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A SPECTRAL CHARACTERIZATION OF GENERALIZED REAL SYMMETRIC CENTROSYMMETRIC AND GENERALIZED REAL SYMMETRIC SKEW-CENTROSYMMETRIC MATRICES*

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Abstract. We show that the only real symmetric matrices whose spectrum is invariant modulo sign changes after either row or column reversal are the centrosymmetric matrices; moreover, we prove that the class of real symmetric centrosymmetric matrices can be completely characterized by this property. We also show that the only real symmetric matrices whose spectrum changes by multiplication by i after either row or column reversal are the skew-centrosymmetric matrices; here, too, we show that the class of real symmetric skew-centrosymmetric matrices can be completely characterized by this property of their eigenvalues. We prove both of these spectral characterizations as special cases of results for what we've called *generalized centrosymmetric K-matrices* and *generalized skew-centrosymmetric K-matrices*. Some results illustrating the application of the generalized centrosymmetric spectral characterization to other classes of real symmetric matrices are also given.

Key words. centrosymmetric matrices, skew-centrosymmetric matrices, eigenvalues

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1. Introduction.

A *centrosymmetric* matrix A of order n is a square matrix whose elements $a_{i,j}$ satisfy the property

$$a_{i,j} = a_{n-i+1,n-j+1} \text{ for } 1 \leq i, j \leq n.$$

A is called *skew-centrosymmetric* if its elements $a_{i,j}$ satisfy the property

$$a_{i,j} = -a_{n-i+1,n-j+1} \text{ for } 1 \leq i, j \leq n.$$

Although they make a brief appearance in [1], centrosymmetric matrices received their first serious treatment in the 1962 work of Collar [4]. Collar's paper also introduces the notion of skew-centrosymmetric matrices (he uses the term *centroskew*).

The symmetric Toeplitz matrices form an important subclass of the class of symmetric centrosymmetric (sometimes called *doubly symmetric*) matrices. An $n \times n$ matrix T is said to be Toeplitz if there exist numbers $r_{-n+1}, \ldots, r_0, \ldots, r_{n-1}$ such that $t_{i,j} = r_{j-i}$ for $1 \leq i, j \leq n$. As such, Toeplitz matrices are sometimes described as being "constant along the diagonals." Toeplitz matrices occur naturally in digital signal processing applications as well as other areas [7]. Centrosymmetric matrices appear in their own right, for example, in the numerical solution of certain differential equations [2], in the study of some Markov processes [8], and in various physics and engineering problems [6].

In this paper, we establish spectral characterizations for both real symmetric centrosymmetric and real symmetric skew-centrosymmetric matrices as special cases of

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results for what we have called *generalized centrosymmetric K-matrices* and *generalized skew-centrosymmetric K-matrices* (defined below).[1] To emphasize the elementary nature of the techniques involved, all results used regarding centrosymmetric and skew-centrosymmetric matrices are developed within this paper.

2. Notation and terminology.

Let J represent the *exchange matrix* of order n defined by $J_{i,j} = \delta_{i,n-j+1}$ for $1 \leq i, j \leq n$, where $\delta_{i,j}$ is the Kronecker delta (i.e., J is a matrix with ones on the cross-diagonal and zeros elsewhere). Left-multiplication by J against a matrix A reverses the row order of A. Right-multiplication by J against A reverses the column order of A. The properties of centrosymmetry and skew-centrosymmetry for a matrix can be written succinctly as $AJ = JA$ (equivalently, $A = JAJ$) and $AJ = -JA$ (equivalently, $A = -JAJ$), respectively.

We use K to denote an involutory (i.e., $K^2 = I$) matrix. The exchange matrix J belongs to the set of involutory matrices. We shall refer to matrices A satisfying $AK = KA$ as *generalized centrosymmetric K-matrices*,[2] and matrices A satisfying $AK = -KA$ as *generalized skew-centrosymmetric K-matrices*.

Following the terminology used in Andrew [2], when $x = Jx$ we say that the vector x is *symmetric*. When $x = -Jx$, we say that the vector x is *skew-symmetric*. We extend this terminology to the situation where J is replaced by an involutory matrix K by saying that when $x = Kx$ the vector x is *K-symmetric* and that when $x = -Kx$ the vector x is *K-skew-symmetric*.

Let R and S be multisets (i.e., elements can appear more than once in the collection). We write $R = \pm S$ if the elements of R are the same as those of S up to sign. We write $R = iS$ if $R = \{is \mid s \in S\}$, where $i = \sqrt{-1}$.

Let $\Lambda(A)$ denote the spectrum (eigenvalues) of A, $\{\lambda_i(A)\}_{1 \leq i \leq n}$. Our primary focus will be on the multisets $\Lambda(A)$, $\pm\Lambda(A)$, and $i\Lambda(A)$.

In what follows, $\|x\| = \sqrt{x^T x}$ will denote the Euclidean vector norm of a vector x.

3. Generalized centrosymmetric matrices.

Although our focus is primarily on real symmetric matrices, we relax that restriction in the following proposition about generalized centrosymmetric K-matrices.

PROPOSITION 3.1. *Suppose $A \in F^{n\times n}$ and $K \in F^{n\times n}$, where F is a field of characteristic not equal to 2 and K is an involutory matrix. If $AK = KA$, then $\Lambda(A) = \pm\Lambda(KA)$.*

Note: The same theorems and proofs hold *mutatis mutandis* when $\Lambda(KA)$ is replaced with $\Lambda(AK)$ in this proposition and all subsequent results of this article.

Proof. Except for the trivial cases $K = \pm I$, the matrix K has minimal polynomial $m(x) = x^2 - 1$. Since the zeros of $m(x)$ have multiplicity one, there exists a matrix $X \in F^{n\times n}$ such that conjugation of K by X yields the block diagonal form

$$K' \equiv X^{-1}KX = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where I represents a block identity matrix, and the sum of the dimensions of the I and $-I$ blocks is n (see [5], for example).

[1] A. Andrew obtained an eigenspace characterization for Hermitian centrosymmetric matrices in [2].

[2] A. Andrew also investigated this generalization in [2].

Conjugation of A by the same matrix X yields a matrix

$$A' \equiv X^{-1}AX = \begin{pmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix},$$

where we assume the same partitioning as that for K'. A simple calculation shows that $AK = KA$ if and only if $A'K' = K'A'$ if and only if A'_{12} and A'_{21} are both zero matrices. Consequently,

$$A' = \begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \text{ and } K'A' = \begin{pmatrix} A'_{11} & 0 \\ 0 & -A'_{22} \end{pmatrix}.$$

Since A is similar to A' and KA is similar to $K'A'$, the result is proved. ∎

Remark 3.2. When $K = J$, we can explicitly construct the eigenvector matrix X as follows. Denote the jth column of X by x_j. For $j \leq \lfloor \frac{n}{2} \rfloor$, let the vector x_j have components of 0 everywhere except for a 1 in components j and $n-j+1$. For $j > \lceil \frac{n}{2} \rceil$, let the vector x_j have components of 0 everywhere except for a 1 in component j and a -1 in component $n-j+1$. If n is odd, we let $x_{\lceil \frac{n}{2} \rceil}$ have components of 0 everywhere except for a 1 in component $\lceil \frac{n}{2} \rceil$. Note that the first $\lceil \frac{n}{2} \rceil$ eigenvectors are symmetric, while the remaining $\lfloor \frac{n}{2} \rfloor$ are skew-symmetric.

A centrosymmetric example. Consider the matrices

$$A_1 = \begin{pmatrix} 3 & -2 & -1 & 0 & 1 \\ -2 & 1 & -3 & 1 & 0 \\ -1 & -3 & 5 & -3 & -1 \\ 0 & 1 & -3 & 1 & -2 \\ 1 & 0 & -1 & -2 & 3 \end{pmatrix} \text{ and } JA_1 = \begin{pmatrix} 1 & 0 & -1 & -2 & 3 \\ 0 & 1 & -3 & 1 & -2 \\ -1 & -3 & 5 & -3 & -1 \\ -2 & 1 & -3 & 1 & 0 \\ 3 & -2 & -1 & 0 & 1 \end{pmatrix}.$$

A_1 is centrosymmetric and, consequently, so is JA_1.

$$\Lambda(A_1) = \left\{ -2, \quad 1-\sqrt{5}, \quad 1+\sqrt{5}, \quad 5, \quad 8 \right\}$$

and

$$\Lambda(JA_1) = \left\{ -1-\sqrt{5}, \quad -2, \quad -1+\sqrt{5}, \quad 5, \quad 8 \right\}.$$

A generalized centrosymmetric example. Let

$$A_2 = \begin{pmatrix} 8 & 2 & -5 \\ 2 & -4 & 1 \\ -5 & 1 & 2 \end{pmatrix} \text{ and } K = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{-2}{3} \\ \frac{-1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{-2}{3} & \frac{-1}{3} \end{pmatrix}.$$

Since $A_2K = KA_2$ and $K^2 = I$, we say that A_2 is a generalized centrosymmetric K-matrix.

$$\Lambda(A_2) = \Lambda(KA_2) = \left\{3 - 3\sqrt{7}, \quad 0, \quad 3 + 3\sqrt{7}\right\}.$$

Before proving the converse of the real symmetric case of Proposition 3.1, we establish a useful lemma.

LEMMA 3.3. *Let $A \in \mathbb{R}^{n \times n}$ be symmetric and nonzero, let $K \in \mathbb{R}^{n\times n}$ be a symmetric involutory matrix, and assume that the largest eigenvalue $\tilde{\lambda}(A)$ of A in magnitude equals the largest eigenvalue $\tilde{\lambda}(KA)$ of KA in magnitude up to sign (i.e.,*

$|\tilde{\lambda}(A)| \equiv \max_i \{|\lambda_i(A)|\}$ and $|\tilde{\lambda}(KA)| \equiv \max_i \{|\lambda_i(KA)|\}$ satisfy $\tilde{\lambda}(A) = \pm\tilde{\lambda}(KA)$). Then there is a nontrivial K-invariant subspace of the eigenspace of A corresponding to $\tilde{\lambda}(A)$. This subspace is also a subspace of the eigenspaces of KA and AK corresponding to their eigenvalues of largest magnitude.

Proof. Let x be a unit eigenvector of KA corresponding to λ, where $|\lambda| = \tilde{\lambda}(KA)$. Then $\lambda = x^T KAx$, and transposing this equation gives $\lambda = x^T AKx$. By the Cauchy–Schwarz inequality, we have that

$$|\lambda| = \left|x^T AKx\right| \le \|AKx\|.$$

Since $|\lambda|$ is extremal for A, $\|AKx\| \le |\lambda|$ and therefore $\|AKx\| = |\lambda|$.

Because the Cauchy–Schwarz inequality $|x^T(AKx)| \le \|AKx\| \cdot \|x\|$ yields an equality only when vectors x and AKx have the same direction up to sign, we may write

$$AKx = \pm\lambda x. \tag{1}$$

Multiplying the equation $KAx = \lambda x$ by K gives

$$Ax = \lambda Kx. \tag{2}$$

Using (1) and (2), we obtain $A^2Kx = \pm\lambda Ax = \pm\lambda^2 Kx$. Since the eigenvalues of A^2 are all nonnegative, we can rewrite (1) as $AKx = \lambda x$.

If $x = \pm Kx$, we're clearly done. Assume this is not the case, so that $x \pm Kx \neq \vec{0}$. Adding and subtracting $AKx = \lambda x$ against $Ax = \lambda Kx$ gives the equations

$$A(x + Kx) = \lambda(x + Kx), \tag{3}$$

$$A(x - Kx) = -\lambda(x - Kx). \tag{4}$$

Observe that the A eigenvector $(x + Kx)$ is K-symmetric, while the A eigenvector $(A - Kx)$ is K-skew-symmetric (i.e., K invariance). Also, we have that the eigenvalue-eigenvector pair $(\lambda, x + Kx)$ of matrix A is simultaneously an eigenpair of the matrix KA (multiply (3) by K) and AK (factor out a K from (3)). Finally, we note that the eigenpair $(-\lambda, x - Kx)$ of matrix A corresponds to an eigenpair $(\lambda, x - Kx)$ of the matrix KA (multiply (4) above by K) and AK (factor out a K from (4) above). This completes the proof. ∎

Remark 3.4. In addition to establishing the lemma, we've also demonstrated that the K-invariant subspace above has a basis consisting of only K-symmetric and K-skew-symmetric vectors.

PROPOSITION 3.5. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric, and suppose $\Lambda(A) = \pm\Lambda(KA)$, where $K \in \mathbb{R}^{n\times n}$ is a symmetric involutory matrix. Then $AK = KA$.*

Proof. Since the proposition holds trivially when A is the zero matrix, we may assume that A is nonzero in the following argument. Hence, A will have at least one nonzero eigenvalue and Lemma 3.3 will apply. Since A is symmetric, we are also guaranteed a full set of n independent eigenvectors.

Let S_0 be the nontrivial K invariant subspace of the eigenspace of A corresponding to the eigenvalue $\tilde{\lambda} \equiv \tilde{\lambda}(A)$ as defined in the statement of Lemma 3.3. Then for any $w_0 \in S_0$, we set $\tilde{w}_0 \equiv Kw_0 \in S_0$. If $w_1 \in S_0^\perp$, then

$$w_0^T Kw_1 = (K\tilde{w}_0)^T Kw_1 = \tilde{w}_0^T w_1 = 0.$$

Therefore, $S_0^\perp$ is also invariant under K. Since A is symmetric and maps S_0 to itself,

$$w_0^T A w_1 = (Aw_0)^T w_1 = 0.$$

This shows that A also maps $S_0^\perp$ to itself. Therefore, so will KA.

If $x = Kx$ is an eigenvector of A corresponding to the eigenvalue $\tilde{\lambda}$, then

$$(KA - AK)x = (KAx - AKx) = (\tilde{\lambda}Kx - Ax) = (\tilde{\lambda}x - \tilde{\lambda}x) = \vec{0}.$$

Similarly, we can show that $(KA - AK)x = \vec{0}$ when $x = -Kx$. Making use of Remark 3.4, we conclude that $(KA - AK)\,w = \vec{0}$ for any $w \in S_0$.

Since $\mathbb{R}^n = S_0 \oplus S_0^\perp$, if $S_0^\perp$ is trivial, we're done. Otherwise, we apply the above argument to A restricted to $S_0^\perp$. That is, let $S_0^\perp = S_1 \oplus S_1^\perp$, where S_1 is the nontrivial K invariant subspace of the eigenspace corresponding to the largest eigenvalue in magnitude for A restricted to $S_0^\perp$. Then, just as before, we can show that $(KA - AK)\,w = \vec{0}$ for any $w \in S_1$. Continuing in this manner, we establish that $KA - AK$ maps each of the (say m total) nontrivial invariant subspaces S_j associated with A's nonzero eigenvalues to $\vec{0}$.

From above, we know that the eigenspace $S_m = S_{m-1}^\perp$ corresponding to A's 0 eigenvalues is K invariant ($S_m = \{\vec{0}\}$ if A is nonsingular). Therefore KA and AK will both be zero when restricted to S_m. Since $KA - AK$ maps $\mathbb{R}^n = \oplus_{j=0}^m S_j$ to zero, we conclude that $KA = AK$. ∎

Remark 3.6. In the course of proving Proposition 3.5, we have shown that the eigenspace of A corresponding to its nonzero eigenvalues has a basis consisting of K-symmetric and K-skew-symmetric eigenvectors. This is also true for the eigenspace corresponding to the eigenvalue 0.

Proof. As noted above, if $Ax = \vec{0}$, then $\vec{0} = \pm KAx = \pm AKx$. If $x = \pm Kx$, we're done, so assume $x \neq \pm Kx$. As in the proof of Lemma 3.3, we finish by noting that $A(x \pm Kx) = \vec{0}$ and that $x + Kx$ is K-symmetric and $x - Kx$ is K-skew-symmetric. ∎

Combining the real symmetric case of Proposition 3.1 with Proposition 3.5, we arrive at the following characterization of real symmetric generalized centrosymmetric matrices.

THEOREM 3.7. *Suppose $A \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$ are symmetric, and $K^2 = I$. Then $AK = KA$ if and only if $\Lambda(A) = \pm\Lambda(KA)$.*

COROLLARY 3.8. *Let $J \in \mathbb{R}^{n\times n}$ be the exchange matrix. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is centrosymmetric if and only if $\Lambda(A) = \pm\Lambda(JA)$.*

Proof. Let $K = J$ in the statement of Theorem 3.7. ∎

It is convenient at this stage to quantify the number of eigenvalues of A which differ (by sign) from those of KA, where $A \in \mathbb{R}^{n\times n}$ is symmetric generalized K-centrosymmetric. We begin by making the following observations.

LEMMA 3.9. *Suppose $K \in \mathbb{R}^{n\times n}$ is a symmetric involutory matrix, and $A \in \mathbb{R}^{n\times n}$ is a symmetric generalized centrosymmetric K-matrix. Assume that we have expressed A's eigenvector basis in terms of K-symmetric and K-skew-symmetric eigenvectors (Remark 3.6 guarantees that this can be done), and assume that*

$$\text{(I)} \qquad \textit{for any } \{\lambda_i, \lambda_j\} \in \Lambda(A),\ |\lambda_i| = |\lambda_j| \ \textit{ implies } \lambda_i = \lambda_j.$$

Then the nonzero eigenvalues of A which differ by a sign from the eigenvalues of KA are precisely those corresponding to A's K-skew-symmetric eigenvectors.

Proof. If $Ax = \lambda x$, then $KAx = \lambda Kx$. If x is K-symmetric, then $KAx = \lambda x$. If x is K-skew-symmetric, we have that $KAx = -\lambda x$. Condition (I) precludes $-\lambda \in \Lambda(A)$ for $\lambda \neq 0$. ∎

Remark 3.10. Examples of matrices satisfying condition (I) include the positive definite and semidefinite matrices.

LEMMA 3.11. *Let $K \in \mathbb{R}^{n\times n}$ be a symmetric involutory matrix and let $A \in \mathbb{R}^{n\times n}$ be symmetric generalized K-centrosymmetric. Assume that K's eigenvalue* 1 *has multiplicity n_1 and that K's eigenvalue -1 has multiplicity n_2, where $n_1+n_2 = n$. If V is a basis for the eigenspace of A consisting entirely of K-symmetric and K-skew-symmetric eigenvectors, then V must contain precisely n_1 K-symmetric eigenvectors and n_2 K-skew-symmetric eigenvectors.*

Proof. The lemma is clearly true for $K = \pm I$, so assume this is not the case.

Let x be a K-symmetric eigenvector of A, and let X be the eigenvector matrix of K used in the proof of Proposition 3.1. Then we may express the K-symmetry of x as

$$KXy = Xy, \tag{5}$$

where $y = X^{-1}x$. Since $X^{-1}KX = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$, we may rewrite (5) as

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} y = y. \tag{6}$$

Equation (6) holds only if the last n_2 components of y are zero. Therefore, V cannot consist of more than n_1 K-symmetric eigenvectors without violating linear independence. Similarly, we can show that V cannot consist of more than n_2 K-skew-symmetric eigenvectors. Since $n = n_1 + n_2$, the basis V must consist of precisely n_1 K-symmetric eigenvectors and n_2 K-skew-symmetric eigenvectors. ∎

Remark 3.12. Lemma 3.11's quantification of the breakdown of V into K-symmetric eigenvectors and K-skew-symmetric eigenvectors generalizes a result in [3]. For real symmetric centrosymmetric matrices, Cantoni and Butler showed that V is composed of $\lceil \frac{n}{2} \rceil$ symmetric eigenvectors and $\lfloor \frac{n}{2} \rfloor$ skew-symmetric eigenvectors. This result follows from Lemma 3.11 applied to $K = J$, together with observations made in Remark 3.2.

PROPOSITION 3.13. *Let $K \in \mathbb{R}^{n\times n}$ be a symmetric involutory matrix and let $A \in \mathbb{R}^{n\times n}$ be symmetric generalized K-centrosymmetric. Assume that K's eigenvalue -1 has multiplicity n_2. If we let $d(X, Y)$ equal the number of eigenvalues of X which differ from those of Y, then $d(A, KA) \leq n_2$. If we further stipulate condition* (I) *above, we also have the lower bound* $\max\{n_2 - m, 0\} \leq d(A, KA)$*, where m is the multiplicity of A's zero eigenvalue.*

Proof. The proposition clearly holds for $K = \pm I$, so assume this is not the case.

The proof of Lemma 3.9 shows that the eigenpairs of A associated with the K-symmetric eigenvectors are also eigenpairs of KA. From Lemma 3.11, A has n_2 K-skew-symmetric eigenvectors and so it follows that $d(A, KA) \leq n_2$. Under the additional constraint of condition (I), Lemma 3.9 shows that the maximum amount by which $d(A, KA)$ can differ from n_2 is equal to the multiplicity of A's zero eigenvalue. ∎

Remark 3.14. The lower bound in Proposition 3.13 is sharp. For example, the spectrum of the rank 2 centrosymmetric matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix}$$

satisfies condition (I), $\Lambda(A) = \Lambda(JA)$, and $n_2 - m = \lfloor \frac{n}{2} \rfloor - m = 0$. Here, the nonzero eigenvalues correspond to symmetric eigenvectors while the zero eigenvalues correspond to skew-symmetric eigenvectors. Of course, when condition (I) holds and the matrix A is nonsingular, the upper and lower bounds in Proposition 3.13 coincide.

We next prove a result that holds for *any* real symmetric matrix satisfying condition (I).

PROPOSITION 3.15. *Suppose $A \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$ are symmetric, with $K^2 = I$. Assume further that if $\{\lambda_i, \lambda_j\} \in \Lambda(A)$ that $|\lambda_i| = |\lambda_j|$ implies $\lambda_i = \lambda_j$. Then $\Lambda(A) = \Lambda(KA)$ if and only if $A = KA$.*

Proof. The $\Leftarrow$ direction is obvious.

$\Rightarrow$ From Proposition 3.5, $\Lambda(A) = \Lambda(KA)$ implies that A is a generalized centrosymmetric K-matrix. Therefore, Remark 3.6 shows that we can construct a basis for the eigenspace of A consisting entirely of K-symmetric and K-skew-symmetric eigenvectors. Assume we have done so.

If x is any K-symmetric eigenvector of A, then $Ax = \lambda x$ if and only if $KAx = \lambda x$. From Lemma 3.9, we know that all of the K-skew-symmetric eigenvectors of A correspond to an eigenvalue of 0 (a sign change would arise for any nonzero eigenvalue corresponding to a K-skew-symmetric eigenvector), so $Ay = KAy = \vec{0}$ for any K-skew-symmetric eigenvector y.

Since A and KA agree on a basis (e.g., the eigenvectors of A), they must in fact represent the same operator. ∎

Remark 3.16. In the case where $K = J$, Proposition 3.15 states that if one row reverses any real symmetric matrix satisfying condition (I), then the spectrum of the resulting matrix will always differ from that of the original matrix unless the original matrix is unchanged from the row reversal.

Remark 3.17. The reader may wish to confirm (if he or she has not already done so) that $A_2 = KA_2$ in the generalized centrosymmetric example given earlier.

Using the same type of argument as in the proof of Proposition 3.15, we can also show the following.

PROPOSITION 3.18. *Suppose $A \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$ are symmetric, with $K^2 = I$. Assume further that if $\{\lambda_i, \lambda_j\} \in \Lambda(A)$ that $|\lambda_i| = |\lambda_j|$ implies $\lambda_i = \lambda_j$. Then $\Lambda(A) = \Lambda(-KA)$ if and only if $A = -KA$.*

4. Generalized skew-centrosymmetric matrices.

The following result is the generalized skew-centrosymmetric analogue of Proposition 3.1.

PROPOSITION 4.1. *Suppose $A \in F^{n\times n}$ and $K \in F^{n\times n}$, where F is a field of characteristic not equal to 2 and K is an involutory matrix. If $AK = -KA$, then $\Lambda(A) = i\Lambda(KA)$.*

Proof. If $K = \pm I$, then A must be the zero matrix and the result clearly holds.

Assume $K \neq \pm I$ and let $X \in F^{n\times n}$ be the same matrix used to diagonalize the matrix K in the proof of Proposition 3.1:

$$K' = X^{-1}KX = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Notationally, let the I block be $n_1 \times n_1$ and let the $-I$ block be $n_2 \times n_2$ where $n_1 + n_2 = n$. Proceeding as we did in Proposition 3.1, we can show that conjugation of A by X yields a matrix of the form

$$A' \equiv X^{-1}AX = \begin{pmatrix} 0 & A'_{12} \\ A'_{21} & 0 \end{pmatrix}$$

and that

$$K'A' = \begin{pmatrix} 0 & A'_{12} \\ -A'_{21} & 0 \end{pmatrix},$$

where we have the same block partitioning for these matrices as for K'.

Consider the case where $n_1 \geq n_2$. Using elementary row operations on the matrices $A' - \lambda I$ and $KA' - \lambda I$, we can construct the block upper triangular matrices

$$\begin{pmatrix} -\lambda I & A'_{12} \\ 0 & \frac{1}{\lambda}A'_{21}A'_{12} - \lambda I \end{pmatrix} \tag{7}$$

and

$$\begin{pmatrix} -\lambda I & A'_{12} \\ 0 & -\frac{1}{\lambda}A'_{21}A'_{12} - \lambda I \end{pmatrix}.$$

Taking determinants, we obtain the characteristic polynomials for $A' - \lambda I$ and $KA' - \lambda I$ as

$$(-1)^{n_1}\lambda^{n_1-n_2}\det\left(A'_{21}A'_{12} - \lambda^2 I\right) \tag{8}$$

and

$$(-1)^{n_1}\lambda^{n_1-n_2}\det\left(A'_{21}A'_{12} + \lambda^2 I\right), \tag{9}$$

respectively. From (8) and (9), the similarity of A to A', and the similarity of KA to $K'A'$, we conclude that $\lambda \in \Lambda(A)$ if and only if $i\lambda \in \Lambda(KA)$.

When $n_1 < n_2$, one can apply elementary row operations to the matrices $A' - \lambda I$ and $KA' - \lambda I$ to obtain block lower triangular matrices analogous to those in (7). Taking determinants, one obtains the characteristic polynomials

$$(-1)^{n_2}\lambda^{n_2-n_1}\det\left(A'_{12}A'_{21} - \lambda^2 I\right)$$

for $A' - \lambda I$ and

$$(-1)^{n_2}\lambda^{n_2-n_1}\det\left(A'_{12}A'_{21} + \lambda^2 I\right)$$

for $K'A' - \lambda I$. Again, we conclude that $\lambda \in \Lambda(A)$ if and only if $i\lambda \in \Lambda(KA)$. $\square$

A skew-centrosymmetric example. Consider the matrices

$$A_3 = \begin{pmatrix} 2 & -1 & 1 & -1 & 0 \\ -1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -2 \end{pmatrix}$$

and

$$JA_3 = \begin{pmatrix} 0 & 1 & -1 & 1 & -2 \\ -1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & 1 \\ 2 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

A_3 is skew-centrosymmetric and, consequently, so is JA_3.

$$\Lambda\left(A_3\right) = \left\{-\sqrt{10}, -\sqrt{3}, 0, \sqrt{3}, \sqrt{10}\right\}$$

and

$$\Lambda\left(JA_3\right) = \left\{-\sqrt{10}i, -\sqrt{3}i, 0, \sqrt{3}i, \sqrt{10}i\right\}.$$

A generalized skew-centrosymmetric example. Let

$$A_4 = \begin{pmatrix} -3\sqrt{2} & -\sqrt{2} & 2 \\ -\sqrt{2} & \sqrt{2} & -2 \\ 2 & -2 & 2\sqrt{2} \end{pmatrix}$$

and

$$K = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{\sqrt{2}}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix}.$$

Since $A_4K = -KA_4$ and $K^2 = I$, we say that A_4 is a generalized skew-centrosymmetric K-matrix.

$$\Lambda\left(A_4\right) = \left\{-2\sqrt{6}, 0, 2\sqrt{6}\right\}$$

and

$$\Lambda\left(KA_4\right) = \left\{-2\sqrt{6}i, 0, 2\sqrt{6}i\right\}$$

We end this article by establishing the real symmetric converse to Proposition 4.1.

PROPOSITION 4.2. *Suppose $A \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$ are symmetric, with $K^2 = I$. If $\Lambda(A) = i\Lambda(KA)$, then $AK = -KA$.*

Proof. We can assume that A is nonzero, as the proposition clearly holds when A is the zero matrix.

Since A is real symmetric and $\Lambda(A) = i\Lambda(KA)$, the eigenvalues of KA must be imaginary. As noted earlier, symmetry of A guarantees a full set of n independent eigenvectors. Let x be an eigenvector of the matrix KA corresponding to the eigenvalue $i\lambda$, where $\lambda \in \mathbb{R}$ has the largest magnitude of KA's eigenvalues. We shall write x as $u + iv$ and $\bar{x} = u - iv$, where u and v are real n-vectors so that $u = \frac{x+\bar{x}}{2}$ and $v = \frac{x-\bar{x}}{2i}$.

Since $KAx = i\lambda x$, we have that $KA\bar{x} = -i\lambda\bar{x}$. Therefore $KA(x + \bar{x}) = i\lambda(x - \bar{x})$ or, equivalently,

$$Au = -\lambda Kv. \tag{10}$$

Using the symmetry of A, we obtain

$$u^T AKv = (Au)^T Kv = (-\lambda Kv)^T Kv = -\lambda \|v\|^2 . \tag{11}$$

Similarly, the equation $KA(x - \bar{x}) = i\lambda(x + \bar{x})$ yields

$$Av = \lambda Ku, \tag{12}$$

and so

$$v^T AKu = (Av)^T Ku = (\lambda Ku)^T Ku = \lambda \|u\|^2 . \tag{13}$$

If $\|u\| < \|v\|$, then (11) implies that $\|AKv\| > |\lambda| \cdot \|v\|$, which is impossible due to the extremality of λ. A similar argument applied to (13) demonstrates that $\|u\| > \|v\|$ is impossible. Hence, $\|u\| = \|v\|$.

Applying the Cauchy–Schwarz inequality to (11), we have that

$$\|u\| \cdot \|AKv\| \geq \left\|u^T AKv\right\| = |\lambda| \cdot \|v\|^2 .$$

We may freely take $\|u\| = \|v\| = 1$ and therefore write $\|AKv\| \geq |\lambda|$. Again, because of the extremality of λ, this must in fact be an equality. Therefore, we have shown that

$$\|u\| \cdot \|AKv\| = \left|u^T (AKv)\right| .$$

Since Cauchy–Schwarz implies equality only if the vectors in question have the same direction up to sign, we have

$$AKv = \pm\lambda u. \tag{14}$$

The same argument applied to (13) shows that

$$AKu = \pm\lambda v. \tag{15}$$

Multiplication of A against (14) and (15) and then using (10) and (12) gives $A^2Kv = \mp\lambda^2 Kv$ and $A^2Ku = \pm\lambda^2 Ku$. As A^2 has only nonnegative eigenvalues, we can dispense with the sign ambiguities in (14) and (15):

$$AKv = -\lambda u, \tag{16}$$

$$AKu = \lambda v. \tag{17}$$

Utilizing the relations (10), (12), (16), and (17), we obtain

$$A(Ku + v) = \lambda(Ku + v),$$

$$A(Ku - v) = -\lambda(Ku - v),$$

$$A(Kv + u) = -\lambda(Kv + u),$$

$$A(Kv - u) = \lambda(Kv - u).$$

So, starting with two eigenvectors of KA ($x = u + iv$ and $\bar{x} = u - iv$) corresponding to $\pm i\lambda$, we have obtained four (not necessarily independent) eigenvectors of A corresponding to $\pm\lambda$. More manipulation with (16) and (17) will show that x and $\bar{x}$ also generate two additional eigenvectors of KA corresponding to $\pm i\lambda$:

$$KA(Kv + iKu) = i\lambda(Kv + iKu),$$

$$KA(Ku + iKv) = -i\lambda(Ku + iKv).$$

Note that the real span of the eigenvectors of A obtained from x and $\bar{x}$ is the same as the real span of the eigenvectors of KA also obtained from x and $\bar{x}$, namely, the vector space

$$T_0 \equiv span_{\mathbb{R}} \{u, v, Ku, Kv\}.$$

Let $y_1 \in span_{\mathbb{R}} \{u + v, u - v\}$ and $y_2 \in span_{\mathbb{R}} \{Ku + Kv, Ku - Kv\}$. Then using (10), (12), (16), and (17), it is easy to see that $(KA+AK)y_1 = \vec{0}$ and $(KA+AK)y_2 = \vec{0}$. In other words, $KA + AK$ maps the space T_0 to zero.

T_0 is clearly K invariant. Using the same method as in the proof of Proposition 3.5, we can show that $T_0^{\perp}$ is also K invariant. Therefore, we can apply the same argument as above to the space $T_0^{\perp}$ and continue doing so (as needed) to show that $KA + AK$ maps each of the (say m total) eigenspaces corresponding to A's nonzero eigenvalues to zero. The last of these repeated arguments shows that the eigenspace $T_m = T_{m-1}^{\perp}$ corresponding to A's 0 eigenvalues (if any) is K invariant, so KA and AK will both be zero when restricted to T_m. Therefore $KA + AK$ maps each of the invariant subspaces T_j associated with A's eigenvalues to zero. Since $\mathbb{R}^n = \oplus_{j=0}^{m} T_j$, we conclude that $KA = -AK$. □

Together, Proposition 4.1 and Proposition 4.2 yield the following characterization of real symmetric generalized skew-centrosymmetric matrices.

THEOREM 4.3. *Suppose $A \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$ are symmetric, with $K^2 = I$. Then $AK = -KA$ if and only if $\Lambda(A) = i\Lambda(KA)$.*

COROLLARY 4.4. *Let $J \in \mathbb{R}^{n\times n}$ be the exchange matrix. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is skew-centrosymmetric if and only if $\Lambda(A) = i\Lambda(JA)$.*

Proof. Let $K = J$ in the statement of Theorem 4.3. □

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