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A Spectral-Element Method for Transmission Eigenvalue Problems

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Abstract We develop an efficient spectral-element method for computing the transmission eigenvalues in two-dimensional radially stratified media. Our method is based on a dimension reduction approach which reduces the problem to a sequence of one-dimensional eigenvalue problems that can be efficiently solved by a spectral-element method. We provide an error analysis which shows that the convergence rate of the eigenvalues is twice that of the eigenfunctions in energy norm. We present ample numerical results to show that the method convergences exponentially fast for piecewise stratified media, and is very effective, particularly for computing the few smallest eigenvalues.

Keywords Spectral method · Transmission eigenvalue · Helmholtz equation

Mathematics Subject Classification (1991) 78M22 · 78A46 · 65N35 · 35J05 · 41A58

1 Introduction

Since the transmission eigenvalues can be used to estimate the material properties of the scattering field, the transmission eigenvalue problems play a very important role in inverse scattering problems and have received much attention recently [2–5,9].

We consider in this paper the interior transmission eigenvalue problem for the scattering of acoustic waves by a bounded inhomogeneous medium $D \subset R^2$:

Find $k \in C,\ w, v \in L^2(D),\ w - v \in H^2(D)$ such that

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$$\Delta w + k^2 n(x) w = 0, \quad in\ D, \tag{1.1}$$
$$\Delta v + k^2 v = 0, \quad in\ D, \tag{1.2}$$
$$w - v = 0, \quad on\ \partial D, \tag{1.3}$$
$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0, \quad on\ \partial D, \tag{1.4}$$

where ν is the unit outward normal to the boundary ∂D, and the index of refraction $n(x)$ is positive. A nonzero value of k is called a transmission eigenvalue if for such k there exists a nontrivial solution to (1.1)–(1.4).

The existence and computation of transmission eigenvalues have been studied recently by many researchers [6–8,13,14,18,19]. In particular, several finite-element methods are presented in [8]. However, all of them are all based on low-order finite element methods, so they become very difficult and expensive if high accuracy is needed. The aim of this paper is to develop a high-order spectral method, for solving the transmission eigenvalue problem, which is stable, efficient and spectrally accurate.

In this paper, we shall restrict ourselves to the case where D is a circular disk and $n(x)$ is radially stratified. When combined with a suitable boundary perturbation method (cf. [1,17]) in which an essential step is to solve the problem in a regular domain, we will be able to deal with more general domains and media. Thus, as a first and important step towards developing robust and accurate algorithm for general transmission eigenvalue problems, we shall develop a fast and accurate algorithm for regular domains which allow us to perform dimension reduction through separation of variables. More precisely, it will enable us to reduce the two-dimensional problem to a sequence of one-dimensional problems that can be effectively solved by a spectral-element method. Note that the approach we present in this paper can be directly extended to the three-dimensional spherical case, thus providing an efficient algorithm for computing transmission eigenvalues for some three-dimensional problems as well.

We now briefly describe the contents in the remainder of the paper. In §2, we first reduce the problem to a sequence of one-dimensional problems, and prove existence of transmission eigenvalue(s) for these one-dimensional problems under a reasonable condition on $n(r)$. Then we formulate a weak formulation for these one-dimensional problems and construct their spectral-element approximations, and derive an error estimate for the transmission eigenvalues in terms of the errors of the corresponding eigenfunctions in energy norm. we believe, to the best of our knowledge, that this is the first result of its kind for transmission eigenvalue problems. In §3, we describe in detail an efficient implementation of the spectral-element method. We present several numerical results in §4 to demonstrate the accuracy and efficiency of our algorithm. We conclude with a few remarks on how to extend the algorithm to more general problems.

2 Weak Formulation and Error Estimation of Transmission Eigenvalues

We shall restrict our attention to the case where D is a disk of radius R. In this case, we can employ a classical technique, separation of variable, to reduce the problem to a sequence of one-dimensional problems. Then, we shall formulate a weak formulation for these one-dimensional eigenvalue problems and construct a spectral-element method. We shall also provide an error estimate for the transmission eigenvalue in terms of the eigenfunction errors in energy norm.

2.1 Dimension Reduction

Applying the polar transformation $x = r\cos\theta,\ \ y = r\sin\theta$ to (1.1)–(1.4), and denoting

$$\widetilde{w}(r,\theta) = w(r\cos\theta, r\sin\theta),\ \ \widetilde{v}(r,\theta) = v(r\cos\theta, r\sin\theta), \widetilde{n}(r,\theta) = n(r\cos\theta, r\sin\theta),$$

We obtain that for all $\theta \in [0, 2\pi)$,

$$\widetilde{w}_{rr} + \frac{1}{r}\widetilde{w}_r + \frac{1}{r^2}\widetilde{w}_{\theta\theta} + k^2\widetilde{n}\widetilde{w} = 0, \quad r \in (0, R), \tag{2.1}$$

$$\widetilde{v}_{rr} + \frac{1}{r}\widetilde{v}_r + \frac{1}{r^2}\widetilde{v}_{\theta\theta} + k^2\widetilde{v} = 0, \quad r \in (0, R), \tag{2.2}$$

$$\widetilde{w} = \widetilde{v}, \quad r = R, \tag{2.3}$$

$$\widetilde{w}_r = \widetilde{v}_r, \quad r = R. \tag{2.4}$$

We shall assume that the index of reflection n is stratified along the radial direction, namely, $\widetilde{n}(r,\theta) = n_0(r)$. Since $\widetilde{w}$ and $\widetilde{v}$ are 2π-periodic in θ, we can write

$$\widetilde{w}(r,\theta) = \sum_{|m|=0}^{\infty} w_m(r)e^{im\theta},\ \ \widetilde{v}(r,\theta) = \sum_{|m|=0}^{\infty} v_m(r)e^{im\theta}. \tag{2.5}$$

Substituting these expansions in (2.1)–(2.4), we obtain a sequence of one-dimensional problems for each Fourier mode m.

- **Case** $m \neq 0$**:**

$$\frac{d^2w_m}{dr^2} + \frac{1}{r}\frac{dw_m}{dr} - \frac{m^2}{r^2}w_m + k^2n_0(r)w_m = 0, \quad r \in (0, R), \tag{2.6}$$

$$\frac{d^2v_m}{dr^2} + \frac{1}{r}\frac{dv_m}{dr} - \frac{m^2}{r^2}v_m + k^2v_m = 0, \quad r \in (0, R), \tag{2.7}$$

$$w_m(0) = v_m(0) = 0; \quad w_m(R) = v_m(R),\ \frac{dw_m}{dr}(R) = \frac{dv_m}{dr}(R). \tag{2.8}$$

- **Case** $m = 0$**:**

$$\frac{d^2w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr} + k^2n_0(r)w_0 = 0, \quad r \in (0, R), \tag{2.9}$$

$$\frac{d^2v_0}{dr^2} + \frac{1}{r}\frac{dv_0}{dr} + k^2v_0 = 0, \quad r \in (0, R), \tag{2.10}$$

$$w_0(R) = v_0(R),\ \frac{dw_0}{dr}(R) = \frac{dv_0}{dr}(R). \tag{2.11}$$

It is clear that the above problem is not self-adjoint, so the existence of transmission eigenvalues can not be established using a classical theory. Recently, Kirsch [14] proved some existence results for the transmission eigenvalue problem (1.1)–(1.4) (see also [6] for some related results). While the result in [14] can not be directly applied to the one dimensional problems (2.6)–(2.8) and (2.9)–(2.11), we can adapt the method in [14,18] to prove an existence result.

Let $n_0(r) = 1 + q(r)$ with $q(r) \geq q_0 > 0$, and $L_m := \partial_r(r\partial_r) - \frac{m^2}{r}$. We denote by $\rho_0 > 0$ the smallest eigenvalue of the problem: Find $u \in H^1_*(0, R)$, $\rho > 0$ such that

$$< -L_m u, v > := \int_0^R \left(r\partial_r u \partial_r v + \frac{m^2}{r}uv \right) dr = \rho \int_0^R ruvdr, \forall v \in H^1_*(0, R), \tag{2.12}$$

where

$$H^1_*(0,R):=\{u : u(0)=0 \text{ if } m\neq 0,\ u(R)=0;\ \int_0^R \left[r(\partial_r u)^2+\frac{m^2}{r}u^2\right]dr<\infty\},$$

and denote by $\{\mu_l\}, \cdots \geq \mu_j \geq \mu_{j-1} \geq \cdots \geq \mu_1 > 0$, the eigenvalues (counted according to their multiplicity) of the problem: Find $u \in H^2_*(0,R)$, $\mu > 0$ such that

$$\int_0^R \frac{1}{r} L_m u L_m v dr = \mu \int_0^R ruvdr, \quad \forall v \in H^2_*(0,R), \tag{2.13}$$

where

$$H^2_*(0,R):=\{u : u(0)=0 \text{ if } m\neq 0;\ u(R)=u'(R)=0,\ \int_0^R \frac{1}{r}(L_m u)^2 dr<\infty\}.$$

Then, we have the following result:

Theorem 2.1 *Under the condition*

$$q_0 > 2\left[\left(\frac{\mu_j}{\rho_0^2}-1\right)+\frac{\sqrt{\mu_j}}{\rho_0}\sqrt{\frac{\mu_j}{\rho_0^2}-1}\right], \tag{2.14}$$

there exists at least j transmission eigenvalues (counted according to their multiplicity) for (2.6)–(2.8)*, and for* (2.9)–(2.11) *if $m=0$.*

Proof The proof follows essentially the same arguments in [14] (Theorem 2.2 and Corollary 2.4).

To fix the idea, we consider only (2.6)–(2.8). Let us denote $u_m = v_m - \omega_m$. Multiply (2.6) and (2.7) by r and subtract the difference, we find

$$L_m u_m + k^2 n_0 r u_m = k^2 r(n_0-1)v_m.$$

Dividing $r(n_0-1)$ on both side, and taking the inner product with $L_m\psi + k^2 r\psi$, we obtain a weak formulation for (2.6)–(2.8): Find $u_m \in H^2_*(0,R)$ $k \in \mathbb{C}$ such that

$$a_k(u_m,\psi)=0, \forall \psi \in H^2_*(0,R), \tag{2.15}$$

where the bilinear form

$$a_k(\phi,\psi):=\int_0^R (L_m\phi + k^2 n_0 r\phi)(L_m\psi + k^2 r\psi)\frac{1}{r(n_0-1)}dr. \tag{2.16}$$

Then, we can see that k is a transmission eigenvalue of (2.6)–(2.8), where (2.7) is understood in the following ultra weak form

$$\int_0^R v_m[L_m\psi + k^2 r\psi]dr = 0, \forall \psi \in H^2_*(0,R), \tag{2.17}$$

if and only if, there exists a non-trivial $u_m \in H^2_*(0,R)$ with $a_k(u_m,\psi)=0,\ \forall \psi \in H^2_*(0,R)$.

Indeed, if k is a transmission eigenvalue with corresponding eigenpair (ω_m, v_m), then $u_m = v_m - \omega_m \in H_*^2(0, R)$ satisfies (2.15). On the other hand, if u_m is a solution of (2.15), then

$$v_m = \frac{1}{k^2 r(n_0 - 1)}(L_m u_m + k^2 n_0 r u_m), \quad a.e. \in (0, R),$$

and satisfies (2.17). Hence, k is a transmission eigenvalue of (2.6)–(2.8) with (2.7) in the sense of (2.17).

In order to use the argument in [14,18] to prove the existence of at least one transmission eigenvalue, an essential step is to construct $\hat{v} \in H_*^2(0, R)$ and $\hat{k} > 0$ such that $a_{\hat{k}}(\hat{v}, \hat{v}) < 0$.

From (2.16) we have

$$\begin{aligned} a_k(u_m, u_m) &\leq \frac{1}{q_0}\int_0^R (L_m u_m + k^2 r u_m)^2 \frac{1}{r} dr + k^2 \int_0^R u_m L_m u_m dr + k^4 \int_0^R r u_m^2 dr \\ &= \frac{1}{q_0}\int_0^R \left[\frac{1}{r}(L_m u_m)^2 + k^2(2+q_0) u_m L_m u_m\right] dr + \frac{q_0+1}{q_0} k^4 \int_0^R r u_m^2 dr \\ &= \frac{1}{q_0}\int_0^R \left[\frac{1}{r}(L_m u_m)^2 - k^2(2+q_0)\left(r(\partial_r u_m)^2 + \frac{m^2}{r} u_m^2\right)\right] dr \\ &\quad + \frac{q_0+1}{q_0} k^4 \int_0^R r u_m^2 dr \end{aligned}$$

Let $\hat{v}$ be an eigenfunction corresponding to the smallest eigenvalue μ_1 of (2.13), then we have $\int_0^R \frac{1}{r}(L_m \hat{v})^2 dr = \mu_1 \int_0^R r\hat{v}^2 dr$ and thus

$$a_k(\hat{v}, \hat{v}) \leq \frac{\mu_1 + k^4(1+q_0)}{q_0}\int_0^R r\hat{v}^2 dr - \frac{k^2(2+q_0)}{q_0}\int_0^R (r(\partial_r \hat{v})^2 + \frac{m^2}{r}\hat{v}^2) dr.$$

Since ρ_0 is the smallest eigenvalue of (2.12), we have

$$\int_0^R r u^2 dr \leq \frac{1}{\rho_0}\int_0^R \left(r(\partial_r u)^2 + \frac{m^2}{r} u^2\right) dr, \quad \forall u \in H_*^1(0, R).$$

Hence, $\hat{v} \in H_*^2(0, R)(\subset H_*^1(0, R))$ satisfies

$$a_k(\hat{v}, \hat{v}) \leq \frac{1}{\rho_0 q_0}(\mu_1 + k^4(1+q_0) - k^2\rho_0(2+q_0))\int_0^R \left(r(\partial_r \hat{v})^2 + \frac{m^2}{r}\hat{v}^2\right) dr.$$

We can now show that, under the condition (2.14) with $j = 1$, the term on the right-hand side is negative. Indeed, we write

$$\mu_1 + k^4(1+q_0) - k^2\rho_0(2+q_0) = (k^2\sqrt{1+q_0} - \frac{(1+q_0/2)\rho_0}{\sqrt{1+q_0}})^2 + \mu_1 - \frac{(1+q_0/2)^2\rho_0^2}{1+q_0},$$

and choose $k = \hat{k}$ such that the first square term vanishes. Then the expression is negative if

$$\mu_1 < \frac{(1+q_0/2)^2\rho_0^2}{1+q_0}.$$

The above condition is equivalent to (note that one can easily show $\mu_1 \geq \rho_0^2$ by a standard procedure):

$$q_0 > 2\left[\left(\frac{\mu_1}{\rho_0^2} - 1\right) + \frac{\sqrt{\mu_1}}{\rho_0}\sqrt{\frac{\mu_1}{\rho_0^2} - 1}\right],$$

which is the condition (2.14) with $j = 1$.

The rest of the proof for the existence of at least one transmission eigenvalue is exactly the same as in Theorem 2.2 of [14]. The existence of at least j transmission eigenvalues under the condition (2.14) follows from the proof of Corollary 2.4 in [14]. We refer to [14] for more details. □

Next, we show a simple result linking the transmission eigenvalues of the original problem (1.1)–(1.4) to that of the one-dimensional problems (2.6)–(2.8) or (2.9)–(2.11).

Proposition 2.1 *Let $D = \{(x, y) : x^2 + y^2 < R\}$. Then,*

(1) any transmission eigenvalue k, of the one-dimensional problems (2.6)–(2.8) *or* (2.9)–(2.11)*, is a transmission eigenvalue of the problem* (1.1)–(1.4)*;*
(2) for any transmission eigenvalue k of the the problem (1.1)–(1.4)*, there exists at least one m such that k is a transmission eigenvalue of* (2.6)–(2.8) *or* (2.9)–(2.11)*.*

Proof It is clear that if (w_m, v_m, k), with $\|w_m\| + \|v_m\| \neq 0$ and $k \neq 0$, solves (2.6)–(2.8) or (2.9)–(2.11), then it also solves (1.1)–(1.4). Hence, k is a transmission eigenvalue of the original problem (1.1)–(1.4).

Conversely, if (w, v, k), with $\|w\| + \|v\| \neq 0$ and $k \neq 0$, solves the original problem (1.1)–(1.4), then, there exists at least one m such that (w_m, v_m, k), with $\|w_m\| + \|v_m\| \neq 0$ where (w_m, v_m) are the m-th Fourier expansion coefficient of (w, v), solves (2.6)–(2.8) or (2.9)–(2.11). Hence, all transmission eigenvalues of the original problem (1.1)–(1.4) can be obtained by solving (2.6)–(2.8) for all $m \neq 0$ and (2.9)–(2.11). □

2.2 The Weak Formulation

We describe below a weak formulation which is convenient for numerical computation of the transmission eigenvalues. To fix the idea, we shall only describe our approach for the case $m \neq 0$ in detail, since the case with $m = 0$ can be treated similarly. We first transform the domain $r \in (0, R)$ to $t \in I:=(-1, 1)$ in which a spectral method is usually applied to.

Let $r = \frac{R(t+1)}{2}$, $\widetilde{w}_m(t) = w_m(\frac{R(t+1)}{2})$, $\widetilde{v}_m(t) = v_m(\frac{R(t+1)}{2})$, $\widetilde{n}_0(t) = n_0(\frac{R(t+1)}{2})$. Then, (2.6)–(2.8) become

$$\widetilde{w}_m'' + \frac{1}{t+1}\widetilde{w}_m' - \frac{m^2}{(t+1)^2}\widetilde{w}_m + \frac{R^2}{4}k^2\widetilde{n}_0(t)\widetilde{w}_m = 0, \qquad t \in (-1, 1), \tag{2.18}$$

$$\widetilde{v}_m'' + \frac{1}{t+1}\widetilde{v}_m' - \frac{m^2}{(t+1)^2}\widetilde{v}_m + \frac{R^2}{4}k^2\widetilde{v}_m = 0, \qquad t \in (-1, 1), \tag{2.19}$$

$$\widetilde{w}_m(-1) = \widetilde{v}_m(-1) = 0; \quad \widetilde{w}_m(1) = \widetilde{v}_m(1), \ \widetilde{w}_m'(1) = \widetilde{v}_m'(1). \tag{2.20}$$

We now formulate a suitable weak formulation for (2.18)–(2.20) using an approach similar to that in [8,14].

Let $Y = \{v \in H^1(I) : v(-1) = 0\}$. Writing $\widetilde{w}_m = w_{m0} + h, \widetilde{v}_m = v_{m0} + h$ with $w_{m0},\ v_{m0} \in X{:=}H_0^1(I)$ and $h \in X_b$, where X_b is the complement of $H_0^1(I)$ in Y, namely $Y = X \oplus X_b$. By definition, we have $\widetilde{w}_m(1) = \widetilde{v}_m(1)$.

Taking the inner product of (2.18) and (2.19) with $\tilde{w}_{m0},\ \tilde{v}_{m0} \in H_0^1(I)$ respectively, we obtain

$$
\begin{aligned}
((t+1)(w_{m0}+h)', \tilde{w}'_{m0}) + m^2\left(\frac{1}{t+1}(w_{m0}+h), \tilde{w}_{m0}\right)&\\
= \frac{R^2}{4}k^2(\widetilde{n}_0(t)(t+1)(w_{m0}+h), \tilde{w}_{m0}),&\\
((t+1)(v_{m0}+h)', \tilde{v}'_{m0}) + m^2\left(\frac{1}{t+1}(v_{m0}+h), \tilde{v}_{m0}\right)&\\
= \frac{R^2}{4}k^2((t+1)(v_{m0}+h), \tilde{v}_{m0}).&
\end{aligned}
$$

Next, taking the inner product of (2.18) and (2.19) with $\tilde{h} \in X_b$ respectively, we find

$$
\begin{aligned}
((t+1)(w_{m0}+h)', \tilde{h}') + m^2\left(\frac{1}{t+1}(w_{m0}+h), \tilde{h}\right) - 2\tilde{w}'_m(1)\tilde{h}(1)&\\
= \frac{R^2}{4}k^2(\widetilde{n}_0(t)(t+1)(w_{m0}+h), \tilde{h}),&\\
((t+1)(v_{m0}+h)', \tilde{h}') + m^2\left(\frac{1}{t+1}(v_{m0}+h), \tilde{h}\right) - 2\tilde{v}'_m(1)\tilde{h}(1)&\\
= \frac{R^2}{4}k^2((t+1)(v_{m0}+h), \tilde{h}).&
\end{aligned}
$$

We derive from the last two equations and the condition $\widetilde{w}'_m(1) = \widetilde{v}'_m(1)$ that

$$
\begin{aligned}
&((t+1)(w_{m0}+h)', \tilde{h}') + m^2\left(\frac{1}{t+1}(w_{m0}+h), \tilde{h}\right)\\
&-\left(((t+1)(v_{m0}+h)', \tilde{h}') + m^2\left(\frac{1}{t+1}(v_{m0}+h), \tilde{h}\right)\right)\\
&= \frac{R^2}{4}k^2(\widetilde{n}_0(t)(t+1)(w_{m0}+h) - (t+1)(v_{m0}+h), \tilde{h}).
\end{aligned}
$$

Hence, a weak formulation of (2.18)–(2.20) is: find $\widetilde{w}_m = w_{m0} + h, \widetilde{v}_m = v_{m0} + h$ with $w_{m0}, v_{m0} \in X$ and $h \in X_b, k \in C$, such that $\forall \tilde{w}_{m0},\ \tilde{v}_{m0} \in X$ and $\tilde{h} \in X_b$, we have

$$
\begin{aligned}
&((t+1)(w_{m0}+h)', \tilde{w}'_{m0}) + m^2\left(\frac{1}{t+1}(w_{m0}+h), \tilde{w}_{m0}\right)\\
&= \frac{R^2}{4}k^2(\widetilde{n}_0(t)(t+1)(w_{m0}+h), \tilde{w}_{m0}), \qquad (2.21)\\
&((t+1)(v_{m0}+h)', \tilde{v}'_{m0}) + m^2\left(\frac{1}{t+1}(v_{m0}+h), \tilde{v}_{m0}\right)\\
&= \frac{R^2}{4}k^2((t+1)(v_{m0}+h), \tilde{v}_{m0}), \qquad (2.22)
\end{aligned}
$$

$$
\begin{aligned}
&((t+1)(w_{m0}+h)', \tilde{h}') + m^2 \left(\frac{1}{t+1}(w_{m0}+h), \tilde{h} \right) \\
&\quad - \left(\left((t+1)(v_{m0}+h)', \tilde{h}' \right) + m^2 \left(\frac{1}{t+1}(v_{m0}+h), \tilde{h} \right) \right) \\
&\quad = \frac{R^2}{4} k^2 (\widetilde{n}_0(t)(t+1)(w_{m0}+h) - (t+1)(v_{m0}+h), \tilde{h}).
\end{aligned}
\tag{2.23}
$$

2.3 Spectral-Element Method and Eigenvalue Error Analysis

Let $I_i = (t_{i-1}, t_i)$, $1 \le i \le M$ with $-1 = t_0 < t_1 < \cdots < t_M = 1$. We now construct a spectral-element method for (2.21)–(2.23).

Let P_N be the set of polynomials of degree less than or equal to N, and define the spectral-element approximation to X:

$$
X_N = \{ v \in C(I) : v(\pm 1) = 0,\ v|_{I_i} \in P_N, 1 \le i \le M \}.
$$

To deal with X_b, we define

$$
h_M(t) = \begin{cases} 0, & t \in [-1, t_{M-1}], \\ \frac{t - t_{M-1}}{1 - t_{M-1}}, & t \in (t_{M-1}, 1]. \end{cases} \tag{2.24}
$$

Then, the spectral-element method for (2.21)–(2.23) is: Find $w_N^m = w_N^{m0} + \alpha h_M(t)$, $v_N^m = v_N^{m0} + \alpha h_M(t)$ with $w_N^{m0}, v_N^{m0} \in X_N$, $\alpha \in \mathbb{R}$, and $k_N \in \mathbb{C}$, such that for $\forall \tilde{w}_N^{m0}, \tilde{v}_N^{m0} \in X_N$, we have

$$
\begin{aligned}
&((t+1)(w_N^{m0}+\alpha h_M)', (\tilde{w}_N^{m0})') + m^2 \left(\frac{1}{t+1}(w_N^{m0}+\alpha h_M), \tilde{w}_N^{m0} \right) \\
&\quad = \frac{R^2}{4} k_N^2 (\widetilde{n}_0(t)(t+1)(w_N^{m0}+\alpha h_M), \tilde{w}_N^{m0}),
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
&((t+1)(v_N^{m0}+\alpha h_M)', (\tilde{v}_N^{m0})') + m^2 \left(\frac{1}{t+1}(v_N^{m0}+\alpha h_M), \tilde{v}_N^{m0} \right) \\
&\quad = \frac{R^2}{4} k_N^2 ((t+1)(v_N^{m0}+\alpha h_M), \tilde{v}_N^{m0}),
\end{aligned}
\tag{2.26}
$$

and

$$
\begin{aligned}
&((t+1)(w_N^{m0}+\alpha h_M)', h_M') + m^2 \left(\frac{1}{t+1}(w_N^{m0}+\alpha h_M), h_M \right) \\
&\quad - \left(\left((t+1)(v_N^{m0}+\alpha h_M)', h_M' \right) + m^2 \left(\frac{1}{t+1}(v_N^{m0}+\alpha h_M), h_M \right) \right) \\
&\quad = \frac{R^2}{4} k_N^2 (\widetilde{n}_0(t)(t+1)(w_N^{m0}+\alpha h_M) - (t+1)(v_N^{m0}+\alpha h_M), h_M).
\end{aligned}
\tag{2.27}
$$

We shall establish below an error estimate for the transmission eigenvalues in terms of the errors for the corresponding eigenfunctions. In particular, we show that the convergence rate of the eigenvalue is twice of that of the eigenfunctions in the energy norm, as in the case of usual eigenvalue problems.

For any positive weight function ω, we denote the weighted L^2-norm and weighted H^1 semi-norm by

$$\|u\|_\omega^2 = \int_{-1}^{1} u^2\,\omega dt,\ \ |u|_{1,\omega}^2 = \int_{-1}^{1} (u_t)^2\,\omega dt. \tag{2.28}$$

Denoting $w_1 = t+1$, $w_2 = \frac{1}{t+1}$ and $w_3 = \tilde{n}_0(t)(t+1)$, we define two energy norms associated with (2.21)–(2.23) by

$$\begin{aligned}\|w\|_{E_1,m}^2 &= |w|_{1,\omega_1}^2 + m^2\|w\|_{\omega_2}^2 + \frac{R^2}{4}k^2\|w\|_{\omega_3}^2,\\ \|v\|_{E_2,m}^2 &= |v|_{1,\omega_1}^2 + m^2\|v\|_{\omega_2}^2 + \frac{R^2}{4}k^2\|v\|_{\omega_1}^2.\end{aligned} \tag{2.29}$$

In order to describe errors more precisely, we also define two related pseudo norms by

$$\begin{aligned}\|w\|_{\triangle,m}^2 &= \left| |w|_{1,\omega_1}^2 + m^2\|w\|_{\omega_2}^2 - \frac{R^2}{4}k^2\|w\|_{\omega_3}^2 \right|,\\ \|v\|_{\square,m}^2 &= \left| |v|_{1,\omega_1}^2 + m^2\|v\|_{\omega_2}^2 - \frac{R^2}{4}k^2\|v\|_{\omega_1}^2 \right|.\end{aligned} \tag{2.30}$$

Theorem 2.2 *Let $(k, \widetilde{w}_m, \widetilde{v}_m)$, (k_N, w_N^m, v_N^m) be the solutions of (2.21)–(2.23) and (2.25)–(2.27) respectively. Then the following inequality holds:*

$$\begin{aligned}|k_N - k| &\le C_1(\|\widetilde{w}_m - w_N^m\|_{\triangle,m}^2 + \|\widetilde{v}_m - v_N^m\|_{\square,m}^2)\\ &\le C_1(\|\widetilde{w}_m - w_N^m\|_{E_1,m}^2 + \|\widetilde{v}_m - v_N^m\|_{E_2,m}^2),\end{aligned}$$

where $C_1 = 1/|\frac{R^2}{4}(k_N+k)(||w_N^m||_{w_3}^2 - ||v_N^m||_{w_1}^2)|$.

Proof Taking $\tilde{w}_{m0} = w_{m0}$, $\tilde{v}_{m0} = v_{m0}$, $\tilde{h} = h$ in (2.21)–(2.23) respectively, we obtain

$$\begin{aligned}&((t+1)(w_{m0}+h)', w_{m0}') + m^2\left(\frac{1}{t+1}(w_{m0}+h\right), w_{m0}\right)\\ &= \frac{R^2}{4}k^2(\widetilde{n}_0(t)(t+1)(w_{m0}+h), w_{m0}),\end{aligned} \tag{2.31}$$

$$\begin{aligned}&((t+1)(v_{m0}+h)', v_{m0}') + m^2\left(\frac{1}{t+1}(v_{m0}+h), v_{m0}\right)\\ &= \frac{R^2}{4}k^2((t+1)(v_{m0}+h), v_{m0}),\end{aligned} \tag{2.32}$$

$$\begin{aligned}&((t+1)(w_{m0}+h)', h') + m^2\left(\frac{1}{t+1}(w_{m0}+h), h\right)\\ &- \left(\left((t+1)(v_{m0}+h)', h'\right) + m^2\left(\frac{1}{t+1}(v_{m0}+h), h\right)\right)\\ &= \frac{R^2}{4}k^2(\widetilde{n}_0(t)(t+1)(w_{m0}+h) - (t+1)(v_{m0}+h), h).\end{aligned} \tag{2.33}$$

We derive by (2.31)−(2.32)+(2.33) that

$$
\begin{aligned}
&((t+1)(w_{m0}+h)', (w_{m0}+h)') + m^2\left(\frac{1}{t+1}(w_{m0}+h), (w_{m0}+h)\right)\\
&-\left(((t+1)(v_{m0}+h)', (v_{m0}+h)') + m^2\left(\frac{1}{t+1}(v_{m0}+h), (v_{m0}+h)\right)\right)\\
&=\frac{R^2}{4}k^2(\widetilde{n}_0(t)(t+1)(w_{m0}+h), (w_{m0}+h))\\
&-\frac{R^2}{4}k^2((t+1)(v_{m0}+h), (v_{m0}+h)).
\end{aligned}
\tag{2.34}
$$

Since $\widetilde{w}_m = w_{m0}+h$, $\widetilde{v}_m = v_{m0}+h$, we find

$$
\begin{aligned}
((t+1)\tilde{w}_m', \tilde{w}_m') &+ m^2\left(\tfrac{1}{t+1}\tilde{w}_m, \tilde{w}_m\right) - \left(((t+1)\tilde{v}_m', \tilde{v}_m') + m^2\left(\tfrac{1}{t+1}\tilde{v}_m, \tilde{v}_m\right)\right)\\
&= \tfrac{R^2}{4}k^2(\widetilde{n}_0(t)(t+1)\tilde{w}_m, \tilde{w}_m) - \tfrac{R^2}{4}k^2((t+1)\tilde{v}_m, \tilde{v}_m).
\end{aligned}
\tag{2.35}
$$

Similarly, we can derive, by taking $\tilde{w}_{m0} = w_N^{m0}$, $\tilde{v}_{m0} = v_N^{m0}$, $\tilde{h} = \alpha h_M$ in (2.21)–(2.23) respectively, that

$$
\begin{aligned}
((t+1)\tilde{w}_m', (w_N^m)') &+ m^2\left(\tfrac{1}{t+1}\tilde{w}_m, w_N^m\right) - \left(((t+1)\tilde{v}_m', (v_N^m)') + m^2\left(\tfrac{1}{t+1}\tilde{v}_m, v_N^m\right)\right)\\
&= \tfrac{R^2}{4}k^2(\widetilde{n}_0(t)(t+1)\tilde{w}_m, w_N^m) - \tfrac{R^2}{4}k^2((t+1)\tilde{v}_m, v_N^m).
\end{aligned}
\tag{2.36}
$$

Subtracting (2.36) from (2.35), we find

$$
\begin{aligned}
&((t+1)(\tilde{w}_m' - (w_N^m)'), \tilde{w}_m') + m^2\left(\frac{1}{t+1}(\tilde{w}_m - w_N^m), \tilde{w}_m\right)\\
&-\left(((t+1)(\tilde{v}_m' - (v_N^m)'), \tilde{v}_m') + m^2\left(\frac{1}{t+1}(\tilde{v}_m - v_N^m), \tilde{v}_m\right)\right)\\
&=\frac{R^2}{4}k^2(\widetilde{n}_0(t)(t+1)(\tilde{w}_m - w_N^m), \tilde{w}_m)\\
&-\frac{R^2}{4}k^2((t+1)(\tilde{v}_m - v_N^m), \tilde{v}_m).
\end{aligned}
\tag{2.37}
$$

Similar to the derivation of (2.35), we can derive from (2.25)–(2.27) that

$$
\begin{aligned}
&((t+1)(w_N^m)', (w_N^m)') + m^2\left(\frac{1}{t+1}w_N^m, w_N^m\right)\\
&-\left(((t+1)(v_N^m)', (v_N^m)') + m^2\left(\frac{1}{t+1}v_N^m, v_N^m\right)\right)\\
&=\frac{R^2}{4}k_N^2(\widetilde{n}_0(t)(t+1)w_N^m, w_N^m) - \frac{R^2}{4}k_N^2((t+1)v_N^m, v_N^m).
\end{aligned}
\tag{2.38}
$$

Subtracting (2.38) from (2.36), we obtain

$$
\begin{aligned}
&((t+1)(\tilde{w}_m' - (w_N^m)'), (w_N^m)') + m^2\left(\frac{1}{t+1}(\tilde{w}_m - w_N^m), w_N^m\right) \\
&\quad - \left(((t+1)(\tilde{v}_m' - (v_N^m)'), (v_N^m)') + m^2\left(\frac{1}{t+1}(\tilde{v}_m - v_N^m), v_N^m\right)\right) \\
&\quad = \frac{R^2}{4}k^2(\widetilde{n}_0(t)(t+1)(\tilde{w}_m - w_N^m), w_N^m) + \frac{R^2}{4}(k^2 - k_N^2)(\widetilde{n}_0(t)(t+1)w_N^m, w_N^m) \\
&\qquad - \frac{R^2}{4}k^2((t+1)(\tilde{v}_m - v_N^m), v_N^m) - \frac{R^2}{4}(k^2 - k_N^2)((t+1)v_N^m, v_N^m).
\end{aligned}
\tag{2.39}
$$

Then, subtracting (2.39) from (2.37), we find

$$
\begin{aligned}
&\frac{R^2}{4}(k_N^2 - k^2)((\widetilde{n}_0(t)(t+1)w_N^m, w_N^m) - ((t+1)v_N^m, v_N^m)) \\
&\quad = ((t+1)(\widetilde{w}_m' - (w_N^m)'), (\widetilde{w}_m' - (w_N^m)')) + m^2\left(\frac{1}{t+1}(\widetilde{w}_m - w_N^m), (\widetilde{w}_m - w_N^m)\right) \\
&\quad - \frac{R^2}{4}k^2(\widetilde{n}_0(t)(t+1)(\widetilde{w}_m - w_N^m), (\widetilde{w}_m - w_N^m)) - (((t+1)(\widetilde{v}_m' - v_N^{m\prime}), (\widetilde{v}_m' - v_N^{m\prime})) \\
&\quad + m^2(\frac{1}{t+1}(\widetilde{v}_m - v_N^m), (\widetilde{v}_m - v_N^m)) - \frac{R^2}{4}k^2((t+1)(\widetilde{v}_m - v_N^m), (\widetilde{v}_m - v_N^m))),
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
\frac{R^2}{4}(k_N^2 - k^2)(||w_N^m||_{w_3}^2 - ||v_N^m||_{w_1}^2) &= |\widetilde{w}_m - w_N^m|_{1,w_1}^2 \\
&\quad + m^2||\widetilde{w}_m - w_N^m||_{w_2}^2 - \frac{R^2}{4}k^2\|\widetilde{w}_m - w_N^m\|_{w_3}^2 \\
&\quad - (|\widetilde{v}_m - v_N^m|_{1,w_1}^2 + m^2||\widetilde{v}_m - v_N^m||_{w_2}^2 - \frac{R^2}{4}k^2||\widetilde{v}_m - v_N^m||_{w_1}^2).
\end{aligned}
$$

We can then derive from the above that

$$
\begin{aligned}
&|\frac{R^2}{4}(k_N + k)(||w_N^m||_{w_3}^2 - ||v_N^m||_{w_1}^2)(k_N - k)| \\
&\quad \le \|\widetilde{w}_m - w_N^m\|_{\Delta,m}^2 + \|\widetilde{v}_m - v_N^m\|_{\Box,m}^2 \\
&\quad \le \|\widetilde{w}_m - w_N^m\|_{E_1,m}^2 + \|\widetilde{v}_m - v_N^m\|_{E_2,m}^2,
\end{aligned}
$$

which implies the desired result. □

Similar results can be derived for the case $m = 0$. We briefly outline the approach and the results for (2.9)–(2.11) below.

Let $\widetilde{w}_0(t) = w_0(\frac{R(t+1)}{2})$, $\widetilde{v}_0(t) = v_0(\frac{R(t+1)}{2})$, $\widetilde{n}_0(t) = n_0(\frac{R(t+1)}{2})$. Then (2.9)–(2.11) become

$$
\widetilde{w}_0'' + \frac{1}{t+1}\widetilde{w}_0' + \frac{R^2}{4}k^2\widetilde{n}_0(t)\widetilde{w}_0 = 0, \qquad t \in (-1, 1), \tag{2.40}
$$

$$
\widetilde{v}_0'' + \frac{1}{t+1}\widetilde{v}_0' + \frac{R^2}{4}k^2\widetilde{v}_0 = 0, \qquad t \in (-1, 1), \tag{2.41}
$$

$$
\widetilde{w}_0(1) = \widetilde{v}_0(1), \quad \widetilde{w}_0'(1) = \widetilde{v}_0'(1). \tag{2.42}
$$

Let $X_0 = \{v \in H^1(I) : v(1) = 0\}$. Writing $\widetilde{w}_0 = w_{00} + h, \widetilde{v}_0 = v_{00} + h$ with $w_{00}, v_{00} \in X_0,\ h \in X_b$ where X_b is the complement of X_0 in $H^1(I)$. By definition, we have $\widetilde{w}_0(1) = \widetilde{v}_0(1)$.

A weak formulation for (2.40)–(2.42) is: find $\widetilde{w}_0 = w_{00} + h, \widetilde{v}_0 = v_{00} + h,\ k \in C$, such that $\forall \tilde{w}_{00}, \tilde{v}_{00} \in X, \tilde{h} \in X_b$, we have

$$((t+1)(w_{00}+h)', \tilde{w}'_{00}) = \frac{R^2}{4} k^2 (\widetilde{n}_0(t)(t+1)(w_{00}+h), \tilde{w}_{00}), \tag{2.43}$$

$$((t+1)(v_{00}+h)', \tilde{v}'_{00}) = \frac{R^2}{4} k^2 ((t+1)(v_{00}+h), \tilde{v}_{00}), \tag{2.44}$$

$$\begin{aligned} &((t+1)(w_{00}+h)', \tilde{h}') - ((t+1)(v_{00}+h)', \tilde{h}') \\ &\quad = \frac{R^2}{4} k^2 (\widetilde{n}_0(t)(t+1)(w_{00}+h) - (t+1)(v_{00}+h), \tilde{h}). \end{aligned} \tag{2.45}$$

Let h_M be as defined in (2.24), and

$$X_{0N} = \{v \in C(I) : v(1) = 0,\ v|_{I_i} \in P_N, 1 \le i \le M\}.$$

Then, the spectral-element method for (2.43)–(2.45) is: Find $w_N^0 = w_N^{00} + \alpha_0 h_M(t), v_N^0 = v_N^{00} + \alpha_0 h_M(t)$ with $w_N^{00}, v_N^{00} \in X_{0N}$, and $\tilde{k}_N \in C$, such that $\forall \tilde{w}_N^{00}, \tilde{v}_N^{00} \in X_{0N}$, we have

$$((t+1)(w_N^{00} + \alpha_0 h_M(t))', (\tilde{w}_N^{00})') = \frac{R^2}{4} \tilde{k}_N^2 (\widetilde{n}_0(t)(t+1)(w_N^{00} + \alpha_0 h_M(t)), \tilde{w}_N^{00}), \tag{2.46}$$

$$((t+1)(v_N^{00} + \alpha_0 h_M(t))', (\tilde{v}_N^{00})') = \frac{R^2}{4} \tilde{k}_N^2 ((t+1)(v_N^{00} + \alpha_0 h_M(t)), \tilde{v}_N^{00}), \tag{2.47}$$

$$\begin{aligned} &((t+1)(w_N^{00} + \alpha_0 h_M(t))', (h_M(t))') - ((t+1)(v_N^{00} + \alpha_0 h_M(t))', (h_M(t))') \\ &\quad = \frac{R^2}{4} \tilde{k}_N^2 (\widetilde{n}_0(t)(t+1)(w_N^{00} + \alpha_0 h_M(t)) - (t+1)(v_N^{00} + \alpha_0 h_M(t)), h_M(t)). \end{aligned} \tag{2.48}$$

Using the same procedure as in the prove of Theorem 2.1, we can also prove the following result:

Theorem 2.3 *Let* $(k, \widetilde{w}_0, \widetilde{v}_0)$, $(\widetilde{k}_N, w_N^0, v_N^0)$ *be the solutions of* (2.43)–(2.45) *and* (2.46)–(2.48)*, respectively. Then, the following inequality holds:*

$$\begin{aligned} |k_N - k| &\le C_2(\|\widetilde{w}_0 - w_N^0\|^2_{\triangle,0} + \|\widetilde{v}_0 - v_N^0\|^2_{\square,0}) \\ &\le C_2(\|\widetilde{w}_0 - w_N^0\|^2_{E_1,0} + \|\widetilde{v}_0 - v_N^0\|^2_{E_2,0}), \end{aligned}$$

where $C_2 = 1/|\frac{R^2}{4}(k_N + k)(||w_N^0||^2_{w_3} - ||v_N^0||^2_{w_1})|$.

3 Efficient Implementation of the Spectral-Element Method

We describe in this section how to solve the problems (2.25)–(2.27) and (2.43)–(2.45) efficiently. We shall only present the algorithm for (2.25)–(2.27), since (2.43)–(2.45) can be treated in a similar fashion.

We start by constructing a set of basis functions for X_N which will consist of interior basis functions and interface basis functions.

Let $\phi_i(x) = L_i(x) - L_{i+2}(x),\ i = 0, 1, \ldots, N-2$, where L_k is the Legendre polynomial of degree k. It is clear that $\{\phi_k\}_{k=0}^{N-2}$ form a basis for $P_N \cap H_0^1(I)$ [20]. For each subinterval $I_j = (t_{j-1}, t_j)$, we define a set of interior basis functions by

$$\varphi_{i+1,j}(t) := \begin{cases} \phi_i(x_j(t)), & t \in I_j, \\ 0, & \text{otherwise}, \end{cases}$$

where $x_j(t) = \frac{2}{t_j - t_{j-1}} t - \frac{t_j + t_{j-1}}{t_j - t_{j-1}},\ t \in I_j$. Then, a set of all interior basis functions in X_N is

$$X_N^0 = \cup_{j=1,\ldots,M} \operatorname{span}\{\varphi_{1,j}, \varphi_{2,j}, \ldots, \varphi_{N-1,j}\}.$$

Next, we define the following interface basis functions

$$\xi_j(t) := \begin{cases} \frac{1}{2}(x_j(t) + 1), & t \in I_j \\ -\frac{1}{2}(x_{j+1}(t) - 1), & t \in I_{j+1} \\ 0, & \text{otherwise} \end{cases}, \ j = 1, 2, \ldots, M-1.$$

Then, it is clear that

$$X_N = X_N^0 \bigoplus \operatorname{span}\{\xi_1, \xi_2, \ldots, \xi_{M-1}\}.$$

Hence, we shall look for

$$w_N^m = \left(\sum_{j=1}^{M} \sum_{i=1}^{N-1} w_{ij}^m \varphi_{ij} + \sum_{j=1}^{M-1} w_j^m \xi_j \right) + \alpha h_M,$$
$$v_N^m = \left(\sum_{j=1}^{M} \sum_{i=1}^{N-1} v_{ij}^m \varphi_{ij} + \sum_{j=1}^{M-1} v_j^m \xi_j \right) + \alpha h_M, \tag{3.1}$$

where h_M is defined in (2.24).

Let us denote

$$\bar{w} = (w_{1,1}^m, \ldots, w_{N-1,1}^m; \ldots; w_{1,M}^m \ldots, w_{N-1,M}^m),$$
$$\bar{v} = (v_{1,1}^m, \ldots, v_{N-1,1}^m; \ldots; v_{1,M}^m \ldots, v_{N-1,M}^m),$$
$$\bar{\xi} = (w_1^m, \ldots, w_{M-1}^m; v_1^m, \ldots, v_{M-1}^m, \alpha).$$

Then, the vector containing all the unknowns is

$$\bar{u} = (\bar{w}, \bar{v}, \bar{\xi})^T := (\bar{u}_0, \bar{\xi})^T. \tag{3.2}$$

Now, plugging the expressions of (3.1) in (2.25)–(2.27), and taking $\tilde{w}_N^{m0}$ and $\tilde{w}_N^{m0}$ through all the basis functions in X_N, we will arrive at the following linear eigenvalue system:

$$A\bar{u} = k_N^2 B\bar{u}, \tag{3.3}$$

where A and B are corresponding stiffness and mass matrices having the following form:

$$A = \begin{pmatrix} A_0, & C \\ D, & A_I \end{pmatrix}, \quad B = \begin{pmatrix} B_0, & E \\ F, & B_I \end{pmatrix}, \tag{3.4}$$

where A_0 and B_0 are the sub-matrices corresponding to the interior unknowns $\bar{u}_0$, and here A_I and B_I are the sub-matrices corresponding to the interface unknowns $\bar{\xi}$; $C,\ D,\ E$ and

F are the interacting matrices. Notice that the size of A_I and B_I is $2M-1$, which is much smaller than the size of A_0 and B_0 which is $2(N-1)M$.

It is clear that A_0 and B_0 consist of M decoupled diagonal blocks of size $2(N-1)$. For general stratified media $\tilde{n}_0(t)$, these blocks are full and can be evaluated by using local Gauss-Legendre quadratures. However, if $\tilde{n}_0(t)$ is piecewise constant, i.e., $\tilde{n}_0(t)=n_i,\ t\in(t_{i-1},t_i)$ for $i=1,2,\ldots,M$, then, each block of A_0 and B_0 is sparse and can be evaluated exactly using the properties of Legendre polynomials (cf. [21,22] for more details).

Since one is mostly interested in a few smallest transmission eigenvalues, it is most efficient to solve (3.3) using a shifted inverse power method (cf., for instance, [11]) which requires solving, repeatedly for different righthand side $\bar{f}$,

$$(A-k_a^2B)\bar{u}=\bar{f}, \tag{3.5}$$

where k_a is some approximate value for the transmission eigenvalue k_N.

The above system can be efficiently solved by the Schur-complement approach, i.e., the block Gaussian elimination, we refer to [15] for a detailed description on a related problem.

In summary, the approximate transmission eigenvalue problem (3.3) can be solved very efficiently, particularly when the medium is radially piecewise constant.

The system for $m=0$, (2.46)–(2.48), can be treated similarly. For the sake of brevity, we omit the detail.

4 Numerical Results

We now perform a sequence of numerical tests to study the convergence behavior and show the effectiveness of our algorithm. In particular, we shall carry out numerical results for those problems reported in [8] so we can compare the accuracy and efficiency of our algorithm with those used in [8].

The results in the Proposition 2.1 indicate that we can obtain all the transmission eigenvalues of the two-dimensional problem (1.1)–(1.4) by solving the one-dimensional problems(2.6)–(2.8) and (2.9)–(2.11). These one-dimensional problems are solving by using the Schur-complement approach for the spectra-element method described in the last section. Our numerical results below indicate that, in most situations, the few smallest transmission eigenvalues can be obtained by solving a few first eigenvalues of (2.9)–(2.11) and (2.6)–(2.8) with the first few small m.

Example 1 (homogeneous medium). We take $R=\frac{1}{2}$ and $n(x)=16$. Since $n(x)$ is a constant, we use the one-domain spectral method, i.e., we take $M=1$ in the algorithm. The smallest eigenvalues with different m and with different N (polynomial order) are listed in Table 1. We observe that the eigenvalues converges exponentially fast as N increases. It achieves at least ten-digit accuracy with $N\leq 10$.

Table 1 The smallest transmission eigenvalue for different m in a disk with R=1/2 and n=16

N	5	10	20
m=0	1.987995161	1.987995124	1.987995124
m=1	2.612930392	2.612929964	2.612929964
m=2	3.226648228	3.226647948	3.226647948
m=3	3.826464576	3.826441449	3.826441449
m=4	4.415500129	4.415390979	4.415390979

Table 2 Smallest transmission eigenvalues in a disk with R=1/2 and n=16

Argyris method ($N = 2074$)	2.0076	2.6382	2.6396	3.2580	3.2598
Continuous method ($N = 256$)	2.0301	2.6937	2.6974	3.3744	3.3777
Mixed method ($N = 398$)	1.9912	2.6218	2.6234	3.2308	3.2397

The total number of unknowns is denoted by N

Table 3 The smallest transmission eigenvalue for different m in a disk with $R = 1$ and $n_0 = 1,\ n_1 = 16$

N	5	10	20
m=0	1.720680058	1.720671074	1.720671074
m=1	1.738231364	1.738230361	1.738230361
m=2	1.887115763	1.887111801	1.887111801
m=3	2.084797251	2.084765339	2.084765339
m=4	2.313381207	2.313274053	2.313274053

Table 4 The smallest transmission eigenvalue for different m in a disk with $R = 1$ and $n_0 = 1/2,\ n_1 = 16$

N	5	10	20
m=0	1.761922977	1.761911074	1.761911074
m=1	1.750465748	1.750464681	1.750464681
m=2	1.893611269	1.893607022	1.893607022
m=3	2.088581731	2.088548843	2.088548843
m=4	2.315649615	2.315540708	2.315540708

As a comparison, we list in Table 2 the results obtained by three different finite element methods in [8]. We observe that the three eigenvalues in Table 1 with $m = 0, 1, 2$ are in fact the three smallest eigenvalues of the two-dimensional problem. However, the results reported in 2 have less than three-digit accuracy despite using a large number of degrees of freedom.

Example 2 (stratified media with two layers).

We take $R = 1$ and

$$n(r) = \begin{cases} n_0, & r \in (0, \frac{1}{2}), \\ n_1, & r \in (\frac{1}{2}, 1). \end{cases}$$

We use our spectral-element method with two-subdomains ($M = 2$). The numerical results with three different sets of n_0 and n_1 are reported in Tables 3, 4 and 5.

We observe once again that the smallest eigenvalues converge exponentially fast, and we have at least ten-digit accuracy with $N \leq 10$ in Tables 3 and 4, while slightly larger N is needed to achieve the same accuracy in Table 5.

We note that the smallest eigenvalue for each m does not necessarily increase with m, see Table 5. In particular, the smallest transmission eigenvalue for the 2-D problem is not always the smallest eigenvalue for $m = 0$, see Table 4.

As a comparison, we list in Table 6 the numerical results in [8] computed by the continuous finite element method with 1074 degrees of freedom. We observe that the eigenvalues listed in Table 6 have only two- to four-digit accuracy, and that we captured all these eigenvalues listed in Table 6 with at least ten-digit accuracy with orders of magnitude less of computational effort.

Table 5 The first three transmission eigenvalues for different m in a disk with $R = 1$ and $n_0 = 1/4,\ n_1 = 9/4$

$N = 10$	1	2	3
m=0	2.58063395	9.890639035	15.13512844
m=1	9.960807228	15.08897438	22.42008063
m=2	14.96412806	19.86837271	26.84739844
m=3	20.18161613	20.5862087	22.32401132
m=4	26.60938245	37.81151066	44.17576616
m=5	7.755958061	8.734272127	10.16429466
m=6	8.142085224	20.61635081	30.40220691
$N = 20$	1	2	3
m=0	2.58063395	9.890639021	15.13520965
m=1	9.960807336	15.08873935	22.5522994
m=2	14.9641851	19.85194193	27.60605847
m=3	22.42020202	24.31848074	35.00829894
m=4	27.44713965	28.51034165	40.02582649
m=5	7.755958061	8.734272131	10.1642946
m=6	8.142085224	20.65026939	22.24707236
$N = 30$	1	2	3
m=0	2.58063395	9.890639021	15.13520965
m=1	9.960807336	15.08873935	22.5522994
m=2	14.9641851	19.85194193	27.60605849
m=3	22.42020202	24.31848074	35.00833985
m=4	27.44713967	28.51034158	40.02665129
m=5	7.755958061	8.734272131	10.1642946
m=6	8.142085224	20.65026939	22.24707236

Table 6 Smallest transmission eigenvalues in the unit disk computed by the continuous finite element method with 1074 unknowns

Case	k_1	k_2	k_3	k_4	k_5	k_6
A	1.7135	1.7343	1.7345	1.8835	1.8837	2.0803
B	1.7462	1.7464	1.7531	1.8898	1.8900	2.0839
C	2.5895	7.5899	7.6041	8.0577	8.0799	8.6780

Case A: $n_0 = 1,\ n_1 = 16$; Case B: $n_0 = 1/2,\ n_1 = 16$; Case C: $n_0 = 1/4, n_1 = 9/4$

Example 3 (stratified media with four layers).

We fix $R = 1$ and set

$$\begin{cases} n = 1/4, & r \in \left(0, \frac{1}{4}\right), \\ n = 1/5, & r \in \left(\frac{1}{4}, \frac{1}{2}\right), \\ n = 9/8, & r \in \left(\frac{1}{2}, \frac{3}{4}\right), \\ n = 3, & r \in \left(\frac{3}{4}, 1\right). \end{cases}$$

We use our spectral-element method with four-subdomains ($M = 4$). The numerical results are reported in Table 7. We observe that our spectral-element method works equally well with four-layered media, and achieves ten-digit accuracy with $N \le 10$ for all m considered.

Table 7 The smallest transmission eigenvalue with different m in a disk with four-layered media

N	5	10	15
m=0	2.152196552	2.152196552	2.152196552
m=1	4.149238018	4.149238019	4.149238019
m=2	11.67762749	11.67918585	11.67918585
m=3	15.81815354	15.85404782	15.85404782

Table 8 The first pair of complex transmission eigenvalues with different m in a disk with $R = 1/2$ and $n = 16$

N	5	10
m=0	$4.901360899 \pm 0.5779215007 * i$	$4.900866276 \pm 0.5780910587 * i$
m=1	$7.607811362 \pm 0.6306469373 * i$	$7.597540164 \pm 0.5961654238 * i$
m=2	$10.04879682 \pm 0.6106637425 * i$	$10.13069137 \pm 0.5096507113 * i$
N	15	20
m=0	$4.900866276 \pm 0.5780910587 * i$	$4.900866276 \pm 0.5780910587 * i$
m=1	$7.597540158 \pm 0.5961654374 * i$	$7.597540158 \pm 0.5961654374 * i$
m=2	$10.13068894 \pm 0.5096395706 * i$	$10.13068894 \pm 0.5096395706 * i$

Table 9 The first pair of complex transmission eigenvalues in a disk with $R = 1/2$ and $n = 16$ computed by three different finite element methods with N degrees of freedom

Argyris method ($N = 2074$)	$4.9495 \pm 0.5795 * i$
Continuous method ($N = 4066$)	$4.9130 \pm 0.5783 * i$
Mixed method ($N = 1557$)	$4.9096 \pm 0.5595 * i$

Example 4 (complex transmission eigenvalues).

Since the problem (1.1)–(1.4) is non self-adjoint, it may have complex transmission eigenvalues.

We take $R = \frac{1}{2}$ and $n = 16$ as in Example 1, and use the one-domain spectral method. We list in Table 8 the first pair (i.e. with the smallest norm) complex transmission eigenvalues with different m. We observe that the convergence of the complex eigenvalues is similar to that of the real eigenvalues. In particular, we achieve at least ten-digit accuracy with $N \leq 10$ for $m = 0$ and with $N \leq 15$ for $m = 1, 2$.

The corresponding numerical results in [8] are listed in Table 9.

As a comparison, the corresponding numerical results in [8] by three different finite element methods are listed in Table 9.

5 Concluding Remarks

We presented in this paper a spectral-element method for computing the transmission eigenvalue problems. The method is based on a dimension reduction approach which reduces the problem to a sequence of one-dimensional eigenvalue problems that can be efficiently solved by a spectral-element method. We derived an error analysis which shows that the convergence rate of the eigenvalues is twice that of the eigenfunctions in energy norm, and presented several numerical examples to show that the method convergences exponentially fast for piecewise stratified media.

While we have restricted our attention in this paper to the cases where the domain is a disk with radially stratified media, The approach presented in this paper can be extended to more general transmission eigenvalue problems. We list below some immediate extensions:

- The method presented above can be applied directly to the case where the scattering medium is a sphere. In this case, one should first transform to the spherical coordinates, and then replace the Fourier expansion in (2.5) by the spherical harmonic expansion. Then, the problem also reduces to a sequence of one-dimensional problems. And for each mode of the spherical harmonic expansion, the corresponding one-dimensional system is very similar to the Fourier case, so similar error analysis can be established and the corresponding linear system can be solved in a similar manner.
- The method can also be easily applied to the transmission eigenvalue problem related to periodic grating, i.e., the domain D in (1.1)–(1.4) is replaced by $D = \{(x, y) : y < f(x)\}$ with $f(x)$ being a quasi-periodic function. In this case, one can apply the Fourier-expansion directly to reduce the 2-D problem to a sequence of one-dimensional problems. The spectral-element scheme for a period grating problem was considered in [12], and it can be used as an essential step for solving the corresponding transmission eigenvalue problem.
- In [6], the authors proved an interesting result, Corollary 2.6, showing that transmission eigenvalues for a general domain Ω can be tightly bounded by corresponding transmission eigenvalues in two disks/spheres B_{r_1} and B_{r_2} ($B_{r_1} \subset \Omega \subset B_{r_2}$) with constant index of refraction n_1 and n_2, respectively, such that $n_1 \le n(x) \le n_2$. Hence, our algorithm can also be used to quickly get some approximate values for transmission eigenvalues in general domains.
- For general domains, we can use the transformed field expansion approach [1,17] to reduce the transmission eigenvalue problems in a general domain to a sequence of transmission eigenvalue problems in a regular domain. It has been shown that this approach is very effective for various scattering problems [10,12,16], and we expect it to be effective for transmission eigenvalue problems as well.
- An essential step in the inverse problem for determining the material property $n(x)$ is to compute the smallest transmission eigenvalue with a given $n(x)$ [4,5]. Since our method can determine the smallest transmission eigenvalue very efficiently and accurately, it can be used to construct an efficient algorithm for the corresponding inverse problem.
- While we have only considered the Helmholtz transmission eigenvalue problems in this paper, it is clear that the same approach can be extended to treat Maxwell transmission eigenvalue problems in regular domains.

In summary, the method developed in this paper is a first but important step towards a robust and accurate algorithm for more general transmission eigenvalue problems which will be the subject of our future endeavor.

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