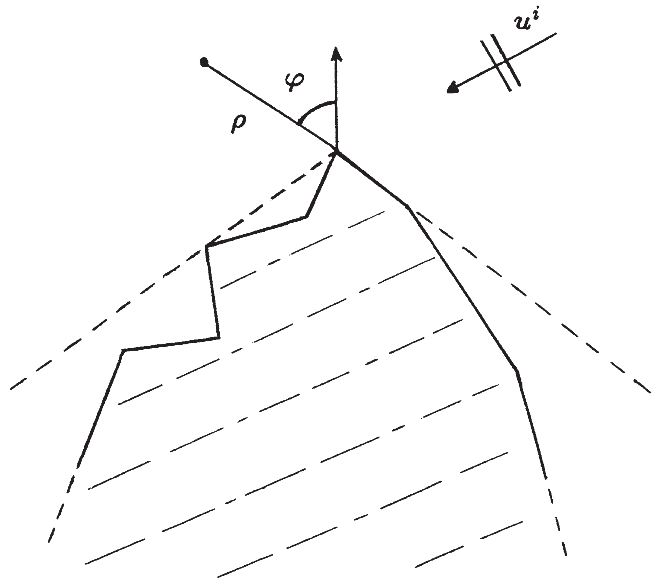

Fig. 1 Geometry: wedge-shaped and polygonal regions

The total field u for $-\Phi_r \leqslant \varphi \leqslant \Phi_l$ can be then represented as a Sommerfeld–Maliuzhinets integral (**10**, **16**)

$$u(\rho, \varphi) = \frac{1}{2\pi i} \int_{\gamma} f(\alpha + \varphi) e^{ik\rho \cos \alpha} d\alpha, \tag{2.3}$$

which satisfies the Helmholtz equation. In this representation, f is an analytic function and the path γ (Fig. 2) consists of two branches: one, named γ_+, going from $(i\infty + \arg(ik) + (a_1 + \frac{1}{2}\pi))$ to $(i\infty + \arg(ik) - (a_2 + \frac{1}{2}\pi))$ with $0 < a_{1,2} < \pi$, as $\mathrm{Im}(\alpha) \geqslant d$, above all the singularities of the integrand, and the other, named γ_-, obtained by inversion of $\gamma_+$with respect to $\alpha = 0$.

This representation is commonly applied for a wedge (**17**). We investigate its use for a scatterer with several discontinuities. This study requires us to express a shift of the origin; for this, we develop and use particular relations of f with radiation by one face only.

2.1 *Some properties of the field and f in the complex plane*

From (**1**, **2**, **10**, **16**, **18**, **24**, **25**), it follows that some elementary properties can be assumed to hold for the field:

(a′) the only incoming plane wave, from the free space sector with origin O, $-\Phi_r \leqslant \varphi \leqslant \Phi_l$, is the incident field;

(b′) the limit of the field u as $\rho \to 0$ is finite and does not depend on φ, while the derivatives $\partial_\rho u$ and $\partial_\varphi u/\rho$ are locally summable with respect to ρ in the vicinity of the origin. This property applies for an origin taken at any point out of or upon the scatterer;

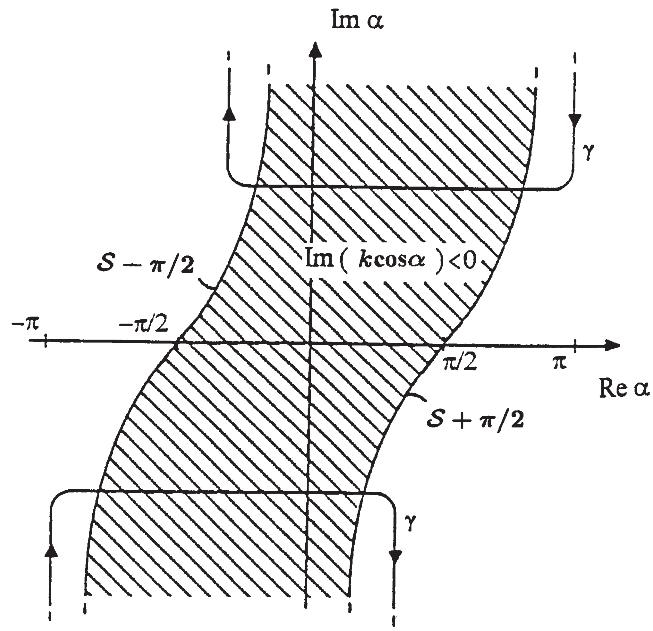

Fig. 2 Description of paths in the complex plane

(c′) the field, except possibly its geometrical-optics part when $\mathrm{Im}(k) \neq 0$, does not grow at infinity. In addition, some bounds on the far field are assumed. We consider here that the field is $O(e^{ik\rho\cos(\varphi-\varphi_\circ)})$ for large ρ, $|\arg(ik)| < \frac{1}{2}\pi$, which is a standard assumption (**1**, **2**) when the scattering object has impedance boundary conditions.

These properties lead us to assume some elementary conditions on f in this case (**16**, **18**, **24**, **25**):

(a) $(f(\alpha) - u^i(O)/(\alpha - \varphi_\circ))$ is regular at points with $\mathrm{Re}(\alpha)$ belonging to the free space angular sector with origin O, $-\Phi_r \leqslant \mathrm{Re}(\alpha) \leqslant \Phi_l$, which ensures (a′);

(b) there exist some constants $g^{\pm}$, some analytic function h, and some Maliuzhinets contour γ such that $|f(\alpha+\varphi) \mp f(-\alpha+\varphi) - g^{\pm}| < |h(\alpha)|$ on and inside the loop formed by the upper branch γ_+ of γ, when $-\Phi_r \leqslant \varphi \leqslant \Phi_l$, the function h being summable on γ_+, regular on and within it. In this respect, we notice that $(f(i|\ln\rho|) - f(-i|\ln\rho|)) = -iu(0,\varphi) + O(\rho\partial u/\partial\rho)$, as $\rho \to 0$, with $\rho\partial u/\partial\rho = o((\ln\rho)^{-1})$ and $u(0,\varphi) = ig^{+}$. Since γ is odd, we can add a constant to f without changing u, which implies that we can define f with $f(i\infty) = -f(-i\infty)$. This ensures (b′);

(c) $f(\alpha+\varphi)$ has no singularity, except possibly those associated to incident, reflected or transmitted plane waves not vanishing at infinity, in the zone defined by $\mathrm{Re}(ik\cos\alpha) > 0$ as $|\mathrm{Re}(\alpha)| < \pi$, $-\Phi_r \leqslant \varphi \leqslant \Phi_l$, $|\arg(ik)| < \frac{1}{2}\pi$. Considering that the far field is $O(e^{ik\rho\cos(\varphi-\varphi_\circ)})$, $f(\alpha+\varphi)$ has no singularity in this region when $\mathrm{Re}(ik\cos(\varphi-\varphi_\circ)) < 0$, that is, $\frac{1}{2}\pi < |\varphi-\varphi_\circ| < \frac{3}{2}\pi$. This ensures (c′).

2.2 *Spectral function corresponding to radiation of a single face*

We now apply Green's theorem (**1**, **2**). To simplify the notation without losing generality, we take $\Phi_r = \Phi_l = \Phi$. From the properties (b′), (c′) on u, the scattered field in free space $u_s = u - u^i$ for $|\varphi| < \Phi$ can be written as the sum $u_+ + u_-$ of the radiations of equivalent surface currents carried by the faces $\varphi = +\Phi$ and $-\Phi$,

$$u_\pm(\rho,\varphi) = \frac{-i}{4} \lim_{\rho_0 \to 0^+} \int_{\rho_0}^{\infty} \left(u(\rho',\varphi') \frac{\partial H_0^{(2)}(kR)}{\partial n} - \frac{\partial u(\rho',\varphi')}{\partial n} H_0^{(2)}(kR) \right) |_{\varphi'=\pm\Phi} d\rho', \quad (2.4)$$

with $R = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi')}$, $\partial(\cdot)/\partial n = \widehat{n}\nabla(\cdot) = \mp\partial(\cdot)/\rho'\partial\varphi'$, $\widehat{n}$ the outward normal to the face $\varphi' = \pm\Phi$, $|\varphi| < \Phi$, $|\arg(ik)| < \frac{1}{2}\pi$. We show that it is possible to express the spectral function corresponding to $u_\pm$ with f. For this, we use a Sommerfeld–Maliuzhinets representation of $H_0^{(2)}(kR)$.

As shown in (**10**, **25**), the spectral function associated with $H_0^{(2)}(kR)$, where $R = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi')}$, $|\varphi'| > \frac{1}{2}\pi$, $|\varphi| < |\varphi'|$, is given by

$$f_{H_0^{(2)}}(\alpha) = \frac{-1}{2\pi} \int_{\mathcal{S}} e^{ik\rho' \cos(\alpha' - \varphi')} \left(\tan\left(\tfrac{1}{2}(\alpha - \alpha')\right) - g_\circ(\alpha') \right) d\alpha' \quad (2.5)$$

for $\alpha \in]\mathcal{S} - \pi, \mathcal{S} + \pi[$ (that is, the domain limited by $\mathcal{S} - \pi$ and $\mathcal{S} + \pi$), where $\mathcal{S}$ is the path from $-i\infty-\arg(ik)$ to $i\infty+\arg(ik)$ with $\mathrm{Im}(k \sin\alpha) = 0$ (see Fig. 2). The term $g_\circ$, normally unnecessary because γ is odd, can be chosen as $g_\circ(\alpha) = -\tan(\frac{1}{2}\alpha)$ in order to ameliorate the convergence of the integral.

By analyticity, the path $\mathcal{S}$ can be deformed continuously, as long as the integrand remains bounded, without changing $f_{H_0^{(2)}}$. Besides, this expression can be continued for $|\varphi'| \leqslant \frac{1}{2}\pi$ by shifting $\mathcal{S}$. Therefore, we can write the spectral function corresponding to $H_0^{(2)}(kR_\epsilon)$, with $R_\epsilon = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \epsilon\Phi)}$, $\epsilon \equiv +$ or $-$, in a more general form

$$f^{\epsilon}_{H_0^{(2)}}(\alpha) = \frac{-1}{2\pi} \int_{\mathcal{S}_\epsilon} e^{ik\rho' \cos(\alpha' - \epsilon\Phi)} \left(\tan\left(\tfrac{1}{2}(\alpha - \alpha')\right) - g_\circ(\alpha') \right) d\alpha', \quad (2.6)$$

where the path $\mathcal{S}$ has been shifted to $\mathcal{S}_\epsilon = \mathcal{S} + a_\epsilon$, with a_ϵ being a constant satisfying $\Phi - \frac{3}{2}\pi < \epsilon a_\epsilon < \Phi - \frac{1}{2}\pi$.

The expression (2.6) allows us to express $H_0^{(2)}(kR_\epsilon)$ and $\partial_n H_0^{(2)}(kR_\epsilon)$ in (2.4) as Sommerfeld–Maliuzhinets integrals, and we then obtain a Sommerfeld–Maliuzhinets representation of $u_\pm$ with f (**10**, **25**):

$$u_\pm(\rho,\varphi) = \frac{1}{2\pi i} \int_\gamma f_\pm(\alpha + \varphi) e^{ik\rho \cos\alpha} d\alpha, \quad (2.7)$$

where

$$f_\epsilon(\alpha) = \frac{1}{4\pi i} \int_{\mathcal{S}_\epsilon} \epsilon f(\epsilon\pi + \alpha') \left(\tan\left(\tfrac{1}{2}(\alpha - \alpha')\right) - g_\circ(\alpha') \right) d\alpha' \quad (2.8)$$

for α between $\mathcal{S}_\epsilon - \pi$ and $\mathcal{S}_\epsilon + \pi$, provided $\frac{1}{2}\pi < \Phi - \epsilon a_\epsilon < \frac{3}{2}\pi$, $\frac{1}{2}\pi < \Phi - \epsilon\varphi_\circ < \frac{3}{2}\pi$ and $g^- = f(i\infty) + f(-i\infty) = 0$ (permitted from (b)), $\epsilon \equiv +$ or $-$. Note that $f_\epsilon(\alpha + \varphi) - f_\epsilon(-\alpha + \varphi)$ is bounded for large α on γ, even if $f_\epsilon(\alpha + \varphi)$ is not when $f(\pm i\infty) \neq 0$.

Since f is an analytic function, (2.8) can be analytically continued so $f_\epsilon(\alpha)$ is a function of f in the whole complex plane. We note, in particular, that taking into account the poles of $\tan(\frac{1}{2}(\alpha-\alpha'))$ which can be captured by $\mathcal{S}_\epsilon$ as α varies, $f_\pm$ satisfies

$$f_\pm(\pi+\alpha) - f_\pm(-\pi+\alpha) = \pm f(\pm\pi+\alpha). \tag{2.9}$$

Concerning the dependence on $\varphi_\circ$ (or Φ), the expression (2.8) has been determined for $\frac{1}{2}\pi < \Phi - \epsilon\varphi_\circ < \frac{3}{2}\pi$, but we can consider $f_\epsilon(\alpha)$ for $\Phi - \epsilon\varphi_\circ \leqslant \frac{1}{2}\pi$ and $\Phi - \epsilon\varphi_\circ \geqslant \frac{3}{2}\pi$ by analytical continuation on $\varphi_\circ$ (or Φ). For (2.8), this corresponds to taking account of the contribution of any singularity that would go through $\mathcal{S}_\epsilon$ as $\varphi_\circ$ (or Φ) goes into these regions. For (2.9), the situation is simpler because it is a functional equation that is not changed by analytic continuation. Also (2.9) agrees with the work of Michaeli (**26**) who considered the case of a wedge with Dirichlet boundary conditions.

2.3 *Far-field radiation of one face and expression of the spectral function f*

Stationary phase methods (**1**, **2**) can be applied to (2.7) to find the far-field radiation of the face $\varphi = \epsilon\Phi$, also written $\varphi = \pm\Phi$. From the regularity of f_ϵ (**10**), we can deform γ to stationary phase points $\alpha = +\pi$ and $-\pi$, when $\frac{1}{2}\pi < \Phi - \epsilon\varphi_\circ < \frac{3}{2}\pi$ and $\frac{1}{2}\pi < \Phi - \epsilon\varphi < \frac{3}{2}\pi$, and thus, out of the reflected and shadowed regions $2\Phi - \epsilon(\varphi+\varphi_\circ) < \pi$ and $|\varphi - \varphi_\circ| > \pi$. Equation (2.7) then gives

$$u_\pm(\rho,\varphi) = \frac{-e^{-i\pi/4}}{\sqrt{2\pi k\rho}} e^{-ik\rho}(f_\pm(\pi+\varphi) - f_\pm(-\pi+\varphi) + O(1/(k\rho))) \tag{2.10}$$

which, from (2.9), is

$$u_\pm(\rho,\varphi) = \frac{-e^{-i\pi/4}}{\sqrt{2\pi k\rho}} e^{-ik\rho}(\pm f(\pm\pi+\varphi) + O(1/(k\rho))). \tag{2.11}$$

In other respects, in (2.4) we can use formulae (**28**, **29**)

$$\begin{aligned} H_0^{(2)}(kR) &= \sqrt{2/(\pi kR)}e^{-ikR+i\pi/4}(1+O(1/(kR))), \\ \partial_n H_0^{(2)}(kR) &= -i\sqrt{2/(\pi kR)}e^{-ikR+i\pi/4}\partial_n(kR)(1+O(1/(kR))) \end{aligned} \tag{2.12}$$

for large R, with $\partial_n(kR) = \mp\partial(kR)/\rho'\partial\varphi'|_{\varphi'=\pm\Phi} = (\pm k\rho\sin(\varphi-\varphi')/R)|_{\varphi'=\pm\Phi}$, $R = \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')}$ and $\cos(\varphi\mp\Phi) < 0$. Taking into account that

$$R = \rho - \rho'\cos(\varphi\mp\Phi) + \frac{(\rho'\sin(\varphi\mp\Phi))^2}{R+\rho-\rho'\cos(\varphi\mp\Phi)}, \tag{2.13}$$

when $R+\rho-\rho'\cos(\varphi\mp\Phi) \neq 0$, and considering the properties (b′), (c′) on u, we then obtain another expression for the far field that, compared with (2.11), gives us

$$f(\pm\pi+\varphi) = \frac{1}{2}\int_0^\infty \left(iku(\rho',\pm\Phi)\sin(\varphi\mp\Phi) \pm \frac{\partial u}{\partial n}(\rho',\pm\Phi)\right) e^{ik\rho'\cos(\varphi\mp\Phi)}d\rho' \tag{2.14}$$

as $\frac{1}{2}\pi < \Phi\mp\varphi_\circ < \frac{3}{2}\pi$ and $\frac{1}{2}\pi < \Phi\mp\varphi < \frac{3}{2}\pi$, $|\arg(ik)| < \frac{1}{2}\pi$.

Using Green's theorem, we note that the contour of integration along $\varphi = \pm\Phi$ in (2.4) can be deformed into any path $L^\pm_{0,\infty}$ without changing the field $u_\pm(M)$ (except at points M captured by the path during its deformation), provided that the integral remains bounded and no source passes through the path during the deformation. Thus, we can write (2.14) in the more general form with the new path $L^{\pm}_{0,\infty}$, following

$$f(\pm\pi+\varphi)=\frac{1}{2}\int_{L^{\pm}_{0,\infty}}\left(iku(\rho',\varphi')\sin(\varphi-\varphi'_t)\pm\frac{\partial u}{\partial n}(\rho',\varphi')\right)e^{ik\rho'\cos(\varphi-\varphi')}dl'(\rho',\varphi'), \quad (2.15)$$

where $dl'(\rho',\varphi')$ is the element of length and φ'_t is the tangent angle along $L^{\pm}_{0,\infty}$.

PROPOSITION 2.1 *If we divide the semi-infinite paths* $L^{\pm}_{0,\infty}$ (*deriving from a deformation of the faces* $\varphi=\pm\Phi$ *enclosing the scatterer, described above*) *into* $L^{\pm}_{0,\Delta^{\pm}}$ (*that is,* $0<l'<\Delta^{\pm}$) *and* $L^{\pm}_{\Delta,\infty}$ (*that is,* $l'>\Delta^{\pm}$), *we have*

$$f(\pm\pi+\varphi)=\frac{1}{2}\int_{L^{\pm}_{0,\Delta^{\pm}}}\left(iku(\rho',\varphi')\sin(\varphi-\varphi'_t)\pm\frac{\partial u}{\partial n}(\rho',\varphi')\right)$$
$$\times e^{ik\rho'\cos(\varphi-\varphi')}dl'(\rho',\varphi')+f_{L^{\pm}_{\Delta^{\pm},\infty}}(\pm\pi+\varphi), \quad (2.16)$$

where $f_{L^{\pm}_{\Delta^{\pm},\infty}}(\alpha)=e^{-ik\rho_{\Delta^{\pm}}\cos(\alpha-\varphi_{\Delta^{\pm}})}f^{\pm}_e(\alpha)$, $f^{\pm}_e(\alpha)$ *is the spectral function related to the Sommerfeld–Maliuzhinets representation of the field in coordinates with origin at* $l'=\Delta^{\pm}$. *We can write, by analytic continuation,*

$$f(\alpha)=\frac{1}{2}\int_{L^{\pm}_{0,\Delta^{\pm}}}\left(-iku(\rho',\varphi')\sin(\alpha-\varphi'_t)\pm\frac{\partial u}{\partial n}(\rho',\varphi')\right)$$
$$\times e^{-ik\rho'\cos(\alpha-\varphi')}dl'(\rho',\varphi')+f_{L^{\pm}_{\Delta^{\pm},\infty}}(\alpha); \quad (2.17)$$

this is called henceforth the single-face expressions of f.

Proof. Dividing the path of integration in (2.15), we obtain (2.16). Since $L^{\pm}_{0,\Delta^{\pm}}$ is a finite path, the integral term in (2.16) is an entire function. Considering the analytical continuation of $f_{L^{\pm}_{\Delta^{\pm},\infty}}$, we then obtain (2.17) in the whole complex plane.

In this expression, the function $f_{L^{\pm}_{\Delta^{\pm},\infty}}(\alpha)$ is simply connected to a change of origin. Indeed, considering the coordinates $(\rho_{\Delta^{\pm}},\varphi_{\Delta^{\pm}})$ of the points of abscissa $l'=\Delta^{\pm}$ on the curve $L^{\pm}_{\Delta^{\pm},\infty}$, and (2.15), we have $f_{L^{\pm}_{\Delta^{\pm},\infty}}(\alpha)=e^{-ik\rho_{\Delta^{\pm}}\cos(\alpha-\varphi_{\Delta^{\pm}})}f^{\pm}_e(\alpha)$, $f^{\pm}_e(\alpha)$ being the spectral function corresponding to the representation of the field in coordinates with origin at $l'=\Delta^{\pm}$.

We note that, considering the function $f(\alpha)-e^{-ik\rho_{\Delta^{\pm}}\cos(\alpha-\varphi_{\Delta^{\pm}})}f^{\pm}_e(\alpha)$ from (2.17), it becomes possible to express, in the Sommerfeld–Maliuzhinets spectral domain, the consequences of particular conditions on the field along any finite part $L^{\pm}_{0,\Delta^{\pm}}$ of the face $L^{\pm}_{0,\infty}$.

REMARK 1 When $L^{\pm}_{\Delta^{\pm},\infty}$ is along the line $\varphi=\pm\Phi^{\pm}_e$ we have, letting $\pm\pi+\varphi=\alpha\pm\Phi^{\pm}_e$,

$$f_{L^{\pm}_{\Delta^{\pm},\infty}}(\alpha\pm\Phi^{\pm}_e)=\frac{1}{2}\int_{\Delta}^{\infty}\left(-iku(\rho',\pm\Phi^{\pm}_e)\sin\alpha\pm\frac{\partial u}{\partial n}(\rho',\pm\Phi^{\pm}_e)\right)e^{-ik\rho'\cos\alpha}d\rho' \quad (2.18)$$

for $\frac{1}{2}\pi<\Phi^{\pm}_e\mp\varphi_\circ<\frac{3}{2}\pi$ and $\frac{1}{2}\pi<\mp\alpha+\pi<\frac{3}{2}\pi$. By analytic continuation, this expression is also valid as $\mathrm{Re}(ik(\cos\alpha-\cos(\Phi^{\pm}_e\mp\varphi_\circ)))>0$, $|\mathrm{Re}\ (\alpha)|<\pi$. Then $f^{\pm}_e$, defined with

$f_{L^{\pm}_{\Delta^{\pm},\infty}}(\alpha \pm \Phi_e^{\pm}) = e^{-ik\Delta\cos\alpha} f_e^{\pm}(\alpha \pm \Phi^{\pm})$, satisfies $f_e^{\pm}(ix \pm \Phi_e^{\pm}) \to -\frac{1}{2}iu(0, \pm\Phi_e^{\pm})$ and $(f_e^{\pm}(ix \pm \Phi_e^{\pm}) - f_e^{\pm}(-ix \pm \Phi_e^{\pm})) = -iu(0, \pm\Phi_e^{\pm}) + O(\partial u/\partial \ln \rho_e|_{|\ln \rho_e| = x})$ as $x \to \infty$, which is in accordance with the property (b) on spectral functions listed in section 2.1.

2.4 *Single-face expression of f concerning a surface with two polygonal faces*

PROPOSITION 2.2 *Consider a polygonal surface located inside the domain $|\varphi| > \Phi$ enclosing a scatterer (Fig. 1). This surface is composed of two joined semi-infinite polygonal faces, denoted $+$ and $-$ respectively, with $m^{\pm}$ segments of lengths $d_j^{\pm}$ with tangent angles $\pm\Phi_j^{\pm}$, $j = 1, 2, \ldots, m^{\pm}$ and a semi-infinite plane with tangent angles $\pm\Phi_e^{\pm}$. Then, the single-face expression (2.17) of the spectral function f becomes*

$$f(\alpha) = \frac{1}{2} \sum_{1 \leqslant j \leqslant m^{\pm}} e^{-ik\sum_{1\leqslant i<j} d_i^{\pm} \cos(\alpha \mp \Phi_i^{\pm})}$$
$$\times \int_0^{d_j^{\pm}} \left(-iku(\rho_j', \pm\Phi_j^{\pm}) \sin(\alpha \mp \Phi_j^{\pm}) \pm \frac{\partial u}{\partial n}(\rho_j', \pm\Phi_j^{\pm}) \right) e^{-ik\rho_j' \cos(\alpha \mp \Phi_j^{\pm})} d\rho_j'$$
$$+ e^{-ik\sum_{1\leqslant i \leqslant m^{\pm}} d_i^{\pm} \cos(\alpha \mp \Phi_i^{\pm})} f_{e,m^{\pm}}^{\pm}(\alpha), \tag{2.19}$$

where $f_{e,m^{\pm}}^{\pm}(\alpha)$ is the analytic continuation of the integral expression

$$f_{e,m^{\pm}}^{\pm}(\alpha' \pm \Phi_e^{\pm}) = \frac{1}{2} \int_0^{\infty} \left(-iku(\rho_e', \pm\Phi_e^{\pm}) \sin\alpha' \pm \frac{\partial u}{\partial n}(\rho_e', \pm\Phi_e^{\pm}) \right) e^{-ik\rho_e' \cos\alpha'} d\rho_e', \tag{2.20}$$

valid as $\mathrm{Re}(ik(\cos\alpha' - \cos(\Phi_e^{\pm} \mp \varphi_\circ))) > 0$, $|\mathrm{Re}\,(\alpha)'| < \pi$, $|\arg(ik)| < \frac{1}{2}\pi$.

The function $f_{e,m^{\pm}}^{\pm}$ is the spectral function corresponding to the Sommerfeld–Maliuzhinets representation of the field in cylindrical coordinates $(\rho_e^{\pm}, \varphi_e^{\pm})$ with origin $Q_{e,m^{\pm}}^{\pm}$ at the edge of the semi-plane $\varphi_e^{\pm} = \pm\Phi_e^{\pm}$, in the free-space sector $\Phi_e^{\pm} - \delta_e^{\pm} \leqslant \pm\varphi_e^{\pm} \leqslant \Phi_e^{\pm}$, where $\delta_e^{\pm}$ is a strictly positive constant determined by the geometry.

Proof. Considering (2.17) for f with $L_{0,\infty}^{\pm}$ being polygonal faces $\pm$, and (2.18), we deduce the expressions (2.19) and (2.20).

PROPOSITION 2.3 *The spectral functions f and $f_{e,m^{\pm}}^{\pm}$ have the following properties:*

(d) *$f_{e,m^{\pm}}^{\pm}(\alpha) - u^i(Q_{e,m^{\pm}}^{\pm})/(\alpha - \varphi_\circ)$ has no singularity in the band $\Phi_e^{\pm} - \delta^{\pm} \leqslant \pm\,\mathrm{Re}\,(\alpha) \leqslant \Phi_e^{\pm}$, where $u^i(Q_{e,m^{\pm}}^{\pm}) = \exp(ik\sum_{1\leqslant i\leqslant m^{\pm}} d_i^{\pm}\cos(\varphi_\circ \mp \Phi_i^{\pm}))$, while $f(\alpha) - 1/(\alpha - \varphi_\circ)$ has no singularity in the band $-\Phi \leqslant \mathrm{Re}\,(\alpha) \leqslant \Phi$, even at infinity.*

(e) *$f(\alpha)$ is regular in the band $-\Phi_e^{-} \leqslant \mathrm{Re}\,(\alpha) \leqslant \Phi_e^{+}$, except at $\alpha = \varphi_\circ$ and possibly as $|\mathrm{Im}(\alpha)| \to \infty$, and thus $f_{e,m^{\pm}}^{\pm}(\alpha)$ too. As $|\mathrm{Im}(\alpha)| \to \infty$, the function $f(\alpha)$ is $O(1)$ when $-\Phi \leqslant \mathrm{Re}\,(\alpha) \leqslant \Phi$, and $O(e^{ik(c_{\pm}\cos\alpha + d_{\pm}\sin\alpha)})$ when $\Phi_e^{\pm} \leqslant \pm\,\mathrm{Re}\,(\alpha) \leqslant \Phi$, $c_{\pm}$ and $d_{\pm}$ being constants.*

Proof. Considering u^i in $(\rho_e^{\pm}, \varphi_e^{\pm})$ coordinates and using the property (a) of section 2.1, we have (d).

The regularity of f and $f_{e,m^{\pm}}^{\pm}$ can be deduced from (2.19) and (d), and so we obtain (e). From (2.19), we notice that $f_{e,m^{\pm}}^{\pm}(\alpha) = e^{A^{\pm}(\alpha)} f(\alpha) + B^{\pm}(\alpha)$, where $A^{\pm}$ and $B^{\pm}$ are entire functions, with

$A^{\pm} = O(\cos\alpha)$ and $B^{\pm} = O(e^{ik(a_{\pm}\cos\alpha + b_{\pm}\sin\alpha)})$, $a_{\pm}$ and $b_{\pm}$ being constants depending on the geometry. Thus, $f(\alpha)$ and $f^{\pm}_{e,m^{\pm}}(\alpha)$ have the same domain of regularity, except possibly at infinity. Now, we can enlarge in (2.19) the surface enclosing the scatterer within the domain $|\mathrm{Re}(\varphi)| \leqslant \Phi$ without changing f, and modify $\delta^{\pm}$ so that $\Phi^{\pm}_e - \delta^{\pm} \leqslant \Phi$ while Φ^{+}_e and Φ^{-}_e remain the same. Using (d), we then deduce (e).

3. Application of the single-face expression in the case of infinite or finite impedance polygons: functional equations and meromorphy

DEFINITION 3.1 The surface enclosing the impedance scatterer is considered to be composed of two joined semi-infinite polygonal faces, defined as in section 2.4. These faces, denoted by $+$ and $-$, are respectively with $m^{\pm}$ segments of lengths $d^{\pm}_j$ with tangent angles $\pm\Phi^{\pm}_j$, $j = 1, 2, \ldots, m^{\pm}$ and a semi-infinite plane with tangent angles $\pm\Phi^{\pm}_e$.

In what follows, $f^{\pm}_{e,m^{\pm}}$ is the spectral function corresponding to a Sommerfeld–Maliuzhinets representation of the field, in cylindrical coordinates $(\rho^{\pm}_e, \varphi^{\pm}_e)$ with origin at the edge $Q^{\pm}_{e,m^{\pm}}$ of the semi-plane $\varphi^{\pm}_e = \pm\Phi^{\pm}_e$, while the functions $f^{\pm}_{a,p}$ and $f^{\pm}_{b,p}$ are the spectral functions associated with a representation of the field in coordinates $(\rho^{\pm}_{a,p}, \varphi^{\pm}_{a,p})$ and $(\rho^{\pm}_{b,p}, \varphi^{\pm}_{b,p})$ with origins at opposite ends $Q^{\pm}_{a,p}$ and $Q^{\pm}_{b,p}$ of an arbitrary finite segment p of the polygonal face $\pm$. From this, we have $Q^{+}_{a,1} \equiv Q^{-}_{a,1}$ and $f^{+}_{a,1} \equiv f^{-}_{a,1}$ $(\equiv f)$ at the junction of the two faces, while $Q^{\pm}_{b,p} \equiv Q^{\pm}_{a,p+1}$ and $f^{\pm}_{b,p} \equiv f^{\pm}_{a,p+1}$ for p $\leqslant m^{\pm} - 1$, $Q^{\pm}_{b,m^{\pm}} \equiv Q^{\pm}_{e,m^{\pm}}$ and $f^{\pm}_{b,m^{\pm}} \equiv f^{\pm}_{e,m^{\pm}}$.

The functions $f^{\pm}_{e,m^{\pm}}$, $f^{\pm}_{a,p}$ and $f^{\pm}_{b,p}$ satisfy the properties (a), (b) and (c) listed in section 2.1, considering the respective origins and cylindrical coordinates attached to them.

The functions $f^{\pm}_{1,p}$ and $f^{\pm}_{2,p}$, are the combinations of functions $f^{\pm}_{a,p}$ and $f^{\pm}_{b,p}$,

$$f^{\pm}_{1,p}(\alpha) = f^{\pm}_{a,p}(\alpha) - e^{-ikd^{\pm}_p\cos(\alpha\mp\Phi^{\pm}_p)} f^{\pm}_{b,p}(\alpha),$$
$$f^{\pm}_{2,p}(\alpha) = f^{\pm}_{b,p}(\alpha) - e^{ikd^{\pm}_p\cos(\alpha\mp\Phi^{\pm}_p)} f^{\pm}_{a,p}(\alpha) = -e^{ikd^{\pm}_p\cos(\alpha\mp\Phi^{\pm}_p)} f^{\pm}_{1,p}(\alpha), \tag{3.1}$$

and are related to the radiation from segment p.

We now derive functional equations on the spectral functions in Propositions 3.1, 3.2 and 3.3, considering boundary conditions on both semi-planes and finite segments which compose the polygonal surface enclosing the finite or infinite impedance scatterer.

3.1 *Functional equations on $f^{\pm}_{e,m^{\pm}}$ due to boundary conditions on semi-infinite planes and meromorphy*

PROPOSITION 3.1 *In the case of an infinite polygonal scatterer with impedance boundary conditions along both semi-infinite planes $\varphi^{\pm}_e = \pm\Phi^{\pm}_e$*

$$\frac{\partial u}{\partial n^{\pm}}(\rho^{\pm}_e, \pm\Phi^{\pm}_e) - ik\sin\theta^{\pm}_p u(\rho^{\pm}_e, \pm\Phi^{\pm}_e) = 0, \tag{3.2}$$

where $\partial(\cdot)/\partial n^{\pm} = \widehat{n}^{\pm}\nabla(\cdot) = \mp\partial(\cdot)/\rho^{\pm}_e\partial\varphi^{\pm}_e$, $\widehat{n}^{\pm}$ is the outward normal to the face $\varphi^{\pm}_e = \pm\Phi^{\pm}_e$, the functions $f^{\pm}_{e,m^{\pm}}$ satisfy the functional equations

$$(\sin\alpha \pm \sin\theta^{\pm}_e) f^{\pm}_{e,m^{\pm}}(\alpha \pm \Phi^{\pm}_e) - (-\sin\alpha \pm \sin\theta^{\pm}_e) f^{\pm}_{e,m^{\pm}}(-\alpha \pm \Phi^{\pm}_e) = 0. \tag{3.3}$$

PROPOSITION 3.2 *If the scatterer is finite, the segments of both faces form a closed surface so that we can take* $Q^{+}_{e,m^{+}} = Q^{-}_{e,m^{-}}$ *with* $\Phi^{+}_{e} + \Phi^{-}_{e} = 2\pi$. *In this case, the fields on both semi-infinite planes* $\varphi^{\pm}_{e} = \pm\Phi^{\pm}_{e}$ *are now equal and their normal derivatives are opposite,*

$$u(\rho^{+}_{e}, \Phi^{+}_{e}) = u(\rho^{-}_{e}, -\Phi^{-}_{e}), \quad \frac{\partial u}{\partial n^{+}}(\rho^{+}_{e}, \Phi^{+}_{e}) = -\frac{\partial u}{\partial n^{-}}(\rho^{-}_{e}, -\Phi^{-}_{e}), \tag{3.4}$$

and we can derive that

$$f^{+}_{e,m^{+}}(\alpha + \pi) = f^{-}_{e,m^{-}}(\alpha - \pi). \tag{3.5}$$

Proof. We obtain the functional equations on $f^{\pm}_{e,m^{\pm}}$, by the use of (3.2) or (3.4) in the expression (2.20) concerning $f^{\pm}_{e,m^{\pm}}(\alpha \pm \Phi^{\pm}_{e})$, as $\mathrm{Re}(ik(\cos\alpha - \cos(\Phi^{\pm}_{e} \mp \varphi_{\circ}))) > 0$, $|\mathrm{Re}(\alpha)| < \pi$. So, if we consider (3.2) and the parity of $\cos\alpha$, we derive (3.3), while, if we consider (3.4) with $\Phi^{+}_{e} + \Phi^{-}_{e} = 2\pi$, we deduce that $f^{+}_{e,m^{+}}(\alpha + \Phi^{+}_{e}) = f^{-}_{e,m^{-}}(\alpha - \Phi^{-}_{e})$ and thus (3.5) holds. Taking account of the analyticity of the spectral functions on α and $\varphi_{\circ}$, (3.3) and (3.5) can be considered, by analytic continuation, for arbitrary α and $\varphi_{\circ}$.

Note that the meromorphy of f and $f^{\pm}_{e,m^{\pm}}$ can be proved from (3.3) or (3.5). From (2.19), $f^{\pm}_{e,m^{\pm}} = e^{A^{\pm}} f + B^{\pm}$, where $A^{\pm}$ and $B^{\pm}$ are entire functions, $A^{\pm}(\alpha) = O(1 + |\cos\alpha|)$ and $B^{\pm}(\alpha) = O(e^{ik(a_{\pm}\cos\alpha + b_{\pm}\sin\alpha)})$, $a_{\pm}$ and $b_{\pm}$ being constants depending on the geometry. If we use this expression of $f^{\pm}_{e,m^{\pm}}$ in (3.3) or (3.5), we see that $e^{A^{\pm}} f$ satisfies functional equations similar to (3.3) or (3.5), except that the second members are then non-nul entire functions. Moreover, the function $f(\alpha)$ is regular in the band $-\Phi^{-}_{e} \leqslant \mathrm{Re}(\alpha) \leqslant \Phi^{+}_{e}$, except for the pole at $\alpha = \varphi_{\circ}$ and possibly at infinity (as an entire function), and $f(\alpha)$ is meromorphic in this strip. Then, we can use equations derived from (3.3) or (3.5) for f and extend the property of meromorphy to the whole complex plane, so that f, and thus $f^{\pm}_{e,m^{\pm}}$, are meromorphic functions with simple poles.

3.2 *Functional equations due to boundary conditions on finite segments*

Considering Definitions 3.1 and the expression (2.19), the spectral functions $f^{\pm}_{a,p}$ and $f^{\pm}_{b,p}$, attached to shifts of the origin at opposite ends of the segment p on the face $\pm$, satisfy

$$\begin{aligned} f^{\pm}_{a,p}(\alpha) = {} & \frac{1}{2} \sum_{p \leqslant j \leqslant m^{\pm}} e^{-ik\sum_{p\leqslant i<j} d^{\pm}_{i}\cos(\alpha\mp\Phi^{\pm}_{i})} \\ & \times \int_{0}^{d^{\pm}_{j}} \left(-iku(\rho'_{a,j}, \pm\Phi^{\pm}_{j})\sin(\alpha \mp \Phi^{\pm}_{j}) \pm \frac{\partial u}{\partial n}(\rho'_{a,j}, \pm\Phi^{\pm}_{j})\right) e^{-ik\rho'_{a,j}\cos(\alpha\mp\Phi^{\pm}_{j})} d\rho'_{a,j} \\ & + e^{-ik\sum_{p\leqslant i\leqslant m^{\pm}} d^{\pm}_{i}\cos(\alpha\mp\Phi^{\pm}_{i})} f^{\pm}_{e,m^{\pm}}(\alpha), \end{aligned} \tag{3.6}$$

and $f^{\pm}_{b,p}(\alpha) = f^{\pm}_{a,p+1}(\alpha)$ for $1 \leqslant p < m^{\pm}$, $f^{\pm}_{b,m^{\pm}}(\alpha) = f^{\pm}_{e,m^{\pm}}(\alpha)$. From this (or (2.17)), we have

$$\begin{aligned} f^{\pm}_{a,p}(\alpha) = {} & \frac{1}{2}\int_{0}^{d^{\pm}_{p}} \left(-iku(\rho'_{a,p}, \pm\Phi^{\pm}_{p})\sin(\alpha \mp \Phi^{\pm}_{p}) \pm \frac{\partial u}{\partial n}(\rho'_{a,p}, \pm\Phi^{\pm}_{p})\right) \\ & \times e^{-ik\rho'_{a,p}\cos(\alpha\mp\Phi^{\pm}_{p})} d\rho'_{a,p} + e^{-ikd^{\pm}_{p}\cos(\alpha\mp\Phi^{\pm}_{p})} f^{\pm}_{b,p}(\alpha), \end{aligned} \tag{3.7}$$

which can be rewritten as

$$f_{b,p}^{\pm}(\alpha) = \frac{1}{2}\int_0^{d_p^{\pm}} \left(-iku(\rho'_{b,p}, \pm(\Phi_p^{\pm}-\pi))\sin(\alpha \mp (\Phi_p^{\pm}-\pi)) \mp \frac{\partial u}{\partial n}(\rho'_{b,p}, \pm(\Phi_p^{\pm}-\pi))\right) \times e^{-ik\rho'_{b,p}\cos(\alpha\mp(\Phi_p^{\pm}-\pi))} d\rho'_{b,p} + e^{-ikd_p^{\pm}\cos(\alpha\mp(\Phi_p^{\pm}-\pi))} f_{a,p}^{\pm}(\alpha). \quad (3.8)$$

PROPOSITION 3.3 *Considering impedance boundary conditions on the segments in the form*

$$\left[\frac{\partial u}{\partial n}(\rho_{a,p}, \pm\Phi_p^{\pm}) - ik\sin\theta_p^{\pm} u(\rho_{a,p}, \pm\Phi_p^{\pm})\right]_{0\leqslant\rho_{a,p}\leqslant d_p^{\pm}} = 0,$$
$$\left[\frac{\partial u}{\partial n}(\rho_{b,p}, \pm(\Phi_p^{\pm}-\pi)) - ik\sin\theta_p^{\pm} u(\rho_{b,p}, \pm(\Phi_p^{\pm}-\pi))\right]_{0\leqslant\rho_{b,p}\leqslant d_p^{\pm}} = 0, \quad (3.9)$$

and letting

$$f_{1,p}^{\pm}(\alpha) = f_{a,p}^{\pm}(\alpha) - e^{-ikd_p^{\pm}\cos(\alpha\mp\Phi_p^{\pm})} f_{b,p}^{\pm}(\alpha),$$
$$f_{2,p}^{\pm}(\alpha) = f_{b,p}^{\pm}(\alpha) - e^{ikd_p^{\pm}\cos(\alpha\mp\Phi_p^{\pm})} f_{a,p}^{\pm}(\alpha) = -e^{ikd_p^{\pm}\cos(\alpha\mp\Phi_p^{\pm})} f_{1,p}^{\pm}(\alpha), \quad (3.10)$$

we have

$$(\sin\alpha \pm \sin\theta_p^{\pm}) f_{1,p}^{\pm}(\alpha \pm \Phi_p^{\pm}) - (-\sin\alpha \pm \sin\theta_p^{\pm}) f_{1,p}^{\pm}(-\alpha \pm \Phi_p^{\pm}) = 0,$$
$$(\sin\alpha \mp \sin\theta_p^{\pm}) f_{2,p}^{\pm}(\alpha \pm (\Phi_p^{\pm}-\pi)) - (-\sin\alpha \mp \sin\theta_p^{\pm}) f_{2,p}^{\pm}(-\alpha \pm (\Phi_p^{\pm}-\pi)) = 0. \quad (3.11)$$

Proof. Equations (3.9) can be used for the expression of $f_{1,p}^{\pm}(\alpha \pm \Phi_p^{\pm})$ and $f_{2,p}^{\pm}(\alpha \pm (\Phi_p^{\pm}-\pi))$ from (3.7) and (3.8). We then obtain (3.11), using the parity of $\cos\alpha$.

We have derived simple functional equations in Propositions 3.1, 3.2 and 3.3 for an impedance polygon. Using the properties of the spectral fonctions and the theory of difference equations, they can be reduced to spectral integral equations. We will illustrate this approach by developing the three-part polygon problem with semi-infinite impedance planes.

4. On the decomposition of the field for an impedance polygon

From properties discussed in sections 2.1, 2.4 and 3, the function $f(\alpha)$ is a meromorphic function with simple poles, regular in the band $-\Phi_e^- \leqslant \mathrm{Re}(\alpha) \leqslant \Phi_e^+$, except for the pole at $\alpha = \varphi_\circ$ and possibly at infinity when $\Phi \leqslant \pm\,\mathrm{Re}(\alpha) \leqslant \Phi_e^{\pm}$. In this latter case, the behaviour is $O(e^{ik(c_\pm\cos\alpha + d_\pm\sin\alpha)})$ as $|\mathrm{Im}(\alpha)| \to \infty$, $c_\pm$ and $d_\pm$ being constants.

In these circumstances, we can modify the integral expression (2.3) of the field, exhibiting the terms of geometrical optics fields, the terms of guided waves, and the term principally radiated cylindrically at large distance from the origin.

For this, we deform the integration path γ in (2.3) to steepest descent path (SDP), summing the contributions of poles encountered during this deformation. The contour SDP consists of two branches $\mathrm{SDP}_\pm$, respectively centred on $\pm\pi$, satisfying the equation $\mathrm{Im}(ik(\cos\alpha + 1)) = 0$ with

$ik(\cos\alpha + 1) \leqslant 0$, that is, in the limit case where k is real positive, $\mathrm{Re}(\alpha \mp \pi)/2 = \arctan(\tanh(\mathrm{Im}(\alpha \mp \pi)/2))$. Thus, we have

$$u(\rho,\varphi) = u_i + \sum_{\pm} u_r^{\pm} + \sum_{\pm} u_s^{\pm} + \frac{e^{-ik\rho}}{2\pi i}\int_{\mathrm{SDP}} f(\alpha+\varphi)e^{ik\rho(\cos\alpha+1)}d\alpha, \qquad (4.1)$$

where

- the term u_i is related to the contribution of the pole $\alpha = \varphi_\circ$, which is the incident field in the illuminated region, and zero in the shadow zone;
- the terms $u_r^{\pm}$ correspond to the fields reflected by the semi-infinite plates at $\varphi = \pm\Phi_e^{\pm}$;
- the terms $u_s^{\pm}$ correspond to the non-uniform leaky waves with complex phase functions guided by the semi-infinite plates. These waves are typically attenuated waves travelling along the faces to infinity;
- the last term u_d has a dependence principally cylindrical from the origin as $\rho \to \infty$ and, using the SDP method,

$$u_d \sim \frac{-e^{-i\pi/4}}{\sqrt{2\pi k\rho}}e^{-ik\rho}(f(\pi+\varphi) - f(-\pi+\varphi)) \quad \text{as } k\rho \to \infty, \qquad (4.2)$$

  except when real poles cross the SDP. The integral term with steepest descent path could also be evaluated asymptotically for ρ large so that the total field expression remains continuous as a pole crosses the SDP (**2**).

Note that we can use (2.19) and (3.6) and express (4.2) with $f^{\pm}_{e,m^{\pm}}$, $f^{\pm}_{a,p}$ or $f^{\pm}_{b,p}$.

5. Formulation of the three-part polygonal problem: spectral functions in Sommerfeld–Maliuzhinets representation and functional equations in the complex plane

5.1 *Position of the problem for a semi-infinite three-part impedance polygon*

DEFINITION 5.1 We consider the diffraction of an incident plane wave by a semi-infinite impedance polygon divided into three parts (Fig. 3), each one characterized by relative surface impedances $\sin\theta_-$, $\sin\theta_1$, and $\sin\theta_+$, with positive real parts (strict passivity).

Using the notation taken of sections 2 and 3, this means that $m^+ = 1$, $m^- = 0$, $\Phi_1^+ = \frac{1}{2}\pi$, $\Phi_e^+ = \frac{1}{2}\pi + \Phi_b$, $\Phi_e^- = -\frac{1}{2}\pi - \Phi_a$, $f_a = f^+_{a,1} = f_e^-$, $f_b = f^+_{b,1} = f_e^+$.

The functions f_a and f_b are the spectral functions associated with the Sommerfeld–Maliuzhinets representation of the field, in cylindrical coordinate systems (ρ_a,φ_a) and (ρ_b,φ_b), with origins at opposite ends of the finite segment (Fig. 3).

We have, in (ρ_a,φ_a) coordinates,

$$\begin{aligned}&\left(\rho_a \in\,]0,\infty[,\ \varphi_a = -\tfrac{1}{2}\pi - \Phi_a\right) \quad \text{with } \partial u/\partial n - ik\sin\theta_- u = 0,\\ &\left(\rho_a \in [0,\Delta],\ \varphi_a = \tfrac{1}{2}\pi\right) \quad \text{with } \partial u/\partial n - ik\sin\theta_1 u = 0,\end{aligned} \qquad (5.1)$$

with the incident field $u^i = e^{ik\rho_a\cos(\varphi_a - \varphi_\circ)}$ and, in coordinates (ρ_b,φ_b),

$$\begin{aligned}&\left(\rho_b \in [0,\Delta],\ \varphi_b = -\tfrac{1}{2}\pi\right) \quad \text{with } \partial u/\partial n - ik\sin\theta_1 u = 0,\\ &\left(\rho_b \in\,]0,\infty[,\ \varphi_b = \tfrac{1}{2}\pi + \Phi_b\right) \quad \text{with } \partial u/\partial n - ik\sin\theta_+ u = 0,\end{aligned} \qquad (5.2)$$

with the incident field $u^i = e^{ik(\rho_b\cos(\varphi_b - \varphi_\circ) + \Delta\sin\varphi_\circ)}$.

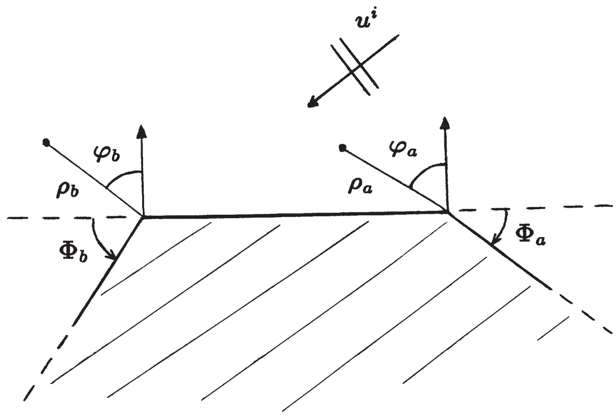

Fig. 3 Geometry: the semi-infinite three-part polygon

Defining two distinct origins has several elementary consequences on f_a and f_b.

The expressions of u^i in these coordinates and the properties (a), (b) of spectral functions in section 2.1) give that

$$(f_a(\alpha) - 1/(\alpha - \varphi_\circ)) \quad \text{and} \quad (f_b(\alpha) - e^{ik\Delta \sin\varphi_\circ}/(\alpha - \varphi_\circ)) \tag{5.3}$$

are regular in the strips $-\frac{1}{2}\pi - \Phi_a \leqslant \mathrm{Re}(\alpha) \leqslant \frac{1}{2}\pi$ and $-\frac{1}{2}\pi \leqslant \mathrm{Re}(\alpha) \leqslant \frac{1}{2}\pi + \Phi_b$ respectively, and that $f_{a(b)}(\alpha) = \mp\frac{1}{2}i\, u(\rho_{a(b)} = 0) + o(\alpha^{-1})$ as $\mathrm{Im}(\alpha) \to \pm\infty$.

Considering (3.7), (3.8) (or (2.14) to (2.17) as $\Phi \to \frac{1}{2}\pi$), we obtain

$$\begin{aligned} f_a(\alpha) &= e^{-ik\Delta\sin\alpha} f_b(\alpha) \\ &\quad - \frac{1}{2}\int_0^\Delta \left(-iku\left(\rho_a, \tfrac{1}{2}\pi\right)\cos\alpha - \frac{\partial u}{\partial n}\left(\rho_a, \tfrac{1}{2}\pi\right)\right) e^{-ik\rho_a \sin\alpha} d\rho_a, \\ f_b(\alpha) &= e^{ik\Delta\sin\alpha} f_a(\alpha) \\ &\quad - \frac{1}{2}\int_0^\Delta \left(iku\left(\rho_b, -\tfrac{1}{2}\pi\right)\cos\alpha + \frac{\partial u}{\partial n}\left(\rho_b, -\tfrac{1}{2}\pi\right)\right) e^{+ik\rho_b \sin\alpha} d\rho_b \end{aligned} \tag{5.4}$$

for any complex angle α.

Besides, we can use (2.15) for $f = f_a$, with $L^-_{0,\infty}$ and $L^+_{0,\infty}$ defined by $\varphi = -\frac{1}{2}\pi - \Phi_a$ and $\varphi = \frac{1}{2}\pi + \Phi_b$ for $f = f_b$ respectively, or directly (2.20) considering that $f^-_{e,1} = f_a$, $f^+_{e,1} = f_b$. We then derive, after analytical continuation, that

$$f_a(\alpha - \varphi_-) = \frac{1}{2}\int_0^\infty \left(-iku(\rho_a, -\varphi_-)\sin\alpha - \frac{\partial u}{\partial n}(\rho_a, -\varphi_-)\right) e^{-ik\rho_a\cos\alpha} d\rho_a \tag{5.5}$$

as $\mathrm{Re}(ik(\cos(\varphi_\circ + \varphi_-) - \cos\alpha)) < 0$, $|\mathrm{Re}(\alpha)| < \pi$, with $\varphi_- = \frac{1}{2}\pi + \Phi_a$, and

$$f_b(\alpha + \varphi_+) = \frac{1}{2}\int_0^\infty \left(-iku(\rho_b, \varphi_+)\sin\alpha + \frac{\partial u}{\partial n}(\rho_b, \varphi_+)\right) e^{-ik\rho_b\cos\alpha} d\rho_b \tag{5.6}$$

as $\mathrm{Re}(ik(\cos(\varphi_\circ - \varphi_+) - \cos\alpha)) < 0$, $|\mathrm{Re}(\alpha)| < \pi$, with $\varphi_+ = \frac{1}{2}\pi + \Phi_b$. Note these domains of regularity contain the strip $|\mathrm{Re}(\alpha)| < \min(\frac{1}{2}\pi - |\arg(ik)|, \frac{1}{2}\pi + \Phi_{a,b} - |\varphi_\circ|)$.

In the next step, we use the boundary conditions defined in (5.1), (5.2) to derive functional equations on f_a and f_b.

5.2 *Elementary functional equations in the complex plane for f_a and f_b*

We now derive functional equations on the spectral functions from the boundary conditions (5.1), (5.2) on the central strip (of finite size) and on the semi-infinite planes of the three-part impedance polygon.

PROPOSITION 5.1 *Using the boundary conditions on the central strip from* (5.1), (5.2), *and letting*

$$\begin{aligned} f_1\Big(\alpha + \tfrac{1}{2}\pi\Big) &= f_a\Big(\alpha + \tfrac{1}{2}\pi\Big) - e^{-ik\Delta\cos\alpha} f_b\Big(\alpha + \tfrac{1}{2}\pi\Big), \\ f_2\Big(\alpha - \tfrac{1}{2}\pi\Big) &= f_b\Big(\alpha - \tfrac{1}{2}\pi\Big) - e^{-ik\Delta\cos\alpha} f_a\Big(\alpha - \tfrac{1}{2}\pi\Big) = -e^{-ik\Delta\cos\alpha} f_1\Big(\alpha - \tfrac{1}{2}\pi\Big) \end{aligned} \tag{5.7}$$

we have

$$\begin{aligned} (\sin\alpha + \sin\theta_1) f_1\Big(\alpha + \tfrac{1}{2}\pi\Big) - (-\sin\alpha + \sin\theta_1) f_1\Big(-\alpha + \tfrac{1}{2}\pi\Big) &= 0, \\ (\sin\alpha - \sin\theta_1) f_2\Big(\alpha - \tfrac{1}{2}\pi\Big) - (-\sin\alpha - \sin\theta_1) f_2\Big(-\alpha - \tfrac{1}{2}\pi\Big) &= 0 \end{aligned} \tag{5.8}$$

while, using the boundary conditions on both semi-infinite planes from (5.1), (5.2), *we have*

$$\begin{aligned} (\sin\alpha + \sin\theta_+) f_b\Big(\alpha + \tfrac{1}{2}\pi + \Phi_b\Big) - (-\sin\alpha + \sin\theta_+) f_b\Big(-\alpha + \tfrac{1}{2}\pi + \Phi_b\Big) &= 0, \\ (\sin\alpha - \sin\theta_-) f_a\Big(\alpha - \tfrac{1}{2}\pi - \Phi_a\Big) - (-\sin\alpha - \sin\theta_-) f_a\Big(-\alpha - \tfrac{1}{2}\pi - \Phi_a\Big) &= 0. \end{aligned} \tag{5.9}$$

Proof. We first consider (5.4) and the boundary conditions on the central strip in (5.1), (5.2). Using evenness of $\cos\alpha$, we then derive (5.8), which is a particular case of (3.11) with $f^+_{a,p} = f_a$, $f^+_{b,p} = f_b$. Besides, we can consider (5.5) and (5.6) for f_a and f_b, use the boundary conditions on both semi-infinite planes from (5.1), (5.2), then evenness of $\cos\alpha$, and obtain (5.9), a particular case of (3.3) with $f^-_{e,1} = f_a$, $f^+_{e,1} = f_b$.

By analytical continuation, these functional equations apply in the whole complex plane. Notice that for $\Delta = 0$ we can write $f_b(\alpha) = f_a(\alpha) = f_0(\alpha + (\Phi_a - \Phi_b)/2)$, and (5.9) then gives the equations for an impedance wedge (**17**).

5.3 *Functional equations for $f_{br}(\alpha) = f_b(\alpha + \frac{1}{2}\Phi_b)$, $f_{ar}(\alpha) = f_a(\alpha - \frac{1}{2}\Phi_a)$*

We now use equations (5.8), (5.9) and (5.7) to derive two sets of functional equations on f_b and f_a, each set corresponding to a discontinuity influenced by the other.

PROPOSITION 5.2 *If $f_{br}(\alpha - \frac{1}{2}\Phi_b) = f_b(\alpha)$ and $\Phi_+ = \frac{1}{2}\pi + \frac{1}{2}\Phi_b$, then we have*

$$(\sin\alpha + \sin\theta_+) f_{br}(\alpha + \Phi_+) - (-\sin\alpha + \sin\theta_+) f_{br}(-\alpha + \Phi_+) = 0, \tag{5.10}$$

and

$$(\sin\alpha - \sin\theta_1) f_{br}(\alpha - \Phi_+) - (-\sin\alpha - \sin\theta_1) f_{br}(-\alpha - \Phi_+)$$
$$= e^{-ik\Delta\cos\alpha}\left((\sin\alpha - \sin\theta_1) f_a\Big(\alpha - \tfrac{1}{2}\pi\Big) - (-\sin\alpha - \sin\theta_1) f_a\Big(-\alpha - \tfrac{1}{2}\pi\Big)\right)$$
$$= 2e^{-ik\Delta\cos\alpha} S_b^-(\alpha). \quad (5.11)$$

If $f_{ar}(\alpha + \frac{1}{2}\Phi_a) = f_a(\alpha)$ *and* $\Phi_- = \frac{1}{2}\pi + \frac{1}{2}\Phi_a$, *then we have*

$$(\sin\alpha + \sin\theta_1) f_{ar}(\alpha + \Phi_-) - (-\sin\alpha + \sin\theta_1) f_{ar}(-\alpha + \Phi_-)$$
$$= e^{-ik\Delta\cos\alpha}\left((\sin\alpha + \sin\theta_1) f_b\Big(\alpha + \tfrac{1}{2}\pi\Big) - (-\sin\alpha + \sin\theta_1) f_b\Big(-\alpha + \tfrac{1}{2}\pi\Big)\right)$$
$$= 2e^{-ik\Delta\cos\alpha} S_a^+(\alpha), \quad (5.12)$$

and

$$(\sin\alpha - \sin\theta_-) f_{ar}(\alpha - \Phi_-) - (-\sin\alpha - \sin\theta_-) f_{ar}(-\alpha - \Phi_-) = 0. \quad (5.13)$$

Proof. We obtain (5.10) from the condition on f_b in (5.9), and we get (5.11) by developing the functional equation on f_2 in (5.8) from (5.7). Similarly, to obtain (5.12), it suffices to develop the functional equation on f_1 in (5.8) from (5.7), and (5.13) is obtained by considering the equation on f_a in (5.9).

The left-hand parts of (5.10), (5.11) (resp. (5.12), (5.13)) concern a two-part discontinuity centred at $\rho_b = 0$ (resp. $\rho_a = 0$) whereas the right-hand term in (5.11) (resp. (5.12)) corresponds to the influence of the second discontinuity at $\rho_a = 0$ (resp. $\rho_b = 0$).

From the properties of f_a and f_b given in section 5.1, the functions

$$(f_{ar}(\alpha) - 1/(\alpha - \varphi_{\circ,a})) \quad \text{and} \quad (f_{br}(\alpha) - e^{ik\Delta\sin\varphi_\circ}/(\alpha - \varphi_{\circ,b})) \quad (5.14)$$

are respectively regular in the bands $|\mathrm{Re}\,\alpha| \leqslant \Phi_-$ and $|\mathrm{Re}\,\alpha| \leqslant \Phi_+$, with $\varphi_{\circ,a} = \varphi_\circ + \frac{1}{2}\Phi_a$ and $\varphi_{\circ,b} = \varphi_\circ - \frac{1}{2}\Phi_b$, $-\frac{1}{2}\pi - \Phi_a < \varphi_\circ < \frac{1}{2}\pi + \Phi_b$, and, in these regions

$$f_{ar}(\alpha) = \mp\tfrac{1}{2}iu(\rho_a = 0) + o(\alpha^{-1}) \quad \text{and} \quad f_{br}(\alpha) = \mp\tfrac{1}{2}iu(\rho_b = 0) + o(\alpha^{-1}) \quad (5.15)$$

as $\mathrm{Im}(\alpha) \to \pm\infty$. Besides, we remark that, for $\Delta = 0$, the three-part polygon is reduced to a wedge with exterior angle $2\Phi_d = (2(\Phi_+ + \Phi_-) - \pi)$ and face impedances $\sin\theta_\pm$. In this case, we have $f_{br}(\alpha - \Phi_- + \frac{1}{2}\pi) = f_{ar}(\alpha + \Phi_+ - \frac{1}{2}\pi) = f_0(\alpha)$, where f_0 (see (**17**) or Appendix A) is given by

$$f_0(\alpha, \varphi_\circ) = \frac{\pi}{2\Phi_d}\frac{\Psi_{+-}(\alpha)\cos(\pi\varphi_{\circ,d}/2\Phi_d)}{\Psi_{+-}(\varphi_{\circ,d})(\sin(\pi\alpha/2\Phi_d) - \sin(\pi\varphi_{\circ,d}/2\Phi_d))}, \quad (5.16)$$

where $\varphi_{\circ,d} = \varphi_\circ + (\Phi_a - \Phi_b)/2$, $-\Phi_d < \varphi_{\circ,d} < \Phi_d$.

6. The integral expressions and equations for the three-part impedance polygon

6.1 *The integral expressions deriving from the functional equations*

Considering the theory of functional equations, the analytic function $\chi(\alpha)$ satisfying

$$\chi(\alpha \pm \Phi) - \chi(-\alpha \pm \Phi) = \vartheta^\pm(\alpha), \quad (6.1)$$

and regular as $|\mathrm{Re}(\alpha)| \leqslant \Phi$, is given in the strip $|\mathrm{Re}(\alpha)| < \Phi$ (**24**, **27**), as

$$\chi(\alpha) = \frac{\chi(i\infty) + \chi(-i\infty)}{2} + \frac{-i}{8\Phi}\int_{-i\infty}^{+i\infty} d\alpha' \left(\vartheta^{+}(\alpha')\tan\left(\frac{\pi}{4\Phi}(\alpha + \Phi - \alpha')\right)\right.$$
$$\left. - \vartheta^{-}(\alpha')\tan\left(\frac{\pi}{4\Phi}(\alpha - \Phi - \alpha')\right)\right) \tag{6.2}$$

when the functions $\vartheta^{\pm}(\alpha)$ are regular and summable on the imaginary axis.

PROPOSITION 6.1 *We can use the solutions* $\Psi_{+1}(\alpha, \Phi_+)$ (*resp.* $\Psi_{1-}(\alpha, \Phi_-)$), *without pole or zero and* $O(\cos(\pi\alpha/2\Phi_+))$ (*resp.* $O(\cos(\pi\alpha/2\Phi_-))$) *in the band* $|\mathrm{Re}(\alpha)| \leqslant \Phi_+$ (*resp.* $|\mathrm{Re}(\alpha)| \leqslant \Phi_-$) (**17**, **34**) *of the equations* (5.10), (5.11) (*resp.* (5.12), (5.13)) *without second members* (*Appendix* A) *and reduce the problem to equations of the type* (6.1).

Then we can write, for $-\Phi_+ < \mathrm{Re}(\alpha) < 3\Phi_+$,

$$\frac{f_{br}(\alpha)}{\Psi_{+1}(\alpha)} = \frac{i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{S_b^{-}(\alpha')e^{-ik\Delta\cos\alpha'}\tan\left(\frac{\pi}{4\Phi_+}(\alpha - \Phi_+ - \alpha')\right)}{(\sin\alpha' - \sin\theta_1)\Psi_{+1}(\alpha' - \Phi_+)} d\alpha' + \chi_b^i(\alpha)$$

$$= \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{S_b^{-}(\alpha')/(\sin\alpha' - \sin\theta_1)}{\Psi_{+1}(\alpha' - \Phi_+)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\pi\alpha'/2\Phi_+)}{\cos\left(\frac{\pi}{2\Phi_+}(\alpha - \Phi_+)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_+}\right)} d\alpha' + \chi_b^i(\alpha)$$

$$= \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{f_{ar}\left(\alpha' - \frac{1}{2}\pi + \frac{1}{2}\Phi_a\right)}{\Psi_{+1}(\alpha' - \Phi_+)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\pi\alpha'/2\Phi_+)}{\cos\left(\frac{\pi}{2\Phi_+}(\alpha - \Phi_+)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_+}\right)} d\alpha' + \chi_b^i(\alpha), \tag{6.3}$$

where the closed form expression of the source term χ_b^i *is given by*

$$\chi_b^i(\alpha) = e^{ik\Delta\sin\varphi_\circ}\chi_{+1}^i(\alpha) = \frac{\pi}{2\Phi_+}\left(\frac{e^{ik\Delta\sin\varphi_\circ}\cos(\pi\varphi_{\circ,b}/2\Phi_+)}{\Psi_{+1}(\varphi_{\circ,b})(\sin(\pi\alpha/2\Phi_+) - \sin(\pi\varphi_{\circ,b}/2\Phi_+))}\right), \tag{6.4}$$

with $\varphi_{\circ,b} = \varphi_\circ - \frac{1}{2}\Phi_b$, *as* $-\frac{1}{2}\pi < \varphi_\circ < \frac{1}{2}\pi + \Phi_b$ *and, for* $-3\Phi_- < \mathrm{Re}(\alpha) < \Phi_-$,

$$\frac{f_{ar}(\alpha)}{\Psi_{1-}(\alpha)} = \frac{-i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{S_a^{+}(\alpha')e^{-ik\Delta\cos\alpha'}\tan\left(\frac{\pi}{4\Phi_-}(\alpha + \Phi_- - \alpha')\right)}{(\sin\alpha' + \sin\theta_1)\Psi_{1-}(\alpha' + \Phi_-)} d\alpha' + \chi_a^i(\alpha)$$

$$= \frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{S_a^+(\alpha')/(\sin\alpha'+\sin\theta_1)}{\Psi_{1-}(\alpha'+\Phi_-)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\pi\alpha'/2\Phi_-)}{\cos\left(\frac{\pi}{2\Phi_-}(\alpha+\Phi_-)\right)+\cos\left(\frac{\pi\alpha'}{2\Phi_-}\right)} d\alpha' + \chi_a^i(\alpha)$$

$$= \frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{f_{br}\left(\alpha'+\frac{1}{2}\pi-\frac{1}{2}\Phi_b\right)}{\Psi_{1-}(\alpha'+\Phi_-)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\pi\alpha'/2\Phi_-)}{\cos\left(\frac{\pi}{2\Phi_-}(\alpha+\Phi_-)\right)+\cos\left(\frac{\pi\alpha'}{2\Phi_-}\right)} d\alpha' + \chi_a^i(\alpha), \tag{6.5}$$

where the source term χ_a^i is given in closed form by

$$\chi_a^i(\alpha) = \frac{\pi}{2\Phi_-}\left(\frac{\cos(\pi\varphi_{\circ,a}/2\Phi_-)}{\Psi_{1-}(\varphi_{\circ,a})(\sin(\pi\alpha/2\Phi_-)-\sin(\pi\varphi_{\circ,a}/2\Phi_-))}\right), \tag{6.6}$$

with $\varphi_{\circ,a} = \varphi_\circ + \frac{1}{2}\Phi_a$, as $-\frac{1}{2}\pi - \Phi_a < \varphi_\circ < \frac{1}{2}\pi$.

Proof. Considering $\Psi_{+1}(\alpha)\chi_b^i(\alpha)$ (resp. $\Psi_{1-}(\alpha)\chi_a^i(\alpha)$) the solution of the equations without second members with the incident field singularity in the band $|\mathrm{Re}(\alpha)| \leqslant \Phi_+$ (resp. $|\mathrm{Re}(\alpha)| \leqslant \Phi_-$) (**17**), we derive, from (5.10) to (5.14), functional equations for the term $\chi_{br}(\alpha) = f_{br}(\alpha)/\Psi_{+1}(\alpha) - \chi_b^i$ (resp. $\chi_{ar}(\alpha) = f_{ar}(\alpha)/\Psi_{1-}(\alpha) - \chi_a^i$) of the type (6.1).

As $1/\Psi_{1-,+1}(\alpha)$, the functions $\chi_{ar,br}(\alpha)$ vanish when $|\mathrm{Im}(\alpha)| \to \infty$, $|\mathrm{Re}(\alpha)| \leqslant \Phi_{-,+}$. We then use (6.2) and obtain (6.3) to (6.6).

In (6.3) to (6.5), we have used that $f_a(\alpha) = f_{ar}(\alpha + \frac{1}{2}\Phi_a)$ and $f_b(\alpha) = f_{br}(\alpha - \frac{1}{2}\Phi_b)$. When $\Phi_b > 0$ (resp. $\Phi_a > 0$), the equations (6.3) to (6.6) can be continued analytically for $\varphi_\circ \geqslant \frac{1}{2}\pi$ (resp. $\varphi_\circ \leqslant -\frac{1}{2}\pi$), considering the poles of $f_{ar}(\alpha)$ (resp. $f_{br}(\alpha)$) at $\alpha = \varphi_{\circ,a}$ (resp. $\alpha = \varphi_{\circ,b}$) captured by the integration path.

Integral equations can be derived for $f_{br}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)$ and $f_{ar}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)$ on imaginary axis, since (6.3) to (6.5) with the functional equations (5.8) to (5.13) (or the analytical continuation of the integral terms in (6.3) to (6.5)) permit us to express $f_{ar}(\alpha)$ and $f_{br}(\alpha)$ in the whole complex plane. In what follows, we restrict ourselves to the case where $\Phi_{a,b} > -\frac{1}{2}\pi$ to simplify the analysis.

6.2 *Integral equations when $\Phi_{a,b} > -\frac{1}{2}\pi$: definitions and properties*

PROPOSITION 6.2 *We can write*

$$\frac{f_{br}(\alpha+\frac{1}{2}\pi-\frac{1}{2}\Phi_b)}{\Psi_{+1}(\alpha+\frac{1}{2}\pi-\frac{1}{2}\Phi_b)} = \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{S_b^-(\alpha')}{(\sin\alpha'-\sin\theta_1)} \frac{1}{\Psi_{+1}(\alpha'-\Phi_+)}$$
$$\times \frac{e^{-ik\Delta\cos\alpha'}\sin(\pi\alpha'/2\Phi_+)}{\cos\left(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b)\right)+\cos\left(\frac{\pi\alpha'}{2\Phi_+}\right)} d\alpha'$$
$$+\chi_b^i\left(\alpha+\frac{1}{2}\pi-\frac{1}{2}\Phi_b\right), \tag{6.7}$$

as $-\Phi_+ < \mathrm{Re}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b) < 3\Phi_+$, *and*

$$\frac{f_{ar}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)}{\Psi_{1-}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)} = \frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{S_a^+(\alpha')}{(\sin\alpha' + \sin\theta_1)}\frac{1}{\Psi_{1-}(\alpha' + \Phi_-)}$$

$$\times \frac{e^{-ik\Delta\cos\alpha'}\sin(\pi\alpha'/2\Phi_-)}{\cos\left(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_-}\right)}d\alpha'$$

$$+ \chi_a^i\left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\right), \qquad (6.8)$$

as $-3\Phi_- < \mathrm{Re}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a) < \Phi_-$, *with* $-\min(\frac{1}{2}\pi, \frac{1}{2}\pi + \Phi_a) < \varphi_\circ < \min(\frac{1}{2}\pi, \frac{1}{2}\pi + \Phi_b)$. *When* $\Phi_{a,b} > -\frac{1}{2}\pi$, *we can take* α *purely imaginary in* (6.7), (6.8).

Proof. Using the integral expressions given by (6.3) to (6.6), we obtain (6.7), (6.8).

PROPOSITION 6.3 *The functions*

$$\frac{S_b^-(\alpha)}{(\sin\alpha - \sin\theta_1)} = \frac{1}{2}\left(f_{ar}\left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\right) - \frac{(-\sin\alpha - \sin\theta_1)\, f_{ar}(-\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)}{(\sin\alpha - \sin\theta_1)}\right),$$

$$\frac{S_a^+(\alpha)}{(\sin\alpha + \sin\theta_1)} = \frac{1}{2}\left(f_{br}\left(\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right) - \frac{(-\sin\alpha + \sin\theta_1)\, f_{br}(-\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)}{(\sin\alpha + \sin\theta_1)}\right) \qquad (6.9)$$

vanish at infinity, contrary to $f_{ar}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)$ *and* $f_{br}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)$ *in general. Choosing these functions as unknowns in* $L^2(iR)$, *we derive the integral equations for imaginary* α

$$\frac{S_a^+(\alpha)}{(\sin\alpha + \sin\theta_1)} = \frac{-i}{8\Phi_+}\int_{-i\infty}^{+i\infty} \frac{S_b^-(\alpha')}{(\sin\alpha' - \sin\theta_1)}\frac{e^{-ik\Delta\cos\alpha'}\sin(\pi\alpha'/2\Phi_+)}{\Psi_{+1}(\alpha' - \Phi_+)}$$

$$\times\left[\frac{\Psi_{+1}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)}{\cos\left(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_+}\right)}\right.$$

$$\left. + \frac{(\sin\alpha - \sin\theta_1)}{(\sin\alpha + \sin\theta_1)}\frac{\Psi_{+1}(-\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)}{\cos\left(\frac{\pi}{2\Phi_+}(\alpha + \Phi_b)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_+}\right)}\right] d\alpha'$$

$$+ \left[\Psi_{+1}\left(\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right)\chi_b^i\left(\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right)\right.$$

$$\left. + \frac{(\sin\alpha - \sin\theta_1)\Psi_{+1}(-\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)}{(\sin\alpha + \sin\theta_1)}\chi_b^i\left(-\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right)\right] \times \frac{1}{2} \qquad (6.10)$$

and

$$
\begin{aligned}
\frac{S_b^-(\alpha)}{(\sin\alpha - \sin\theta_1)} = \frac{i}{8\Phi_-} \int_{-i\infty}^{+i\infty} \frac{S_a^+(\alpha')}{(\sin\alpha' + \sin\theta_1)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\pi\alpha'/2\Phi_-)}{\Psi_{1-}(\alpha' + \Phi_-)} \\
\times \left[\frac{\Psi_{1-}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)}{\cos\left(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_-}\right)} \right. \\
\left. + \frac{(\sin\alpha + \sin\theta_1)}{(\sin\alpha - \sin\theta_1)} \frac{\Psi_{1-}(-\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)}{\cos\left(\frac{\pi}{2\Phi_-}(\alpha - \Phi_a)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_-}\right)} \right] d\alpha' \\
+ \left[\Psi_{1-}\left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\right) \chi_a^i \left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\right) \right. \\
\left. + \frac{(\sin\alpha + \sin\theta_1)\Psi_{1-}(-\alpha - \frac{1}{2}\pi + {}_{2}\Phi_a)}{(\sin\alpha - \sin\theta_1)} \chi_a^i \left(-\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\right) \right] \times \frac{1}{2}.
\end{aligned} \tag{6.11}
$$

Proof. The new unknowns described in (6.9) vanish at infinity and belong to $L^2(iR)$ from (5.15). Then, by simple combinations, we can derive (6.10), (6.11) from (6.7), (6.8).

Once given a solution of the new system (6.10), (6.11) in $L^2(iR)$, we can define f_{br} (resp. f_{ar}) satisfying (6.7) (resp. (6.8)). By simple combinations and comparison with (6.10), (6.11), we notice that these functions necessarily satisfy (6.9) on the imaginary axis.

Since $\Psi_{+1,1-}(\alpha) = A_0\cos(\pi\alpha/2\Phi_{+,-})(1 + O(\alpha^\nu e^{-|\mu \mathrm{Im}(\alpha)|}))$ for $|\mathrm{Im}(\alpha)|$ large, we have

$$
\begin{aligned}
\Psi_{+1}\left(-\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right) &= -\Psi_{+1}\left(\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right) e^{\mp i\pi\Phi_b/\Phi_+}(1 + O(\alpha^\nu e^{-|\mu\mathrm{Im}(\alpha)|})), \\
\Psi_{1-}\left(-\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\right) &= -\Psi_{1-}\left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\right) e^{\pm i\pi\Phi_a/\Phi_-}(1 + O(\alpha^\nu e^{-|\mu\mathrm{Im}(\alpha)|}))
\end{aligned} \tag{6.12}
$$

when $\mathrm{Im}(\alpha) \to \pm\infty$, so that the kernels are $O(e^{-ik\Delta\cos\alpha'}/\cos(a\alpha'))$ as $|\mathrm{Im}(\alpha')| \to \infty$, and $O(1/\cos(a\alpha))$ as $|\mathrm{Im}(\alpha)| \to \infty$, $a > 0$, like the source terms. Then we notice that (6.10), (6.11) is a system of Fredholm equations of the second kind in $L^2(iR)$ as $\mathrm{Im}(k\Delta) < 0$.

Spectral integral equations are currently encountered in diffraction theory (see for example (**1**, **9**, **10**, **31**, **32**)), and they can be solved numerically (**29**, **30**) or analytically by approximations. In our case, the approximations principally depend on $k\Delta$.

We now discuss existence and properties of solutions, and some features concerning calculus. Uncoupling and approximations for small or large $k\Delta$ are detailed in section 8.

6.3 *Existence and properties of the solution in* $L^2(iR)$

We have shown that the spectral functions satisfying the functional equations defined in (5.10) to (5.13), and the properties of regularity (5.14), verify the equations (6.7), (6.8) and the integral equations derived from them on the imaginary axis.

Reciprocally, it is important to study the solutions of these integral equations, and their properties. For this, we can consider (6.10), (6.11) in L^2, or (6.7), (6.8) along the imaginary axis when

$$\int_{-\infty}^{+\infty}\left|\frac{S_b^-(ix)}{(\sin(ix)-\sin\theta_1)}\right|^2 dx<\infty,\quad \int_{-\infty}^{+\infty}\left|\frac{S_a^+(ix)}{(\sin(ix)+\sin\theta_1)}\right|^2 dx<\infty, \tag{6.13}$$

and (6.9) is satisfied.

We notice that (6.13) is permitted by the conditions (5.14), (5.15) on the spectral functions. Reciprocally, the analyticity and the behaviour of solutions of (6.7) to (6.11) with (6.13) can be studied from Schwarz inequality (**29**, **30**), and (5.10) to (5.13) are satisfied. The integral equations imply that (5.14) is satisfied, and

$$f_{br}\Big(\alpha+\tfrac{1}{2}\pi-\tfrac{1}{2}\Phi_b\Big)\quad\text{and}\quad f_{ar}\Big(\alpha-\tfrac{1}{2}\pi+\tfrac{1}{2}\Phi_a\Big)\quad\text{are } O(1)\text{ and regular} \tag{6.14}$$

for imaginary α, when $\mathrm{Im}(k\Delta)\leqslant 0$, and that we have

$$f_{br}(\alpha)\to f_{br}(\pm i\infty)\quad\text{and}\quad f_{ar}(\alpha)\to f_{ar}(\pm i\infty) \tag{6.15}$$

when $\mathrm{Im}(\alpha)\to\pm\infty$, respectively as $-\Phi_+<\mathrm{Re}(\alpha)<3\Phi_+$ and $-3\Phi_-<\mathrm{Re}(\alpha)<\Phi_-$, with $f_{br,ar}(-i\infty)=-f_{br,ar}(+i\infty)$ when $\mathrm{Im}(k\Delta)<0$.

Besides, we can also consider existence and uniqueness of the solution. We notice that, as $\mathrm{Im}(k\Delta)\leqslant 0$, the kernels are regular functions of $k\Delta$ in (6.10), (6.11) while, as $k\Delta=0$, the solution is uniquely defined from the use of Tuzhilin's works (**34**). Considering the theory of integral equations depending on a parameter in L^2 (**29**, **30**), the resolvent for our system of Fredholm integral equations of the second kind is then an analytical function of the parameter $k\Delta$, which defines a unique solution in $L^2(iR)$ as $\mathrm{Im}(k\Delta)<0$ and $k\Delta=0$, except possibly for some discrete values of $k\Delta$ where the resolvent is singular. The existence of solutions satisfying (6.14), (6.15) can be then derived in the limit case $\mathrm{Im}(k\Delta)=0$, except possibly for some discrete values of $k\Delta$.

7. The scattering diagram from the solutions of the integral equations

Previously, we have reduced the problem to a system of non-singular integral equations. When the solutions of the integral equations are known, the different elements of the decomposition of the field given in (4.1) can be evaluated from the expressions (6.3) to (6.5), where the integral terms can be considered as smooth coupling terms between both edges. In this case, the functional equations (5.8), (5.9) have to be used to reduce the calculus in the band of validity of (6.3) to (6.5).

Here, we study the reduction of the term with radial dependence $\exp(-ik\rho_{a,b})/\sqrt{2\pi k\rho_{a,b}}$ in (4.2) for large $\rho_{a,b}$. Its angular dependence $\mathcal{F}_{a,b}$ in the direction $\varphi_a=\varphi_b=\varphi$ is commonly called the scattering diagram (or directivity), given, from (4.2), by

$$\mathcal{F}_{a,b}(\varphi)=-e^{-i\pi/4}(f_{a,b}(\pi+\varphi)-f_{a,b}(-\pi+\varphi)) \tag{7.1}$$

where, from (5.4), $\mathcal{F}_a(\varphi)=e^{ik\Delta\sin\varphi}\mathcal{F}_b(\varphi)$.

We illustrate the development by the one of $\mathcal{F}_b(\varphi_b)$ in the case where $\Phi_b>-\frac{1}{2}\pi$, $\Phi_a>0$, and $-\min(\frac{1}{2}\pi,\frac{1}{2}\pi+\Phi_a)<\varphi_\circ<\min(\frac{1}{2}\pi,\frac{1}{2}\pi+\Phi_b)$, when $-\frac{1}{2}\pi<\varphi_b<\frac{1}{2}\pi+\Phi_b$ (similar to $\mathcal{F}_a(\varphi_a)$ when $-\frac{1}{2}\pi<-\varphi_a<\frac{1}{2}\pi+\Phi_a$, $\Phi_a>-\frac{1}{2}\pi$, $\Phi_b>0$).

We consider at first the reduction of $f_b(\pi+\varphi_b)$ in (7.1). If $\Phi_b>0$, then $-\frac{1}{2}\pi<\pi+\varphi_b<\frac{3}{2}\pi+2\Phi_b$, and we are in the domain of validity of (6.3). If $\Phi_b<0$, then we have a concave part and there are two cases. If $-\frac{1}{2}\pi < \pi + \varphi_b < \frac{3}{2}\pi + 2\Phi_b$, (6.3) applies directly. In contrast, if $\frac{3}{2}\pi + 2\Phi_b < \pi + \varphi_b < \frac{3}{2}\pi + \Phi_b$, we use (5.9) and write

$$f_b(\pi + \varphi_b) = \frac{(-\sin(\frac{1}{2}\pi - \Phi_b + \varphi_b) + \sin\theta_+)}{(\sin(\frac{1}{2}\pi - \Phi_b + \varphi_b) + \sin\theta_+)} f_b(-\varphi_b + 2\Phi_b). \tag{7.2}$$

Then we use (5.8)

$$\begin{aligned} f_b(-\varphi_b + 2\Phi_b) = \frac{(\sin(\frac{1}{2}\pi - \varphi_b + 2\Phi_b) + \sin\theta_1)}{(-\sin(\frac{1}{2}\pi - \varphi_b + 2\Phi_b) + \sin\theta_1)} &(f_b(-\pi + \varphi_b - 2\Phi_b) \\ &- e^{ik\Delta\sin(2\Phi_b - \varphi_b)} f_a(-\pi + \varphi_b - 2\Phi_b)) \\ &+ e^{ik\Delta\sin(2\Phi_b - \varphi_b)} f_a(-\varphi_b + 2\Phi_b), \end{aligned} \tag{7.3}$$

where the terms in the right-hand side of (7.3) can be expressed with (6.3) and (6.5).

Then, we reduce $f_b(-\pi + \varphi_b)$ in (7.1). If $-\frac{1}{2}\pi < -\pi + \varphi_b < \frac{3}{2}\pi + 2\Phi_b$, which implies $\varphi_b > \frac{1}{2}\pi$ and thus $\Phi_b > 0$, we are in the domain of validity of (6.3). In contrast, when $-\frac{1}{2}\pi < \varphi_b < \frac{1}{2}\pi$, which is forced when $\Phi_b < 0$, we first have to use (5.8)

$$\begin{aligned} f_b(-\pi + \varphi_b) = &\frac{(\sin(\frac{1}{2}\pi - \varphi_b) - \sin\theta_1)}{(-\sin(\frac{1}{2}\pi - \varphi_b) - \sin\theta_1)} \\ &\times (f_b(-\varphi_b) - e^{-ik\Delta\sin\varphi_b} f_a(-\varphi_b)) + e^{-ik\Delta\sin\varphi_b} f_a(-\pi + \varphi_b). \end{aligned} \tag{7.4}$$

In this case, since $-\frac{3}{2}\pi - 2\Phi_a < -\varphi_b < \frac{1}{2}\pi$ and $-\frac{3}{2}\pi - 2\Phi_a < -\pi + \varphi_b < \frac{1}{2}\pi$ when $\Phi_a > 0$, and $-\frac{1}{2}\pi < -\varphi_b < \frac{3}{2}\pi + 2\Phi_b$ when $\Phi_b > -\frac{1}{2}\pi$, $f_a(-\pi + \varphi_b)$ and $f_a(-\varphi_b)$ can be expressed with (6.5), and $f_b(-\varphi_b)$ with (6.3), which ends the reduction.

8. Some features of the system of integral equations and their consequences

The system of integral equations given by (6.10), (6.11), or (6.7), (6.8) with (6.9) constitutes an original important step in the reduction by a spectral method of the problem of the diffraction in free space by a three-part polygonal impedance scatterer. These integral equations and the integral expressions (6.3) to (6.8) of the spectral functions have special features concerning

- the decoupling of integral equations in important cases,
- the approximations for $k\Delta$ small or $k\Delta$ large,

that we now illustrate.

8.1 *Decoupling in the case of the three-part impedance plane*

In the case of the three-part impedance plane, we have $\Phi_a = \Phi_b = 0$ and $\Phi_+ = \Phi_- = \frac{1}{2}\pi$. Considering parity and (**10**, equation (4.8)), we can use

$$\begin{aligned} &\frac{\Psi_{1-}(\alpha - \frac{1}{2}\pi)}{2\Psi_{+1}(\alpha - \frac{1}{2}\pi)} - \frac{\Psi_{1-}(-\alpha - \frac{1}{2}\pi)}{2\Psi_{+1}(-\alpha - \frac{1}{2}\pi)} \\ &\quad = \frac{\Psi_{1-}(\alpha - \frac{1}{2}\pi)}{\Psi_{+1}(\alpha - \frac{1}{2}\pi)} \frac{\sin\alpha(\sin\theta_- - \sin\theta_1)}{(\sin\alpha + \sin\theta_-)(\sin\alpha - \sin\theta_1)} = \frac{C(\alpha)\sin\alpha(\sin\theta_- - \sin\theta_1)}{\sin\alpha - \sin\theta_1}, \end{aligned}$$

$$\frac{\Psi_{+1}(\alpha+\frac{1}{2}\pi)}{2\Psi_{1-}(\alpha+\frac{1}{2}\pi)}-\frac{\Psi_{+1}(\frac{1}{2}\pi-\alpha)}{2\Psi_{1-}(\frac{1}{2}\pi-\alpha)}$$

$$=\frac{-\Psi_{+1}(\frac{1}{2}\pi-\alpha)}{\Psi_{1-}(\frac{1}{2}\pi-\alpha)}\frac{\sin\alpha(\sin\theta_{+}-\sin\theta_{1})}{(\sin\alpha+\sin\theta_{+})(\sin\alpha-\sin\theta_{1})}=\frac{-C(\alpha)\sin\alpha(\sin\theta_{+}-\sin\theta_{1})}{\sin\alpha-\sin\theta_{1}} \tag{8.1}$$

in (6.7), (6.8). Letting $N(\alpha')=\dfrac{iC(\alpha')\sin\alpha'/2\pi}{\sin\alpha'-\sin\theta_1}$, we then derive

$$\frac{(\sin\theta_1-\sin\theta_+)^{1/2}f_{br}(\alpha+\frac{1}{2}\pi)}{\Psi_{+1}(\alpha+\frac{1}{2}\pi)}=\eta_m\int_{-i\infty}^{+i\infty}N(\alpha')\frac{(\sin\theta_1-\sin\theta_-)^{1/2}f_{ar}(\alpha'-\frac{1}{2}\pi)}{\Psi_{1-}(\alpha'-\frac{1}{2}\pi)}$$
$$\times\frac{e^{-ik\Delta\cos\alpha'}\sin\alpha'}{\cos\alpha+\cos\alpha'}d\alpha'+(\sin\theta_1-\sin\theta_+)^{1/2}\chi_b^i\Big(\alpha+\tfrac{1}{2}\pi\Big), \tag{8.2}$$

$$\frac{(\sin\theta_1-\sin\theta_-)^{1/2}f_{ar}(\alpha-\frac{1}{2}\pi)}{\Psi_{1-}(\alpha-\frac{1}{2}\pi)}=\eta_m\int_{-i\infty}^{+i\infty}N(\alpha')\frac{(\sin\theta_1-\sin\theta_+)^{1/2}f_{br}(\alpha'+\frac{1}{2}\pi)}{\Psi_{+1}(\alpha'+\frac{1}{2}\pi)}$$
$$\times\frac{e^{-ik\Delta\cos\alpha'}\sin\alpha'}{\cos\alpha+\cos\alpha'}d\alpha'+(\sin\theta_1-\sin\theta_-)^{1/2}\chi_a^i\Big(\alpha-\tfrac{1}{2}\pi\Big), \tag{8.3}$$

where $\eta_m=(\sin\theta_1-\sin\theta_+)^{1/2}(\sin\theta_1-\sin\theta_-)^{1/2}$. In this case, we obtain two uncoupled equations by simple addition and subtraction:

$$(S,D)=\eta_m\int_{-i\infty}^{+i\infty}N(\alpha')((S,-D)e^{-ik\Delta\cos\alpha'}\frac{\sin\alpha'}{\cos\alpha+\cos\alpha'}d\alpha'+(S^i,-D^i), \tag{8.4}$$

similar to the equations obtained in (**10**).

8.2 *Decoupling for symmetric polygons* ($\Phi_b=\Phi_a$ *and* $\sin\theta_+=\sin\theta_-$)

In this case, the system (6.7), (6.8) can also be decoupled. For this, we first express the system of integral equations (6.7), (6.8) in a new form. We observe that $\Psi_{-1}(\alpha,\Phi_-)$ (also denoted by $\Psi_{-1}(\alpha)$) satisfies $\Psi_{-1}(\alpha,\Phi_-)=\Psi_{1-}(-\alpha,\Phi_-)$ (see Appendix A). Then we can write

$$\frac{f_{br}(-\alpha+\frac{1}{2}\pi-\frac{1}{2}\Phi_b)}{\Psi_{+1}(-\alpha+\frac{1}{2}\pi-\frac{1}{2}\Phi_b)}=\frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty}\frac{f_{ar}(\alpha'-\frac{1}{2}\pi+\frac{1}{2}\Phi_a)}{\Psi_{+1}(\alpha'-\Phi_+)}$$
$$\times\frac{e^{-ik\Delta\cos\alpha'}\sin(\pi\alpha'/2\Phi_+)}{\cos\Big(\dfrac{\pi}{2\Phi_+}(\alpha+\Phi_b)\Big)+\cos\Big(\dfrac{\pi\alpha'}{2\Phi_+}\Big)}d\alpha'+\chi_b^i\Big(-\alpha+\tfrac{1}{2}\pi-\tfrac{1}{2}\Phi_b\Big), \tag{8.5}$$

$$\frac{f_{ar}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)}{\Psi_{-1}(-\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_a)} = \frac{-i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{f_{br}(-\alpha' + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)}{\Psi_{-1}(\alpha' - \Phi_-)}$$

$$\times \frac{e^{-ik\Delta\cos\alpha'}\sin(\pi\alpha'/2\Phi_-)}{\cos\left(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_-}\right)} d\alpha' + \chi_a^i\left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\right). \tag{8.6}$$

This form is particularly suitable for the case of a symmetric polygon. In this case $\Phi_+ = \Phi_-$ (that is, $\Phi_b = \Phi_a$) and $\sin\theta_+ = \sin\theta_-$ so that $\Psi_{+1} = \Psi_{-1}$ and the equations (8.5), (8.6) have the same kernels. Thus, by addition and subtraction, we derive a system of decoupled equations

$$\frac{(S, D)(\alpha)}{\Psi_{+1}(-\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)} = \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{(S, -D)(\alpha')}{\Psi_{+1}(\alpha' - \Phi_+)}$$

$$\times \frac{e^{-ik\Delta\cos\alpha'}\sin(\pi\alpha'/2\Phi_+)}{\cos\left(\frac{\pi}{2\Phi_+}(\alpha+\Phi_b)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_+}\right)} d\alpha' + (S^i, D^i)(\alpha), \tag{8.7}$$

where

$$\begin{aligned} S(\alpha) &= f_{br}\left(-\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right) + f_{ar}\left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_b\right), \\ D(\alpha) &= f_{br}\left(-\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right) - f_{ar}\left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_b\right), \\ S^i(\alpha) &= \chi_{br}^i\left(-\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right) + \chi_{ar}^i\left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_b\right), \\ D^i(\alpha) &= \chi_{br}^i\left(-\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right) - \chi_{ar}^i\left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_b\right), \end{aligned} \tag{8.8}$$

for $|\varphi_\circ| < \min(\frac{1}{2}\pi, \frac{1}{2}\pi + \Phi_b)$.

Thus we have reduced the problem of diffraction by a three-part symmetric polygonal with impedance conditions to a system of uncoupled Fredholm integral equations of the second kind. As for (6.3) to (6.8), when $\Phi_{a,b} > 0$ these equations can be continued analytically in the domain $|\varphi_\circ| \geqslant \frac{1}{2}\pi$ if we consider the poles of $f_a(\alpha)$ and $f_b(\alpha)$ at $\alpha = \varphi_\circ$ captured by the integration path.

REMARK 2 The decoupling considered here leads to two scalar equations of the type

$$\mathcal{G}_1 - (\mathcal{M}_1)\mathcal{G}_1 = \mathcal{R}_1 \quad \text{and} \quad \mathcal{G}_2 - (\mathcal{M}_2)\mathcal{G}_2 = \mathcal{R}_2, \tag{8.9}$$

where $(\mathcal{M}_{1,2})$ are simple integral operators and $\mathcal{R}_{1,2}$ are known, while a usual reduction of the coupled integral equations (6.7), (6.8) leads to two scalar equations of the type

$$G_1 - (M_1)((M_2)G_1 + R_2) = R_1 \quad \text{and} \quad G_2 - (M_2)((M_1)G_2 + R_1) = R_2, \tag{8.10}$$

where the products $(M_{1,2})(M_{2,1})$ are double integral operators and $R_{1,2}$ are known.

8.3 *Partial inversion and approximation for small $k\Delta$*

Considering the known solution for $k\Delta = 0$ (**17**), we can apply the identity

$$e^{-ik\Delta\cos\alpha'} = (e^{-ik\Delta\cos\alpha'} - 1) + 1 \tag{8.11}$$

and invert the part corresponding to the unit term (see Appendix B), which results in equations with kernels vanishing as $k\Delta \to 0$, suitable for approximation. By this process, we derived in (**10**) an approximation of the scattering diagram for the three-part impedance plane, that is, the three-part polygon with $\Phi_a = \Phi_b = 0$, which is given by

$$\mathcal{F}(\varphi) \simeq -e^{i\pi/4} \frac{k\Delta \dfrac{2\cos\varphi}{\cos\varphi + \sin\theta_+} \dfrac{2\cos\varphi_\circ}{\cos\varphi_\circ + \sin\theta_+}}{k\Delta \left(1 - i\dfrac{2}{\pi}\ln\dfrac{\gamma' k\Delta}{2e}\right) + 2/(\sin\theta_1 - \sin\theta_+)} \tag{8.12}$$

when $\sin\theta_+ = \sin\theta_-$ that we have validated. For this, we compare the results given by (8.12) with the results derived by a classical numerical moment method on a large object (approximating a plane). We consider in Fig. 4 the values of $U = |\mathcal{F}(\varphi)/\sqrt{2\pi}|$ in db, for an impedance strip on a plane, with $k\Delta = 2\pi/10$, in the cases $\sin\theta_+ = 0{\cdot}625$ and $\sin\theta_1 = 0{\cdot}625i$, $\sin\theta_+ = 1{\cdot}6$ and $\sin\theta_1 = 1{\cdot}6 + 1{\cdot}6i$, $\sin\theta_+ = 0{\cdot}324 - 0{\cdot}226i$ and $\sin\theta_1 = 0{\cdot}595 - 0{\cdot}053i$, when $\varphi = \varphi_\circ$, letting φ vary. We notice the excellent agreement of the results.

8.4 *Asymptotics for $k\Delta$ large*

In section 7, the scattering diagram has been reduced to a combination of values of the spectral functions f_a and f_b in the band of validity of (6.3) to (6.6), with exponential factors depending on angles. So reduced, the oscillatory nature of the diagram comes principally from exponential factors. We give here approximations of the spectral terms in the band of validity of (6.7), (6.8), which have smooth dependence on the angle when k is large.

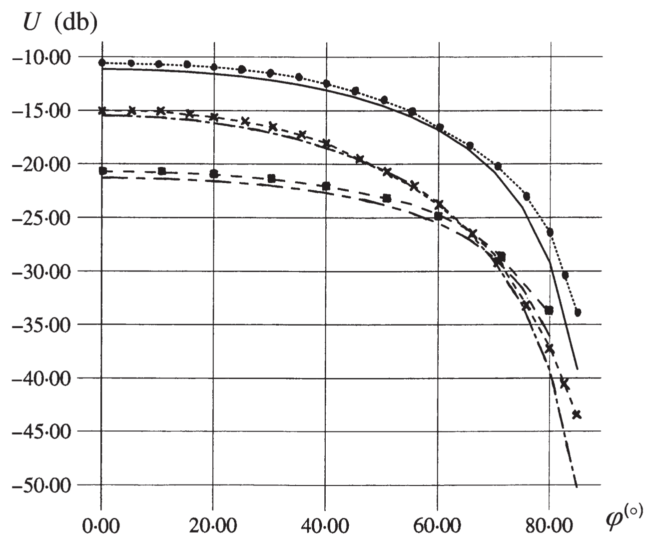

Fig. 4 Comparisons on $U = |\mathcal{F}(\varphi)/\sqrt{2\pi}|$: $\sin\theta_+ = 0{\cdot}625, \sin\theta_1 = 0{\cdot}625i$ (-●- moment method, — analytical expression); $\sin\theta_+ = 1{\cdot}6, \sin\theta_1 = 1{\cdot}6 + 1{\cdot}6i$ (-×- moment method. – – – analytical expression); $\sin\theta_+ = 0{\cdot}324 - 0{\cdot}226i, \sin\theta_1 = 0{\cdot}595 - 0{\cdot}053i$ (-■- moment method, – – – analytical expression)

We can write, from (6.7), (6.8),

$$\mathcal{R}_b(\alpha) = \frac{-i}{4\Phi_+} m_1(\alpha) \int_{-i\infty}^{+i\infty} \mathcal{R}_a(\alpha') \frac{e^{-ik\Delta\cos\alpha'} \sin(\pi\alpha'/2\Phi_+)}{\cos\left(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_+}\right)} d\alpha' + m_1(\alpha)\chi_b^i\left(\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right) \tag{8.13}$$

as $-\Phi_+ < \text{Re}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b) < 3\Phi_+$, and

$$\mathcal{R}_a(\alpha) = \frac{i}{4\Phi_-} m_2(\alpha) \int_{-i\infty}^{+i\infty} \mathcal{R}_b(\alpha') \frac{e^{-ik\Delta\cos\alpha'} \sin(\pi\alpha'/2\Phi_-)}{\cos\left(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_-}\right)} d\alpha' + m_2(\alpha)\chi_a^i\left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\right) \tag{8.14}$$

as $-3\Phi_- < \text{Re}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a) < \Phi_-$, for $-\min(\frac{1}{2}\pi, \frac{1}{2}\pi + \Phi_a) < \varphi_\circ < \min(\frac{1}{2}\pi, \frac{1}{2}\pi + \Phi_b)$, where

$$\begin{aligned} \mathcal{R}_b(\alpha) &= \frac{f_{br}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)}{\Psi_{1-}(\alpha + \Phi_-)}, \quad \mathcal{R}_a(\alpha) = \frac{f_{ar}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)}{\Psi_{+1}(\alpha - \Phi_+)}, \\ m_1(\alpha) &= \frac{\Psi_{+1}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)}{\Psi_{1-}(\alpha + \Phi_-)}, \quad m_2(\alpha) = \frac{\Psi_{1-}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)}{\Psi_{+1}(\alpha - \Phi_+)}, \end{aligned} \tag{8.15}$$

and χ_b^i and χ_a^i are smooth functions defined in (6.4) and (6.6).

In the equations (8.13), (8.14), we can choose to consider or to neglect the influence of complex poles of $\mathcal{R}_{a,b}(\alpha')$ (corresponding to guided waves vanishing when Δ is large) in the vicinity of the integration path. To simplify the presentation, we develop here only the second case. So, we assume that the principal contribution of $\mathcal{R}_{a,b}(\alpha')$ comes from the vicinity of the stationary phase point $\alpha' = 0$. Letting $\cos\alpha' = 1 - x^2/2$ with $x = 2\sin(\alpha'/2)$, and taking into account the parity of the integration path, we have

$$\mathcal{R}_b(\alpha) \sim \frac{m_1(\alpha)(\partial_\alpha \mathcal{R}_a(\alpha)|_{\alpha=0})e^{-ik\Delta}}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{2e^{ik\Delta x^2/2}x^2}{2\left(\frac{\pi}{2\Phi_+}\right)^{-2}\left(1 + \cos\left(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)\right)\right) - x^2} dx + m_1(\alpha)\chi_b^i\left(\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right) \tag{8.16}$$

as $-\Phi_+ < \text{Re}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b) < 3\Phi_+$, and

$$\mathcal{R}_a(\alpha) \sim -\frac{m_2(\alpha)(\partial_\alpha \mathcal{R}_b(\alpha)|_{\alpha=0})e^{-ik\Delta}}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{2e^{ik\Delta x^2/2}x^2}{2\left(\frac{\pi}{2\Phi_-}\right)^{-2}\left(1 + \cos\left(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)\right)\right) - x^2} dx + m_2(\alpha)\chi_a^i\left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\right) \tag{8.17}$$

as $-3\Phi_- < \mathrm{Re}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a) < \Phi_-$. After derivation of previous equations, we obtain

$$\begin{aligned}\partial_\alpha \mathcal{R}_b(\alpha)|_{\alpha=0} &\sim \partial_\alpha\Big(m_1(\alpha)\chi_b^i\left(\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\right)\Big)\Big|_{\alpha=0},\\ \partial_\alpha \mathcal{R}_a(\alpha)|_{\alpha=0} &\sim \partial_\alpha\Big(m_2(\alpha)\chi_a^i\left(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\right)\Big)\Big|_{\alpha=0},\end{aligned} \tag{8.18}$$

so that (8.16) and (8.17) get the form

$$\begin{aligned}\mathcal{R}_b(\alpha) \sim{} & \frac{m_1(\alpha)\partial_\alpha(m_2(\alpha)\chi_a^i(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a))|_{\alpha=0})e^{-ik\Delta}}{2\pi i}\\ & \times \int_{-i\infty}^{+i\infty} \frac{2e^{ik\Delta x^2/2}x^2}{2\left(\frac{\pi}{2\Phi_+}\right)^{-2}\left(1+\cos\left(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)\right)\right) - x^2}dx\\ & + m_1(\alpha)\chi_b^i\Big(\alpha + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\Big)\end{aligned} \tag{8.19}$$

as $-\Phi_+ < \mathrm{Re}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b) < 3\Phi_+$, and

$$\begin{aligned}\mathcal{R}_a(\alpha) \sim{} & -\frac{m_2(\alpha)\partial_\alpha(m_1(\alpha)\chi_b^i(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b))|_{\alpha=0})e^{-ik\Delta}}{2\pi i}\\ & \times \int_{-i\infty}^{+i\infty} \frac{2e^{ik\Delta x^2/2}x^2}{2\left(\frac{\pi}{2\Phi_-}\right)^{-2}\left(1+\cos\left(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)\right)\right) - x^2}dx\\ & + m_2(\alpha)\chi_a^i\Big(\alpha - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\Big)\end{aligned} \tag{8.20}$$

as $-3\Phi_- < \mathrm{Re}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a) < \Phi_-$, for $-\min(\frac{1}{2}\pi, \frac{1}{2}\pi + \Phi_a) < \varphi_\circ < \min(\frac{1}{2}\pi, \frac{1}{2}\pi + \Phi_b)$.

Concerning the scattering diagram we need, from section 7, the evaluation of $\mathcal{R}_{a,b}(\alpha)$ for real $\alpha = \varphi$. Deforming the path of integration to the steepest descent path in (8.19), (8.20) (without capturing any poles), we derive

$$\begin{aligned}\mathcal{R}_b(\varphi) \sim{} & \frac{m_1(\varphi)\partial_\alpha(m_2(\alpha)\chi_a^i(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a))|_{\alpha=0}}{\sqrt{2ik\Delta}} 2e^{-ik\Delta}\left(ve^{v^2}\mathrm{erfc}(v) - \frac{1}{\sqrt{\pi}}\right)\\ & + m_1(\varphi)\chi_b^i\Big(\varphi + \tfrac{1}{2}\pi - \tfrac{1}{2}\Phi_b\Big)\end{aligned} \tag{8.21}$$

as $-\Phi_+ < (\varphi + \frac{1}{2}\pi - \frac{1}{2}\Phi_b) < 3\Phi_+$, $v^2 = ik\Delta(\pi/2\Phi_+)^{-2}(1 + \cos((\pi/2\Phi_+)(\varphi - \Phi_b)))$, and

$$\begin{aligned}\mathcal{R}_a(\varphi) \sim{} & -\frac{m_2(\varphi)\partial_\alpha(m_1(\alpha)\chi_b^i(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b))|_{\alpha=0}}{\sqrt{2ik\Delta}} 2e^{-ik\Delta}\left(we^{w^2}\mathrm{erfc}(w) - \frac{1}{\sqrt{\pi}}\right)\\ & + m_2(\varphi)\chi_a^i\Big(\varphi - \tfrac{1}{2}\pi + \tfrac{1}{2}\Phi_a\Big)\end{aligned} \tag{8.22}$$

as $-3\Phi_- < (\varphi - \frac{1}{2}\pi + \frac{1}{2}\Phi_a) < \Phi_-$, $w^2 = ik\Delta(\pi/2\Phi_-)^{-2}(1 + \cos((\pi/2\Phi_-)(\varphi + \Phi_a)))$, for $-\min(\frac{1}{2}\pi, \frac{1}{2}\pi + \Phi_a) < \varphi_\circ < \min(\frac{1}{2}\pi, \frac{1}{2}\pi + \Phi_b)$, where

$$ae^{a^2}\operatorname{erfc}(a) - \frac{1}{\sqrt{\pi}} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{-t^2e^{-t^2}}{a^2+t^2}dt = \frac{-1}{2\sqrt{\pi}a^2} + O\left(\frac{1}{a^4}\right) \tag{8.23}$$

when a is real. In the right-hand sides of (8.21), (8.22), note that the first term corresponds to double diffraction, and the second term to single diffraction.

REMARK 3 The smooth behaviour of $f_{br}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)$ and $f_{ar}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)$ on the imaginary axis for large k simplifies the calculus of the integral terms with $e^{-ik\Delta\cos\alpha'}$ factor in the general case. So, taking $x' = \cos\alpha'$ as a new variable of integration in (6.7), (6.8) with (6.9) or in (6.10), (6.11), the integral terms can be reduced to sums of elementary terms of the type

$$\int_{a_i}^{a_{i+1}} G(x')e^{-ik\Delta x'}dx', \tag{8.24}$$

where G slowly varies on fixed intervals $[a_i, a_{i+1}]$, which can be calculated by standard methods (**35**, **36**) using polynomial approximation.

9. Conclusion

We have shown that a method using the Sommerfeld–Maliuzhinets representation of fields is not limited to the study of diffraction by isolated wedges; it can be also constructed for complex angular domains. For this purpose, a new representation of spectral functions, that we initially developed in (**10**) for the three-part impedance plane, has been used to reduce the problem of diffraction by a polygonal object. Notably, the spectral functional equations for a general (finite or infinite) n-part impedance polygon in free space has been derived here for the first time.

These equations can be transformed by the theory of difference equations to a system of integral equations. The approach is illustrated in detail in the important case of the three-part semi-infinite polygon, for which the problem is reduced to a system of non-singular Fredholm integral equations of the second kind. We then note the smooth behaviour of the spectral functions on the path of integration, the simplicity of the exponential factor taking account of the length Δ of the central plate, the regularity of the kernels in the spectral integral equations. This permits to apply approximations for $k\Delta$ large (by asymptotic evaluation of integrals) or small (by semi-inversion (**10**)), or to consider classical inversion methods of non-singular Fredholm equations of the second kind (**29**, **30**). Thus, the evaluation of the far field, which depends on the calculus of the spectral functions at some discrete points of the complex plane, is possible in a well-posed manner. Moreover, we show that the system of spectral integral equations can be decoupled in various cases, in particular for the three-part impedance plane where the results given in (**10**) are recovered, and for the symmetric semi-infinite impedance polygon where no exact results were known. Some numerical results when $k\Delta$ is small and some approximations when $k\Delta$ is large are given.

Acknowledgments

I thank R. V. Craster, V. P. Smyshlyaev, B. V. Budaev and the referees for valuable comments which improved the paper.

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APPENDIX A

The function $\Psi_{lr}(\alpha, \Phi)$

The function $\Psi_{lr}(\alpha)$, also denoted by $\Psi_{lr}(\alpha, \Phi)$, has been defined by Maliuzhinets, for the diffraction by an impedance wedge (**17**), when he considered the solution of the equations

$$\begin{aligned}(\sin\alpha + \sin\theta_l)\, f_{lr}(\alpha + \Phi) - (-\sin\alpha + \sin\theta_l)\, f_{lr}(-\alpha + \Phi) = 0,\\ (\sin\alpha - \sin\theta_r)\, f_{lr}(\alpha - \Phi) - (-\sin\alpha - \sin\theta_r)\, f_{lr}(-\alpha - \Phi) = 0,\end{aligned} \tag{A.1}$$

regular in the strip $|\mathrm{Re}(\alpha)| \leqslant \Phi$, except for the pole with unit residue at $\alpha = \varphi'$, and $O(1)$ at infinity in this band, where $\sin\theta_l$ (resp. $\sin\theta_r$) corresponds to the relative impedance attached to the face $\varphi = +\Phi$ (resp. $\varphi = -\Phi$). The solution is expressed in the form $\dfrac{\Psi_{lr}(\alpha)}{\Psi_{lr}(\varphi')}\sigma(\alpha)$, where $\Psi_{lr}(\alpha)$ is the solution of (A.1) without poles or zeros as $|\mathrm{Re}(\alpha)| \leqslant \Phi$ when $\mathrm{Re}(\sin\theta_{l,r}) > 0$, $O(\cos(\pi\alpha/2\Phi))$ in this band, and

$$\sigma(\alpha) = \frac{\pi}{2\Phi}\cos\left(\frac{\pi\varphi'}{2\Phi}\right)\Big/\left(\sin\left(\frac{\pi\alpha}{2\Phi}\right) - \sin\left(\frac{\pi\varphi'}{2\Phi}\right)\right)$$

(**17**, **34**). The function $\Psi_{lr}(\alpha)$ has numerous properties. We have

$$\begin{aligned}\Psi_{lr}(\alpha) = A\Big(&\Psi_\Phi\Big(\alpha + \Phi + \Big(\tfrac{1}{2}\pi - \theta_l\Big)\Big)\,\Psi_\Phi\Big(\alpha - \Phi - \Big(\tfrac{1}{2}\pi - \theta_r\Big)\Big)\\ &\times \Psi_\Phi\Big(\alpha + \Phi - \Big(\tfrac{1}{2}\pi - \theta_l\Big)\Big)\,\Psi_\Phi\Big(\alpha - \Phi + \Big(\tfrac{1}{2}\pi - \theta_r\Big)\Big)\Big),\end{aligned} \tag{A.2}$$

where Ψ_Φ is the Maliuzhinets function (**17**), $0 < \mathrm{Re}(\theta_{l,r}) \leqslant \frac{1}{2}\pi$, A is an arbitrary constant. This function satisfies $\Psi_\Phi(\alpha) = A_0' e^{\mp i\pi\alpha/8\Phi}(1 + O(\alpha^\nu |e^{\pm i\mu\alpha}|))$ when $\mathrm{Im}(\alpha) \to \pm\infty$, where A_0' and ν are constants, $\mu = \min(\pi/2\Phi, 1)$ (see (**33**) for more details), and we can write $\Psi_{lr}(\alpha) = A_0 \cos(\pi\alpha/2\Phi)(1 + O(\alpha^\nu e^{-\mu|\mathrm{Im}(\alpha)|}))$ for $|\mathrm{Im}(\alpha)|$ large, A_0 a constant. Since $\Psi_\Phi(\alpha) = \Psi_\Phi(-\alpha)$ and $\Psi_\Phi(\alpha + \frac{1}{2}\pi)\Psi_\Phi(\alpha - \frac{1}{2}\pi) = \Psi_\Phi^2(\frac{1}{2}\pi)\cos(\pi\alpha/4\Phi)$, we have $\Psi_{lr}(-\alpha) = \Psi_{rl}(\alpha)$ and

$$4\Psi_{lr}\left(\alpha + \tfrac{1}{2}\pi\right)\Psi_{lr}\left(\alpha - \tfrac{1}{2}\pi\right) = A^2\Psi_\Phi^8\left(\tfrac{1}{2}\pi\right)\left(\cos\left(\pi\left(\tfrac{1}{2}\pi - \theta_l\right)\Big/2\Phi\right) - \sin(\pi\alpha/2\Phi)\right) \times \left(\cos\left(\pi\left(\tfrac{1}{2}\pi - \theta_r\right)\Big/2\Phi\right) + \sin(\pi\alpha/2\Phi)\right), \tag{A.3}$$

with $A\Psi_\Phi^4(\frac{1}{2}\pi) = 2A_0$. Besides, from (**24**), we can write, for $|\mathrm{Re}(\alpha)| \leqslant \Phi + (N+1)\pi$,

$$\Psi_{lr}(\alpha) = B_N \prod_{(\pm,(l,r))}\left(\prod_{m=0}^{N}\left[\Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}(\alpha \pm \Phi + \theta_{l,r} + m\pi)\right)\right.\right.$$
$$\times\,\Gamma\left(\frac{1}{2} - \frac{1}{4\Phi}(\alpha \pm \Phi - (\theta_{l,r} + m\pi))\right)\Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}(\alpha \pm \Phi + \pi - \theta_{l,r} + m\pi)\right)$$
$$\left.\left.\times\,\Gamma\left(\frac{1}{2} - \frac{1}{4\Phi}(\alpha \pm \Phi - (\pi - \theta_{l,r} + m\pi))\right)\right]^{(-1)^{m+1}}\right)$$
$$\times\,\exp\left(\int_0^\infty (-e^{-\nu\pi})^{N+1}\frac{e^{-\nu\theta_{l,r}} + e^{-\nu(\pi-\theta_{l,r})}}{(1 + e^{-\nu\pi})}\frac{(1 - \cosh(\nu(\alpha \pm \Phi)))}{\nu\sinh(2\nu\Phi)}d\nu\right), \tag{A.4}$$

where the upper (resp. lower) sign is attached to θ_l (resp. θ_r), N is an arbitrary positive integer and B_N is a constant. For the applications, we choose N fixed and define Ψ_{lr} with $B_N = 1$. This expression is suitable for numerical calculus (with $N = 1$ or 2) or to derive the analytical properties of $\Psi_{lr}(\alpha)$ (with $N \to \infty$) from those of Γ.

The zeros which are the closest to the imaginary axis are $\alpha = \Phi + \theta_l$ and $\alpha = -\Phi - \theta_r$, and the closest poles are $\alpha = \Phi + \theta_l + \pi$ and $\alpha = -\Phi - \theta_r - \pi$.

REMARK 4 From (**17**), the zeros of Ψ_Φ which are the closest to $\alpha = 0$ and the corresponding poles are the points $\alpha = \pm(\frac{1}{2}\pi + 2\Phi)$ and $\alpha = \pm(\frac{3}{2}\pi + 2\Phi)$. In other respects, Ψ_Φ satisfies

$$\Psi_\Phi(\alpha + 2\Phi)/\Psi_\Phi(\alpha - 2\Phi) = \cot\left(\left(\alpha + \tfrac{1}{2}\pi\right)\Big/2\right), \Psi_\Phi(\alpha + \Phi)\Psi_\Phi(\alpha - \Phi) = \Psi_\Phi^2(\Phi)\Psi_{\Phi/2}(\alpha). \tag{A.5}$$

APPENDIX B

Principle of semi-inversion for our system of integral equations

We can modify equations and derive integral equations with kernels vanishing as $k\Delta \to 0$ for the three-part semi-infinite impedance polygon, for approximations when $k\Delta$ is small. For this, we begin with changing the unknowns in the equations (6.7), (6.8). We consider

$$f_{ar0}(\alpha) = \left[f_{ar}(\alpha) - f_0\left(\alpha - \left(\Phi_+ - \tfrac{1}{2}\pi\right)\right)\right],$$
$$f_{br0}(\alpha) = \left[f_{br}(\alpha) - e^{ik\Delta\sin\varphi_\circ} f_0\left(\alpha + \left(\Phi_- - \tfrac{1}{2}\pi\right)\right)\right] \tag{B.1}$$

as new unknowns, where $f_0(\alpha, \varphi_\circ)$, corresponding to the solution for $\Delta = 0$, is given by (5.16). These functions vanish as $\Delta = 0$, and satisfy, from (6.7), (6.8) with (6.9),

$$\frac{f_{br0}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)}{\Psi_{+1}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)} = \frac{-i}{4\Phi_+}\left(\int_{-i\infty}^{+i\infty} \frac{f_{ar0}(\alpha' - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)}{\Psi_{+1}(\alpha' - \Phi_+)} \times \frac{\sin(\pi\alpha'/2\Phi_+)}{\cos\left(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_+}\right)} d\alpha' + \int_{-i\infty}^{+i\infty} \frac{B_{a0}(\alpha')\sin(\pi\alpha'/2\Phi_+)}{\cos\left(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_+}\right)} d\alpha'\right), \tag{B.2}$$

where

$$B_{a0}(\alpha') = \frac{f_{ar0}(\alpha' - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)}{\Psi_{+1}(\alpha' - \Phi_+)}(e^{-ik\Delta\cos\alpha'} - 1) + \frac{f_0(\alpha' - \frac{1}{2}\pi + (\Phi_a - \Phi_b)/2, \varphi_\circ)}{\Psi_{+1}(\alpha' - \Phi_+)}(e^{-ik\Delta(\cos\alpha' + \sin\varphi_\circ)} - 1)e^{ik\Delta\sin\varphi_\circ} \tag{B.3}$$

as $-\Phi_+ < \mathrm{Re}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b) < 3\Phi_+$, and

$$\frac{f_{ar0}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)}{\Psi_{1-}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)} = \frac{i}{4\Phi_-}\left(\int_{-i\infty}^{+i\infty} \frac{f_{br0}(\alpha' + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)}{\Psi_{1-}(\alpha' + \Phi_-)} \times \frac{\sin(\pi\alpha'/2\Phi_-)}{\cos\left(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_-}\right)} d\alpha' + \int_{-i\infty}^{+i\infty} \frac{B_{b0}(\alpha')\sin(\pi\alpha'/2\Phi_-)}{\cos\left(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_-}\right)} d\alpha'\right), \tag{B.4}$$

where

$$B_{b0}(\alpha') = \left(\frac{f_{br0}(\alpha' + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)}{\Psi_{1-}(\alpha' + \Phi_-)}\right)(e^{-ik\Delta\cos\alpha'} - 1) + \left(\frac{f_0(\alpha' + \frac{1}{2}\pi - (\Phi_b - \Phi_a)/2, \varphi_\circ)}{\Psi_{1-}(\alpha' + \Phi_-)}\right)(e^{-ik\Delta(\cos\alpha' - \sin\varphi_\circ)} - 1) \tag{B.5}$$

as $-3\Phi_- < \mathrm{Re}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a) < \Phi_-$, for $-\min(\frac{1}{2}\pi, \frac{1}{2}\pi + \Phi_a) < \varphi_\circ < \min(\frac{1}{2}\pi, \frac{1}{2}\pi + \Phi_b)$.

We then notice some similarity with the equations satisfied by f_0 when $\Delta = 0$. Thus, we let

$$f'_{br0}\left(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}\right) = \int_{-i\infty}^{i\infty} G(\varphi') f_0\left(\alpha + \frac{\pi}{2} + (\Phi_a - \Phi_b)/2, \varphi'\right) d\varphi',$$

$$f'_{ar0}\left(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}\right) = \int_{-i\infty}^{i\infty} G(\varphi') f_0\left(\alpha - \frac{\pi}{2} + (\Phi_a - \Phi_b)/2, \varphi'\right) d\varphi' \tag{B.6}$$

as $|\mathrm{Re}(\alpha)| < \frac{1}{2}\pi$, and search to define $G(\varphi')$ so that f'_{br0} and f'_{ar0} verify (B.2) to (B.5). The functions $f_0(\alpha \pm \frac{1}{2}\pi + (\Phi_a - \Phi_b)/2, \varphi')$ are regular and $O(1/\cos(\pi\varphi'/2\Phi_d))$ on the imaginary axis, and a pole at $\varphi' = \alpha \pm \frac{1}{2}\pi$ ensures that, even if $\Phi_a = \Phi_b = 0$, in general $f'_{br0} \neq f'_{ar0}$.

Using the equations (6.7), (6.8) when $\Delta = 0$ satisfied by f_0, we remark that we can write

$$\begin{aligned}
&\frac{f'_{br0}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)}{\Psi_{+1}(\alpha + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)} \\
&\quad = \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{f'_{ar0}(\alpha' - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)\sin(\pi\alpha'/2\Phi_+)}{\Psi_{+1}(\alpha' - \Phi_+)\left(\cos\left(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_+}\right)\right)} d\alpha' \\
&\qquad + \frac{\pi}{2\Phi_+}\int_{-i\infty}^{i\infty} \frac{G(\varphi')}{\Psi_{+1}(\varphi' - \frac{1}{2}\Phi_b)} \frac{\sin(\pi(\varphi' + \frac{1}{2}\pi)/2\Phi_+)}{(\cos(\pi(\alpha - \Phi_b)/2\Phi_+) + \cos(\pi(\varphi' + \frac{1}{2}\pi)/2\Phi_+))} d\varphi',
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
&\frac{f'_{ar0}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)}{\Psi_{1-}(\alpha - \frac{1}{2}\pi + \frac{1}{2}\Phi_a)} \\
&\quad = \frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{f'_{br0}(\alpha' + \frac{1}{2}\pi - \frac{1}{2}\Phi_b)\sin(\pi\alpha'/2\Phi_-)}{\Psi_{1-}(\alpha' + \Phi_-)\left(\cos\left(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)\right) + \cos\left(\frac{\pi\alpha'}{2\Phi_-}\right)\right)} d\alpha' \\
&\qquad + \frac{\pi}{2\Phi_-}\int_{-i\infty}^{i\infty} \frac{G(\varphi')}{\Psi_{1-}(\varphi' + \frac{1}{2}\Phi_a)} \frac{\sin(\pi(\varphi' - \frac{1}{2}\pi)/2\Phi_-)}{(\cos(\pi(\alpha + \Phi_a)/2\Phi_-) + \cos(\pi(\varphi' - \frac{1}{2}\pi)/2\Phi_-))} d\varphi'.
\end{aligned} \tag{B.8}$$

In the case where $G(\varphi')$ is regular in the band $|\mathrm{Re}(\varphi')| \leqslant \frac{1}{2}\pi$, we can shift the integral paths in the integrals containing $G(\varphi')$. Comparing (B.2) to (B.5) with (B.7), (B.8), we notice that (f'_{br0}, f'_{ar0}) is a solution of the system of equations (B.2) to (B.4) if G satisfies the conditions

$$\begin{aligned}
\frac{G(\alpha' + \frac{1}{2}\pi)}{\Psi_{1-}(\alpha' + \Phi_-)} - \frac{G(-\alpha' + \frac{1}{2}\pi)}{\Psi_{1-}(-\alpha' + \Phi_-)} &= \frac{i}{2\pi}(B_{b0}(\alpha') - B_{b0}(-\alpha')), \\
\frac{G(\alpha' - \frac{1}{2}\pi)}{\Psi_{+1}(\alpha' - \Phi_+)} - \frac{G(-\alpha' - \frac{1}{2}\pi)}{\Psi_{+1}(-\alpha' - \Phi_+)} &= \frac{-i}{2\pi}(B_{a0}(\alpha') - B_{a0}(-\alpha')),
\end{aligned} \tag{B.9}$$

where $\Phi_+ = \frac{1}{2}\pi + \frac{1}{2}\Phi_b$ and $\Phi_- = \frac{1}{2}\pi + \frac{1}{2}\Phi_a$. Taking account of the properties of Ψ_{+1} and Ψ_{1-} (see appendix A), and letting $G(\alpha') = (\cos\alpha' + \sin\theta_1)g(\alpha')$, (B.9) can be written as

$$\begin{aligned}
g\left(\alpha' + \frac{\pi}{2}\right) - g\left(-\alpha' + \frac{\pi}{2}\right) &= \frac{i\Psi_{1-}(\alpha' + \Phi_-)(B_{b0}(\alpha') - B_{b0}(-\alpha'))}{2\pi(-\sin\alpha' + \sin\theta_1)}, \\
g\left(\alpha' - \frac{\pi}{2}\right) - g\left(-\alpha' - \frac{\pi}{2}\right) &= \frac{i\Psi_{+1}(\alpha' - \Phi_+)(B_{a0}(\alpha') - B_{a0}(-\alpha'))}{2\pi(-\sin\alpha' - \sin\theta_1)}.
\end{aligned} \tag{B.10}$$

Since $G(\alpha')$ is regular in the band $|\mathrm{Re}(\alpha')| \leqslant \frac{1}{2}\pi$ and $\mathrm{Re}(\sin\theta_1) > 0$, $g(\alpha')$ is regular in this band. We can then use (6.1), (6.2) and write, as $|\mathrm{Re}(\alpha)| < \frac{1}{2}\pi$,

$$g(\alpha) = \frac{i}{4\pi}\int_{-i\infty}^{+i\infty} d\alpha' \left(\frac{i\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha') - B_{b0}(-\alpha'))}{2\pi(\sin\alpha' - \sin\theta_1)} \tan\left(\frac{1}{2}\left(\alpha + \frac{\pi}{2} - \alpha'\right)\right) \right.$$
$$\left. - \frac{i\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha') - B_{a0}(-\alpha'))}{2\pi(\sin\alpha' + \sin\theta_1)} \tan\left(\frac{1}{2}\left(\alpha - \frac{\pi}{2} - \alpha'\right)\right) \right). \tag{B.11}$$

Using (B.6) and (B.11), we obtain the equations with kernels vanishing as $k\Delta \to 0$:

$$f_{br0}\left(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}\right) = \frac{1}{8\pi^2}\int_{-i\infty}^{+i\infty} d\alpha' \left(\frac{\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha') - B_{b0}(-\alpha'))}{\sin\alpha' - \sin\theta_1} M_+(\alpha,\alpha') \right.$$
$$\left. - \frac{\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha') - B_{a0}(-\alpha'))}{\sin\alpha' + \sin\theta_1} M_-(\alpha,\alpha') \right), \tag{B.12}$$

$$f_{ar0}\left(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}\right) = \frac{1}{8\pi^2}\int_{-i\infty}^{+i\infty} d\alpha' \left(\frac{\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha') - B_{b0}(-\alpha'))}{\sin\alpha' - \sin\theta_1} N_+(\alpha,\alpha') \right.$$
$$\left. - \frac{\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha') - B_{a0}(-\alpha'))}{\sin\alpha' + \sin\theta_1} N_-(\alpha,\alpha') \right), \tag{B.13}$$

where $M_\pm(\alpha,\alpha') = L_\pm(\alpha + \frac{1}{2}\pi + \frac{1}{2}(\Phi_a - \Phi_b), \alpha')$, $N_\pm(\alpha,\alpha') = L_\pm(\alpha - \frac{1}{2}\pi + \frac{1}{2}(\Phi_a - \Phi_b), \alpha')$,

$$L_\pm(\alpha,\alpha') = \frac{\pi\sin\alpha'\Psi_{+-}(\alpha)}{2\Phi_d}\int_{-i\infty}^{i\infty} \frac{\cos\left(\dfrac{\pi(\varphi' + (\Phi_a - \Phi_b)/2)}{2\Phi_d}\right)}{\Psi_{+-}(\varphi' + (\Phi_a - \Phi_b)/2)}$$
$$\times \frac{\cos\varphi' + \sin\theta_1}{\cos\left(\varphi' \pm \dfrac{\pi}{2}\right) + \cos\alpha'} \frac{1}{\left(\sin\left(\dfrac{\pi\alpha}{2\Phi_d}\right) - \sin\left(\dfrac{\pi(\varphi' + (\Phi_a - \Phi_b)/2)}{2\Phi_d}\right)\right)} d\varphi'. \tag{B.14}$$

In the particular case $\Phi_a = \Phi_b = 0$, $\Phi_d = \frac{1}{2}\pi$, the functions $L_\pm$ can be simplified so that we recover the expressions found in (**10**) for the three-part impedance plane (see Remark 5 below).

REMARK 5 There exist analytical expressions of $L_\pm$ when $\Phi_a = \Phi_b = 0$. For this, we consider

$$\begin{aligned}
&(\sin\alpha + \sin\theta_l)\Psi_{lr}(\alpha + \tfrac{1}{2}\pi) - (-\sin\alpha + \sin\theta_l)\Psi_{lr}(-\alpha + \tfrac{1}{2}\pi) = 0,\\
&(\sin\alpha - \sin\theta_l)\Psi_{lr}(\alpha - \tfrac{1}{2}\pi) - (-\sin\alpha - \sin\theta_l)\Psi_{lr}(-\alpha - \tfrac{1}{2}\pi)\\
&\quad = (\sin\theta_r - \sin\theta_l)(\Psi_{lr}(\alpha - \tfrac{1}{2}\pi) - \Psi_{lr}(-\alpha - \tfrac{1}{2}\pi)),
\end{aligned} \tag{B.15}$$

and use (6.1), (6.2) for $\chi = \Psi_{lr}/\Psi_{ll}$ with $\Psi_{ll}(\alpha) = A_0(\cos\alpha + \sin\theta_l)$, $\chi(\pm i\infty) = 1$. We then obtain

$$\frac{\Psi_{lr}(-\alpha)}{A_0(\cos\alpha + \sin\theta_l)} - 1 = \frac{-i(\sin\theta_r - \sin\theta_l)}{4\pi}\int_{-i\infty}^{+i\infty} A_0 H_{lr}(\alpha')\tan\left(\frac{1}{2}\left(\alpha + \frac{\pi}{2} - \alpha'\right)\right) d\alpha' \tag{B.16}$$

for $-\frac{3}{2}\pi < \mathrm{Re}(\alpha) < \frac{1}{2}\pi$, where $H_{lr}(\alpha') = 2\sin\alpha'/(\Psi_{lr}(\alpha' + \frac{1}{2}\pi)(\sin\alpha' + \sin\theta_l))$ and

$$H_{lr}(\alpha') = \frac{1}{\Psi_{lr}(\alpha + \frac{1}{2}\pi)} - \frac{1}{\Psi_{lr}(-\alpha + \frac{1}{2}\pi)} = \frac{\sin\alpha/\sin\theta_l}{\Psi_{lr}(\alpha + \frac{1}{2}\pi)} + \frac{\sin\alpha/\sin\theta_l}{\Psi_{lr}(-\alpha + \frac{1}{2}\pi)}. \tag{B.17}$$

Using that $\Psi_{lr}(-\alpha)\Psi_{lr}(\alpha) = A_0^2(\cos\alpha + \sin\theta_l)(\cos\alpha + \sin\theta_r)$ when $\Phi_a = \Phi_b = 0$, and an analytic continuation by shifting the path of integration, we derive that

$$\int_{-i\infty}^{+i\infty} \frac{1}{\Psi_{lr}(\varphi')} \frac{\cos\varphi'}{\cos\alpha + \sin\varphi'} d\varphi' = -2\pi i \frac{\dfrac{\sin\alpha + \sin\theta_r}{\Psi_{lr}(\alpha - \frac{1}{2}\pi)} - \frac{1}{A_0}}{(\sin\theta_l - \sin\theta_r)},$$
$$\int_{-i\infty}^{+i\infty} \frac{\cos\varphi'}{\Psi_{lr}(\varphi')} \frac{\cos\varphi'}{\cos\alpha + \sin\varphi'} d\varphi' = 2\pi i \sin\theta_l \frac{\dfrac{\sin\alpha + \sin\theta_r}{\Psi_{lr}(\alpha - \frac{1}{2}\pi)} - \frac{1}{A_0}}{(\sin\theta_l - \sin\theta_r)} \tag{B.18}$$

as $|\mathrm{Re}(\alpha)| < \frac{1}{2}\pi$. Similar expressions with $-\sin\varphi'$ in place of $\sin\varphi'$ can be obtained from $\Psi_{lr}(-\alpha) = \Psi_{rl}(\alpha)$ (or by continuation of (B.18) with capture of a pole). This permits, by elementary combinations, analytical expressions of $L_\pm$ terms when $\Phi_a = \Phi_b = 0$.