The Annals of Probability
1984, Vol. 12, No. 4, 1194–1204

A SPECTRAL REPRESENTATION FOR MAX-STABLE PROCESSES[1]

BY L. DE HAAN

University of North Carolina at Chapel Hill
and
Erasmus University, Rotterdam

The elements of an arbitrary max-stable sequence are exhibited as functionals of a 2-dimensional Poisson point process. The result is extended to a continuous time max-stable process that is continuous in probability. We define an analogue of a stochastic integral appropriate for this context.

Introduction. A well known and useful tool in studying stationary Gaussian processes is the spectral representation. It clarifies the structure of the process and makes prediction possible. A similar spectral representation is known for symmetric stable processes, i.e. processes for which each finite-dimensional marginal distribution is symmetric stable (Kuelbs 1973, page 269). Both representations are stochastic integrals with respect to an independent increment stable process. The proof of the representation uses the theory of L_α-spaces $(0 < \alpha \le 2)$.

Now an independent increment symmetric stable process with all marginals of the same type is, at least for $0 < \alpha < 1$, equivalent to the (Poisson) point process of its jumps. Hence the above mentioned integrals can also be considered as weighted sums of these jumps. This interpretation of the spectral representation of symmetric stable processes indicates the possibility of obtaining an analogous spectral representation for *max*-stable processes which we consider now.

DEFINITION. A stochastic process $\{Y_t\}_{t\in T}$ is called a *max-stable process* if the following property holds:

If $\{Y_t^{(i)}\}_{t\in T}$, $i = 1, 2, \cdots, r$, are independent copies of the process then the process $\{\max_{i\le r} Y_t^{(i)}\}_{t\in T}$ has the same distribution as $\{rY_t^{(1)}\}_{t\in T}$.

A consequence of the definition is that all one-dimensional marginal distribution functions are of the form $\exp(-c(t)/x)$, $x > 0$ (for some $c(t) \ge 0$), i.e. they are of type Φ_1 which is one of the extreme-value distributions. All n-dimensional marginal distribution functions of the process satisfy: for $0 < t_1 < t_2 < \cdots < t_n \in T$ with $\underline{t} := (t_1, t_2, \cdots, t_n)$ and $x_i \ge 0$, $i = 1, 2, \cdots, n$,

$$P\{Y_{t_1} \le x_1, \cdots, Y_{t_n} \le x_n\} =: F_{\underline{t}}(x_1, \cdots, x_n) = F_{\underline{t}}^r(rx_1, \cdots, rx_n) \tag{1}$$

Received July 1982; revised March 1984.

[1] This research has been supported by AFOSR Grant No. F49620 82 C 0009.

AMS 1980 *subject classifications.* Primary, 60H05; secondary, 62M20.

Key words and phrases. Max-stability, spectral representation.

from which it follows (de Haan and Resnick, 1977)

$$F_t(x_1, \cdots, x_n) = \exp\left(-\int_\Omega \left(\max_{1\le i\le n} \frac{a_i}{x_i}\right) U_t(da_1, \cdots, da_n)\right)$$

with U_t some finite measure on

$$\Omega = \{(a_1, \cdots, a_n) \mid a_i \ge 0,\ i = 1, 2, \cdots, n,\ \textstyle\sum_{i=1}^n a_i^2 = 1\}.$$

Max-stable processes can be obtained as weak limits as follows. Let $\{Q_n(t)\}_{t\in T}$ be independent copies of some stochastic process on T and define $\psi_n(t) := \max_{k\le n} Q_k(t)$ for $t \in T$. If for some sequence $\{C_n\}$ of positive constants $\{C_n^{-1}\psi_n(t)\}_{t\in T}$ converges weakly to a stochastic process $\{Y_t\}_{t\in T}$, then for some $\alpha > 0$ the process $\{Y_t^\alpha\}_{t\in T}$ is a max-stable process according to our definition.

EXAMPLE. Consider a Poisson point process on $\mathbb{R}_+ \times [0, 1]$ with intensity measure $(dx/x^2)dt$. With probability one there are denumerably many points in the point process. Let $\{X_k, T_k\}_{k=1}^\infty$ be an enumeration of the points in the process.

Consider now a family of nonnegative functions $\{f_t(x)\}_{t\in T}$ defined on $[0, 1]$. Suppose for fixed $t \in T$ the function $f_t(\cdot)$ is measurable and

$$\int_0^1 f_t(x)\, dx < \infty.$$

We claim that the family of random variables $(t \in T)$

$$Y_t := \sup_{k\ge1} f_t(T_k) \cdot X_k$$

form a max-stable process.

Clearly it is sufficient to show that for any $n = 1, 2, \cdots$ and $0 < t_1 < t_2 < \cdots < t_n \in T$ the joint distribution function of $(Y_{t_1}, \cdots, Y_{t_n})$ satisfies (1). Now

$$\begin{aligned}
&P^r\{Y_{t_1} \le ry_1, \cdots, Y_{t_n} \le ry_n\}\\
&\quad = P^r\{f_{t_i}(T_k) \cdot X_k \le ry_i,\ i = 1, \cdots, n;\ k = 1, 2, \cdots\}\\
&\quad = P^r\left\{X_k \le r \min_{i\le n} \frac{y_i}{f_{t_i}(T_k)} \text{ for } k = 1, 2, \cdots\right\}\\
&\quad = P^r\{\text{there are no points of the point process above the graph of the function } g\colon [0,1] \to \mathbb{R}_+ \text{ defined by } g(s) := r \min_{i\le n} y_i/f_{t_i}(s)\}\\
&\quad = \left\{\exp\left(-\int_0^1 \left[\int_{x > r\min_{i\le n} y_i/f_{t_i}(s)} \frac{dx}{x^2}\right] ds\right)\right\}^r\\
&\quad = \left\{\exp\left(-\int_0^1 \left(r^{-1}\max_{i\le n} \frac{f_{t_i}(s)}{y_i}\right) ds\right)\right\}^r = \exp\left(-\int_0^1 \left(\max_{i\le n} \frac{f_{t_i}(s)}{y_i}\right) ds\right)\\
&\quad = P\{Y_{t_1} \le y_1, \cdots, Y_{t_n} \le y_n\}.
\end{aligned}$$

Our aim is to give a spectral representation for max-stable processes. It will be shown that basically any max-stable process can be obtained as in the example above.

The outline of the paper is as follows. We consider a max-stable sequence, i.e. a max-stable process with $T = \mathbb{N}$, and we identify the distribution of this max-stable sequence with a finite measure on the set $\{(x_1, x_2, \cdots) \mid x_i \geq 0, i = 1, 2, \cdots\}$ in a similar way as it has been done in de Haan and Resnick (1977) for the case when the index set T is finite (Section 1). Next we prove a representation theorem for max-stable sequences (Section 2), which is subsequently extended to max-stable processes with $T = \mathbb{R}$ (Section 3). The representation involves a certain functional of a point process that can be considered as the analogue in the present setting of a stochastic integral. Finally an example is considered.

1. Identification of max-stable sequences. Suppose $Y_1, Y_2, \cdots$ is a max-stable sequence. Here, as in the next section, I try to follow the line of Kuelbs' paper (1973). However the proof is quite different.

THEOREM 1. *There exists a finite measure U on the set $S = \mathbb{R}_+ \times \mathbb{R}_+ \times \cdots$ with its Borel sets such that for $n = 1, 2, \cdots$ and $y_i > 0$, $i = 1, \cdots, n$*

$$P\{Y_1 \leq y_1, \cdots, Y_n \leq y_n\} = \exp\left(-\int_S \left(\max_{i \leq n} \frac{x_i}{y_i}\right) U(d\underline{x})\right).$$

PROOF. From (1) it follows that for any $y_1, \cdots, y_n$ as above

$$P^k\{Y_1 \leq ky_1, \cdots, Y_n \leq ky_n\}$$

does not depend on k. Hence

$$\begin{aligned} -\log P\{Y_1 \leq y_1, \cdots, Y_n \leq y_n\} &= \lim_{k\to\infty} - k \log P\{Y_1 \leq ky_1, \cdots, Y_n \leq ky_n\} \\ &= \lim_{k\to\infty} kP\{(Y_1 \leq ky_1, \cdots, Y_n \leq ky_n)^c\}. \end{aligned}$$

The superscript c indicates the complement of the set.

Since the right-hand side is the distribution function of a measure for all k, the limit must be a distribution function too. Hence for $n = 1, 2, \cdots$ and $y_i \geq 0$, $i = 1, \cdots, n$

$$P\{Y_1 \leq y_1, \cdots, Y_n \leq y_n\} = \exp(-\nu\{([0, y_1] \times \cdots \times [0, y_n])^c\})$$

where ν is a measure on $\mathbb{R}_+ \times \mathbb{R}_+ \times \cdots$ (with its Borel sets) such that for $k > 0$, $n = 1, 2, \cdots$ and $y_i > 0$, $i = 1, \cdots, n$

$$\nu\{([0, y_1] \times \cdots \times [0, y_n])^c\} < \infty$$

and (by max-stability)

$$(*) \qquad k\nu\{k([0, y_1] \times \cdots \times [0, y_n])^c\} = \nu\{([0, y_1] \times \cdots \times [0, y_n])^c\}.$$

It follows that $k \cdot \nu(kB) = \nu(B)$ for $k > 0$ and any Borel set $B \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \cdots$. We wish to give a representation for ν. First recall that $P\{Y_i \leq y_i\} = \exp(-c_i^2/y_i)$ for $y_i > 0$, where c_i is some finite constant ($i = 1, 2, \cdots$). Since

$EY_i^{1/2} < \infty$ for $i = 1, 2, \cdots$, there are positive numbers a_i $(i = 1, 2, \cdots)$ such that

$$E \sup_{i\ge1} a_i Y_i^{1/2} \le E \sum_{i\ge1} a_i Y_i^{1/2} < \infty,$$

hence

$$\nu\{(x_1, x_2, \cdots) \mid \sup_{i\ge1} a_i^2 x_i = \infty\} = -\log P\{\sup_{i\ge1} a_i^2 Y_i < \infty\} = 0.$$

Using the transformation L

$$w := \sup_{i\ge1} a_i^2 x_i$$

$$z_k := \begin{cases} x_k/w & \text{if } w > 0 \\ 0 & \text{if } w = 0 \end{cases}$$

$(k = 1, 2, \cdots)$ mapping S into $[0, \infty] \times S$, we get for $c > 0$, $n = 1, 2, \cdots$, $u_0 > 0$, $u_i \ge 0$ $(i = 1, 2, \cdots, n)$.

$$\begin{aligned} c \cdot \nu\{(x_1, x_2, \cdots) \mid w > cu_0, z_1 \le u_1, \cdots, z_n \le u_n\} \\ &= \nu\{(c^{-1}x_1, c^{-1}x_2, \cdots) \mid w > cu_0, z_1 \le u_1, \cdots, z_n \le u_n\} \\ &= \nu\{(x_1, x_2, \cdots) \mid w > u_0, z_1 \le u_1, \cdots, z_n \le u_n\} \\ &\le \nu\{(x_1, x_2, \cdots) \mid \sup_{i\ge1} a_i^2 x_i > u_0\} \le -\log P\{\sup_{i\ge1} a_i^2 Y_1 \le u_0\} < \infty. \end{aligned}$$

With $u_0 = 1$ this gives (with $z_i := 0$ for all i if $w = \infty$)

$$\begin{aligned} \nu\{(x_1, x_2, \cdots) \mid w > c, z_1 \le u_1, \cdots, z_n \le u_n\} \\ &= c^{-1} \cdot \nu\{(x_1, x_2, \cdots) \mid w > 1, z_1 \le u_1, \cdots, z_n \le u_n\} \end{aligned}$$

i.e. the transformed measure $\mu := \nu L^{\leftarrow}$ is a product measure $\mu = \mu_1 \times \mu_2$ on $[0, \infty] \times S$. For $n = 1, 2, \cdots$, $y_i > 0$, $i = 1, 2, \cdots, n$ this gives with

$$\begin{aligned} S_1 &:= \{(w, z_1, z_2, \cdots) \mid wz_i \le y_i, i = 1, 2, \cdots, n\} \\ &= \{(w, z_1, z_2, \cdots) \mid w \le \min_{i\le n} y_i/z_i\} \end{aligned}$$

that

$$\begin{aligned} P\{Y_1 \le y_1, \cdots, Y_n \le y_n\} \\ &= \exp\left(-\int_{S_1^c} \frac{dw}{w^2}\, \mu_2(d\underset{\sim}{z})\right) = \exp\left(-\int_S \left(\int_{w>\min_{i\le n}(y_i/z_i)} \frac{dw}{w^2}\right)\mu_2(d\underset{\sim}{z})\right) \\ &= \exp\left(-\int_S \left(\max_{i\le n} \frac{z_i}{y_i}\right)\mu_2(d\underset{\sim}{z})\right) \end{aligned}$$

where μ_2 is a finite measure on S. Hence the representation of the theorem is true with $U = \mu_2$. Note that μ_2 is concentrated on the "rectangle" $\{(z_1, z_2, \cdots) \in S \mid \sup_{i\ge1} a_i^2 z_i = 1\}$.

REMARK. Note that conversely any finite measure U on S with $\int x_i U(d\underset{\sim}{x}) < \infty$ for $i = 1, 2, \cdots$ corresponds to a probability measure P on S (via the identity in the statement of the theorem) for which $P^r\{rB\} = P\{B\}$ for $r > 0$ and every Borel set $B \subset S$.

EXAMPLE. If U is concentrated on the set

$$\mathbb{R}_+ \times \{0\} \times \{0\} \times \cdots \cup \{0\} \times \mathbb{R}_+ \times \{0\} \times \cdots \cup \cdots,$$

then $Y_1, Y_2, \cdots$ are independent. If U is concentrated on the set $x_1 = x_2 = x_3 = \cdots$, then $Y_1 = Y_2 = Y_3 = \cdots$ a.s.

2. Spectral representation of a max-stable sequence. In order to get a spectral representation for a max-stable sequence $Y_1, Y_2, \cdots$ we again follow the line of Kuelbs' paper.

THEOREM 2. *There exists a finite measure ρ on* $[0, 1]$ *such that, if* (X_k, T_k) *is an enumeration of points in the Poisson process on* $\mathbb{R}_+ \times [0, 1]$ *with intensity measure* $(dx/x^2) \times \rho(dt)$, *then the random variables*

$$Z_n = \max\{f_n(T_k) \cdot X_k \mid (X_k, T_k) \in \textit{the point process}\}$$

$n = 1, 2, \cdots$ *with suitable L_1-functions $f_n \geq 0$, have the same finite dimensional distributions as the* $\{Y_n\}$.

PROOF. First remark that it is sufficient to prove the result for those max-stable sequences for which the measure U from Theorem 1 is a probability measure: this only involves multiplying the entire process by a constant. So in the rest of the proof we shall assume that $U(S) = 1$. Let $D = \{d_1, d_2, \cdots\}$ be the set of atoms of U. Since U has no atoms in $S\backslash D$, there exists a U-null set $S_0 \subset S\backslash D$, a Lebesgue-null set $N_0 \subset [\sum_{i=1}^{\infty} U(d_i), 1]$ and a map L: $[\sum_{i=1}^{\infty} U(d_i), 1]\backslash N_0 \to S\backslash(D \cup S_0)$ which is one-to-one and such that (Royden, 1963, Chapter 15)

(1) both L and its inverse $L^{\leftarrow}$ take Borel sets into Borel sets (this is called Borel equivalence) and

(2) $UL(B) = \lambda(B)$ for every Borel set $B \subset [\sum_{i=1}^{\infty} U(d_i), 1]\backslash N_0$ with λ Lebesgue measure.

We extend L to a function on the whole of $[0, 1]\backslash N_0$ by defining

$$L(s) = d_k \quad \text{if} \quad s \in [\textstyle\sum_{i=1}^{k-1} U(d_i), \sum_{i=1}^{k} U(d_i)] \quad (k = 1, 2, \cdots).$$

Now L, as a *set* function, is a one-to-one mapping of A_0, the σ-field generated by $\{[\sum_{i=1}^{k-1} U(d_i), \sum_{i=1}^{k} U(d_i)]\}_{k=1}^{\infty}$ and the Borel sets of $[\sum_{i=1}^{\infty} U(d_i), 1]\backslash N_0$ to the family of Borel sets of $S\backslash S_0$. It is clear that the measure $\rho := UL$ is just the Lebesgue-measure restricted to A_0. Write $L(s) = (f_1(s), f_2(s), \cdots) \in S$ for $0 \leq s \leq 1$. Define the point process $\{(X_k, T_k)\}_{k=1}^{\infty}$ as in the theorem with $\rho = UL$. We now check that the $\{Z_n\}$ have the same finite-dimensional distributions as the $\{Y_n\}$. For $n =$

$1, 2, \cdots, y_i \geq 0, i = 1, 2, \cdots, n$

$$P\{Z_1 \leq y_i, \cdots, Z_n \leq y_n\}$$
$$= P\{f_i(T_k) \cdot X_k \leq y_i,\ i = 1, \cdots n;\ k = 1, 2, \cdots\}$$
$$= P\left\{X_k \leq \min_{i \leq n} \frac{y_i}{f_i(T_k)} \text{ for } k = 1, 2, \cdots\right\}$$
$$= P\{\text{there are no points in the point process above the graph of } \min_{i\leq n} y_i/f_i(s)\}$$
$$= \exp\left(-\int_0^1 \max_{i\leq n} \frac{f_i(s)}{y_i}\,\rho(ds)\right) = \exp\left(-\int_S \max_{i\leq n} \frac{x_i}{y_i} \cdot U(d\bar{x})\right)$$

in accordance with Theorem 1.

3. Representation of a continuous time max-stable process.

THEOREM 3. *Let* $\{Y(t)\}_{t\in\mathbb{R}}$ *be a max-stable process. If* $\{Y(t)\}$ *is continuous in probability, there exists a finite measure* ρ *on* $[0, 1]$ *such that, if* (X_k, T_k) *is an enumeration of the points in the Poisson process on* $\mathbb{R}_+ \times [0, 1]$ *with intensity measure* $(dx/x^2) \times \rho(ds)$, *then the random variables*

$$Z_t = \max_{k\geq 1} f_t(T_k) \cdot X_k$$

with suitable L_1*-functions* $f_t \geq 0$, *have the same finite-dimensional distributions as the* $\{Y(t)\}$.

PROOF. Apply the previous theorem to $\{Y(r_n)\}_{n\in\mathbb{N}}$ where $\{r_n\}$ is an enumeration of the rationals. The result then follows from Lemma 2 below.

For convenience we introduce some notation that is suggestive of the analogy to the stochastic integral (cf. the second remark after Lemma 2 below).

DEFINITION. Let $\{(X_k, T_k)\}_{k=1}^{\infty}$ be an enumeration of the points of the Poisson point process $P = P_\rho$ on $\mathbb{R}_+ \times [0, 1]$ with intensity measure $(dx/x^2) \times \rho(dt)$ where ρ is finite and let the nonnegative function f on $[0, 1]$ be in $L_1(\rho)$. We write

$$\overset{\vee}{\int_0^1} f(t)P(dt) := \max_{k\geq 1} f(T_k) \cdot X_k$$
$$= \max\{f(T_k) \cdot X_k \mid (X_k, T_k) \in \text{the point process}\}.$$

Since $f \in L_1$, $\vee\int_0^1 f$ is finite a.s. (see the proof of Theorem 2). For a Borel set $A \subset [0, 1]$ one has

$$\overset{\vee}{\int_A} f(t)P(dt) := \overset{\vee}{\int_0^1} f(t)\, \chi_A(t)P(dt)$$
$$= \max\{f(T_k)X_k \mid (X_k, T_k) \in \text{the point process and } T_k \in A\}.$$

Some simple properties are the following:

1. ${}^{\vee}\!\!\int af = a.\ {}^{\vee}\!\!\int f$ for $a > 0$.
2. $f < g$ a.e. $A \Rightarrow {}^{\vee}\!\!\int_A f < {}^{\vee}\!\!\int_A g$ a.s.
3. $\max({}^{\vee}\!\!\int f, {}^{\vee}\!\!\int g) = {}^{\vee}\!\!\int \max(f, g)$ a.s.
4. $d\nu = g d\mu (g \geq 0) \Rightarrow {}^{\vee}\!\!\int f\, dP_\nu = {}^{\vee}\!\!\int fg\, dP_\mu$ a.s. with an obvious interpretation.
5. $A \cap B = \varnothing \Rightarrow {}^{\vee}\!\!\int_A f$ and ${}^{\vee}\!\!\int_B g$ are independent and ${}^{\vee}\!\!\int_{A\cup B} f = \max({}^{\vee}\!\!\int_A f, {}^{\vee}\!\!\int_B f)$ a.s.
6. ${}^{\vee}\!\!\int f$ and ${}^{\vee}\!\!\int g$ *are independent* $\Leftrightarrow fg = 0 \quad \rho -$ a.e. (cf. Schilder, 1970).

LEMMA 1.

$$f = g \quad a.e. \qquad {}^{\vee}\!\!\int f = {}^{\vee}\!\!\int g \quad a.s.$$

PROOF. Define $A := \{t \mid f(t) > g(t)\}$. Now

$$\begin{aligned}
\left\{{}^{\vee}\!\!\int f > {}^{\vee}\!\!\int g\right\} &= \left\{\max\left({}^{\vee}\!\!\int_A f, {}^{\vee}\!\!\int_{A^c} f\right) > \max\left({}^{\vee}\!\!\int_A g, {}^{\vee}\!\!\int_{A^c} g\right)\right\} \\
&= \left\{{}^{\vee}\!\!\int_A f > \max\left({}^{\vee}\!\!\int_{A^c} f, {}^{\vee}\!\!\int_A g, {}^{\vee}\!\!\int_{A^c} g\right)\right\} \\
&\quad \cup \left\{{}^{\vee}\!\!\int_{A^c} f > \max\left({}^{\vee}\!\!\int_A f, {}^{\vee}\!\!\int_A g, {}^{\vee}\!\!\int_{A^c} g\right)\right\} \\
&= \left\{{}^{\vee}\!\!\int_A f > \max\left({}^{\vee}\!\!\int_{A^c} f, {}^{\vee}\!\!\int_{A^c} g\right) = {}^{\vee}\!\!\int_{A^c} g\right\}
\end{aligned}$$

by 2, 3 and the definition of A.

So by 5 (remark that $P\{{}^{\vee}\!\!\int_B f < x\} = \exp(-(1/x) \int_B f(t)\rho(dt))$ for $x > 0$)

$$(*) \qquad \begin{aligned}
P\left\{{}^{\vee}\!\!\int f > {}^{\vee}\!\!\int g\right\} &= \int_0^\infty \exp\left(-\frac{1}{x}\int_{A^c} g\, d\rho\right) d\, \exp\left(-\frac{1}{x}\int_A f\, d\rho\right) \\
&= \frac{\int_A f\, d\rho}{\int_A f\, d\rho + \int_{A^c} g\, d\rho}
\end{aligned}$$

and hence

$$P\left\{{}^{\vee}\!\!\int f > {}^{\vee}\!\!\int g\right\} > 0 \quad \text{if and only if} \quad \rho(A) > 0.$$

LEMMA 2. *f_n converges in $L_1 \Leftrightarrow {}^{\vee}\!\!\int f_n$ converges in probability. Moreover*

$$\lim_{n\to\infty} {}^{\vee}\!\!\int f_n = {}^{\vee}\!\!\int \lim_{n\to\infty} f_n.$$

PROOF. Suppose $\lim_{n\to\infty} f_n = f$ in L_1. Define $A := \{t \mid f(t) = 0\}$. By looking at $\chi_A f_n$ and $\chi_{A^c} f_n$ it is clear that it is sufficient to consider the cases $f \equiv 0$ (which is trivial) and $f > 0$. In the latter case ${}^{\vee}\!\!\int f > 0$ a.s. and (from (*))

$$P\left\{{}^{\vee}\!\!\int f_n > {}^{\vee}\!\!\int f(1+\varepsilon)\right\} = \frac{\int_{f_n > f(1+\varepsilon)} f_n \, d\rho}{\int \max(f_n, f(1+\varepsilon)) \, d\rho}$$

and this tends to zero as $n \to \infty$ since

$$\left| \int_{f_n > f(1+\varepsilon)} f_n \, d\rho - \int_{f_n > f(1+\varepsilon)} f \, d\rho \right| \le \| f_n - f \|_1 \to 0$$

and

$$\int_{f_n > f(1+\varepsilon)} f \, d\rho \le \int_{|f_n - f| > f\varepsilon} f \, d\rho \le \varepsilon^{-1} \cdot \int |f_n - f| \, d\rho \to 0 \quad (n \to \infty).$$

Similarly $P\{{}^{\vee}\!\!\int f_n < {}^{\vee}\!\!\int f(1-\varepsilon)\} \to 0$ $(n \to \infty)$. It now follows from 1 that $\log {}^{\vee}\!\!\int f_n$ converges in probability to $\log {}^{\vee}\!\!\int f$ and hence ${}^{\vee}\!\!\int f_n$ converges in probability to ${}^{\vee}\!\!\int f$.

Next suppose ${}^{\vee}\!\!\int f_n$ converges in probability. The sequence then converges in distribution, hence $\int f_n \, d\rho$ converges to a finite value c $(n \to \infty)$. It follows $P\{\lim_{n\to\infty} {}^{\vee}\!\!\int f_n = 0\} = 0$ or 1. We only consider the case $P\{\lim_{n\to\infty} {}^{\vee}\!\!\int f_n = 0\} = 0$, the other case being obvious. It follows that $\log {}^{\vee}\!\!\int f_n$ converges in probability. Observe that

$$0 = \lim_{n,m\to\infty} P\left\{{}^{\vee}\!\!\int f_n > {}^{\vee}\!\!\int f_m(1+\varepsilon)\right\} = \lim_{n,m\to\infty} \frac{\int_{f_n > f_m(1+\varepsilon)} f_n \, d\rho}{\int \max(f_n, f_m(1+\varepsilon)) \, d\rho}.$$

Since the denominator is bounded,

$$\lim_{n,m\to\infty} \int_{f_n > f_m(1+\varepsilon)} f_n \, d\rho = 0.$$

Similarly

$$\lim_{n,m\to\infty} \int_{(1-\varepsilon) f_m > f_n} f_m \, d\rho = 0.$$

Now

$$\int |f_n - f_m| \, d\rho \le \int_{f_n > f_m(1+\varepsilon)} 2 f_n \, d\rho + \int_{(1-\varepsilon) f_m > f_n} 2 f_m \, d\rho + \int_{|f_n - f_m| \le \varepsilon f_m} |f_n - f_m| \, d\rho.$$

Since the limit $(n, m \to \infty)$ of the latter term is $O(\varepsilon)$ $(\varepsilon \downarrow 0)$, $\{f_n\}$ is a Cauchy sequence in L_1.

REMARK. Note that we did not and could not use the linear structure of L_1.

REMARK. It can be proved that

$$\liminf_{n\to\infty} {}^{\vee}\!\!\int f_n \geq {}^{\vee}\!\!\int \liminf_{n\to\infty} f_n \quad \text{a.s.};$$

$f_n \to f$ monotonically a.e. $\Rightarrow {}^{\vee}\!\int f_n \to {}^{\vee}\!\int f \leq \infty$ monotonically a.s.;

$$f_n\, f \text{ a.e. and } f_n \leq g \in L_1 \Rightarrow \limsup_{n\to\infty} {}^{\vee}\!\!\int f_n = {}^{\vee}\!\!\int \limsup_{n\to\infty} f_n \quad \text{a.s.}$$

EXAMPLE. Take ρ Lebesgue measure and for $t > 0$

$$f_t(s) = \begin{cases} 1 & \text{if } s \leq t \\ 0 & \text{if } s > t. \end{cases}$$

Then $\{{}^{\vee}\!\int f_t\}_{t>0}$ is a so-called extremal process (Dwass, Lamperti 1964).

EXTENSION. Note that it is easy to extend the above results to cover processes for which the marginal distributions are one of the other extreme-value distributions: The processes $\{Y_t^{1/\alpha}\}_{t\in T}$, $\{\log Y(t)\}_{t\in T}$ and $\{-Y_t^{-1/\alpha}\}_{t\in T}$ with $\{Y_t\}_{t\in T}$ the max-stable process treated above are "max-stable" processes (with an adapted definition) based on Φ_α, Λ and Ψ_α, respectively. Their properties are easily derived from the properties of $\{Y_t\}_{t\in T}$. Also one can introduce "min-stable" processes: a stochastic process is called a min-stable process if the following property holds:

If $\{Y_t^i\}_{t\in T}$, $i = 1, 2, \cdots, r$, are independent copies then the process $\{\min_{i=1}^r Y_t^{(i)}\}_{t\in T}$ has the same distribution as $\{r^{-1} Y_t^{(i)}\}_{t\in T}$. Obviously, if $\{Z_t\}_{t\in T}$ is any max-stable process from the previous paragraph, then $\{-Z_t\}_{t\in T}$ is a min-stable processes. From this connection the properties of min-stable processes are derived easily.

4. Example. One is tempted to consider the analogue of the stochastic integral, introduced above, as a functional of the extremal process rather than of the point process, in the same way as for the stochastic integral itself. I shall now show that this is not possible, the difference being that in the extremal case much information from the sample is lost in the limiting procedure.

Consider the following model: Let X_0, $X_{\pm 1}$, $X_{\pm 2}$, $\cdots$ be i.i.d. Φ_1 and take $Y_n := \max_{k\geq 0} e^{-k} X_{n-k}$ for $n = 0, \pm 1, \pm 2, \cdots$. Then for $k = 1, 2, \cdots$

$$Y_n = \max(e^{-k} Y_{n-k}, e^{-(k-1)} X_{n-k+1}, \cdots, X_n) \tag{1}$$

where the random variables at the right-hand side are independent. So the probability that $Y_n = e^{-k} Y_{n-k}$ (i.e. the process has no jump between the epochs $n - k$ and n) given Y_{n-k} is

$$\exp(-(1/Y_{n-k})\{e + e^2 + \cdots + e^k\}).$$

We are going to give a continuous time analogue of the sequence $\{Y_n\}$, similar to the Ornstein–Uhlenbeck process in the partial sum set-up. Consider the sequence of processes $\{Y_n(t)\}_{t>0}$ given by (compare with (1) for $k = n$)

$$Y_n(t) := e^{-t}\max(Y_0, \max_{1\le k\le nt} n^{-1}e^{k/n}X_k) \tag{2}$$

with $Y_0 \in \Phi_1$ independent of the X_k's. Now the probability that $Y_t = e^{-t}Y_0$ (no jump) given Y_0 is

$$\exp(-(1/Y_0)\{\textstyle\sum_{k=1}^{[nt]} n^{-1}e^{k/n}\}).$$

Since the sequence of point processes $(X_k/n, k/n)_k$ converges $(n \to \infty)$ to a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $(dx/x^2) \times dt$ (see e.g. Adler, 1978), we get by the invariance principle that the sequence of processes $\{Y_n(t)\}_{t>0}$ converges $(n \to \infty)$ to (with λ Lebesgue measure)

$$Y(t) := e^{-t}\max\left(Y_0, \overset{\vee}{\int_0^t} e^u P_\lambda(du)\right) \tag{3}$$

and the probability of no jump before time t given Y_0 is $\exp(-(1/Y_0)\int_0^t e^u\,du)$. The process $Y(t)$ of course is Markov and stationary; it decreases in an exponential way except for jumps. The representation (3) for the process $\{Y(t)\}$ is actually too fancy since by 4 of Section 3 (with $\nu[0, u] = e^u - 1$)

$$\left\{\max\left(Y_0, \overset{\vee}{\int_0^t} e^u P_\lambda(du)\right)\right\}_{t\ge0} =_d \left\{\max\left(Y_0, \overset{\vee}{\int_0^t} P_\nu(du)\right)\right\}_{t\ge0} =_d \{M(e^t)\}_{t\ge0}$$

where $\{M(t)\}_{t\ge0}$ is an extremal process based on Φ_1. So

$$\{Y(t)\}_{t\ge0} =_d \{e^{-t}M(e^t)\}_{t\ge0}.$$

This is also an easy way to extend $\{Y_t\}$ to a process with $t \in \mathbb{R}$.

As a byproduct we get the

PROPOSITION.

$$\{\max_{0\le s\le t} e^{-s}M(e^s)\}_{t>0} =_d \{M(t)\}_{t>0}.$$

PROOF. From (2) we get

$$\begin{aligned}\max_{0\le s\le t} Y_n(s) &= \max_{0\le s\le t}\max_{1\le k\le ns} e^{-s}\max(Y_0, n^{-1}e^{k/n}X_k).\\ &= \max_{k\le nt}\max_{k/n\le s\le t} e^{-s}\max(Y_0, n^{-1}e^{k/n}X_k)\\ &= \max_{k\le nt}(e^{-(k/n)}Y_0, n^{-1}X_k) = \max(e^{-(1/n)}Y_0, \max_{k\le nt} n^{-1}X_k).\end{aligned}$$

Taking $n \to \infty$ we get the result.

It is now clear that it is not possible to exhibit $\{Y(t)\}$ as a functional of $\{M(t)\}$ using some invariance principle as above since the jump structure is different and can be compared only using a time transformation.

Acknowledgements. I would like to acknowledge fruitful discussions on the concepts above with members of the Center of Stochastic Processes at the University of North Carolina, Chapel Hill, especially Stamatis Cambanis and Rajeeva Karandikar. I am grateful to Professor Marjorie Hahn who pointed out an error in the original Theorem 1. After completing the revision of Theorem 1 I learned that Pirooz Vatan had obtained the results of Theorems 1 and 2 independently.

REFERENCES

ADLER, R. J. (1978). Weak convergence results for extremal processes generated by dependent random variables. *Ann. Probab.* **6** 660–667.

DWASS, M. (1964). Extremal processes. *Ann. Math. Statist.* **35** 1718–1725.

DE HAAN, L. and RESNICK, S. I. (1977). Limit theory for multivariate sample extremes. *Z. Wahrsch. verw. Gebiete* **40** 317–333.

KUELBS, J. (1973). A representation theorem for symmetric stable processes and stable measures on *H. Z. Wahrsch. verw. Gebiete* **26** 259–271.

LAMPERTI, J. (1964). On extreme order statistics. *Ann. Math. Statist.* **35** 1726–1737.

ROYDEN, H. L. (1963). *Real Analysis.* Macmillan, New York.

SCHILDER, M. (1970). Some structure theorems for the symmetric stable laws. *Ann. Math. Statist.* **41** 412–421.

ERASMUS UNIVERSITEIT ROTTERDAM
POSTBUS 1738
3000 DR ROTTERDAM
THE NETHERLANDS