# A Spectral Study of the Boundary Controllability of the Linear 2-D Wave Equation in a Rectangle

Sorin Micu *  Luz de Teresa †

March 20, 2009

**Abstract**

The paper studies the controllability properties of the linear 2-D wave equation in the rectangle $\Omega = (0, a) \times (0, b)$. We consider two types of action, on an edge or on two adjacent edges of the boundary. Our analysis is based on Fourier expansion and explicit construction and evaluation of biorthogonal sequences. This method allows us to measure the magnitude of the control needed for each eigenfrequency. In both analyzed cases we give a Fourier characterization of the controllable spaces of initial data and we construct particular controls for them.

**Keywords**: wave equation, control, Fourier expansion, biorthogonal

## 1 Introduction

Fourier techniques have been used for a long time to study the controllability properties of linear differential equations (see, for instance the books [2, 19, 38] or the survey articles [29, 39]). Important and interesting results have been obtained for the heat equation [8, 9, 21, 29], the wave equation [7, 18, 14, 15], the beam and plate equations [10, 33] and so on. In the recent years there were many applications to discrete equations too [27, 23, 34].

The main idea is to equivalently transform the controllability problem into a moments problem of the form: find $v \in L^2((0,T) \times \Gamma)$ such that

$$\int_0^T \int_\Gamma e^{\lambda_k t} \varphi_k(s)\, v(t,s) ds dt = \beta_k, \quad \forall k \in K \tag{1}$$

*Facultatea de Matematica si Informatica, Universitatea din Craiova, 200585, Romania. E-mail: sd_micu@yahoo.com. Partially supported by Grant MTM2008-03541 funded by MICINN (Spain) and Grant 420/2008 of CNCSIS (Romania).

†Corresponding author. Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510, D.F. México. E-mail: deteresa@matem.unam.mx. Tel: (5255) 56224766, FAX 5616-0348. Partially supported by CONACyT and Intercambio Académico, UNAM, México and CEAL-UAM, Spain.

where $K$ is a family of indices, $(\lambda_k)_{k\in K}$ is the sequence of eigenvalues of the corresponding differential operator, $(\varphi_k)_{k\in K}$ are functions defined on $\Gamma$ given by the eigenvectors and $(\beta_k)_{k\in K}$ is a sequence depending on the Fourier coefficients of the initial data to be controlled by $v$. Here $T$ is the controllability time and $\Gamma$ a part of the boundary or of the domain of our equation where the control acts.

The easiest cases are the one-dimensional problems with boundary control when $\Gamma$ reduces to a point and $K$ is $\mathbb{N}^*$ or $\mathbb{Z}^*$. In this situation (1) is only time depending problem and the well-developed theory of exponential functions may be successfully used. Applications of the moments problem in several dimensions are somehow more complicated and less frequent. Nevertheless many of the above cited works contain fine results in this context too.

The following particular case has special relevance. If the initial datum has only one eigenmode (for instance, the l-th), then all the coefficients $(\beta_k)_{k\in K}$ are zero, except $\beta_l$. If $\beta_l = 1$, problem (1) becomes: find $\psi_l \in L^2((0,T)\times\Gamma)$ such that

$$\int_0^T \int_\Gamma e^{\lambda_k t}\,\varphi_k(s)\,\psi_l(t,s)dsdt = \delta_{kl}, \quad \forall k \in K. \tag{2}$$

A sequence $(\psi_l)_{l\in K}$ which verifies (2) for every $l \in K$ is called *biorthogonal* to the family $\left(e^{\lambda_k t}\,\varphi_k(s)\right)_{k\in K}$ in $L^2((0,T)\times\Gamma)$. Note that the element $\psi_l$ may be view as a control for the $l-$th eigenmode. Clearly, controls for arbitrary initial data may be obtained from linear combinations of $(\psi_l)_{l\in K}$. Consequently, the study of the biorthogonal sequence's properties allows to obtain information about the control of any frequency or range of frequencies. Hence, one may deduce what frequencies are more difficult to control, estimate the magnitude of the control for any of them and characterize the spaces of controllable initial data. This is one of the advantages of the Fourier method's application in control theory.

In this paper we shall consider the linear wave equation in the rectangular domain $\Omega = (0,a)\times(0,b)\subset\mathbb{R}^2$. The control will act on a part of the boundary which could be one edge or two adjacent edges. It is known that, in the former case no Sobolev space of initial data may be controlled with controls in $L^2$ (see, for instance, [5]). On the contrary, in the later case any initial data in $L^2(\Omega)\times H^{-1}(\Omega)$ is controllable (see, for instance, [20]).

The aim of this article is to use the Fourier technique for study of the following two problems:

(P1) to give the space of initial data that can be controlled from one edge

(P2) to give the space of initial data controllable from two adjacent edges.

For problem (P1) we improve a result obtained in [14] by showing the controllability of a larger space of initial data. Moreover, we give bounds for the control of each frequency and show which initial data are more difficult to control from one side of the boundary. For problem (P2) we obtain the space of controllable data $L^2(\Omega)\times H^{-1}(\Omega)$, but the control time needs to be larger than the known optimal value. Although a somehow weaker result is obtained, the study reveals at least two interesting facts: how the controls on each edge may be used alternatively and how their norms change, depending on the range of frequencies we want to control.

The eigenfrequencies corresponding to the wave operator in $\Omega$ are

$$\lambda_{mn}^{\pm} = \pm\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}, \quad \forall m, n \in \mathbb{N}^*.$$

The control problems (P1) and (P2) are reduced to find, for each $m \in \mathbb{N}^*$, a sequence $(\Psi_{mn}^{1,\pm})_{n\geq 1}$ and, for each $n \in \mathbb{N}^*$, a sequence $(\Psi_{mn}^{2,\pm})_{m\geq 1}$ such that

$$\int_0^T \Psi_{mn}^{1,\pm}(t)e^{-i\lambda_{ml}^{\pm}t}dt = \delta_{nl}, \quad \int_0^T \Psi_{mn}^{1,\mp}(t)e^{-i\lambda_{ml}^{\pm}t}dt = 0, \qquad \forall n, l \in \mathbb{N}^*,$$

$$\int_0^T \Psi_{mn}^{2,\pm}(t)e^{-i\lambda_{kn}^{\pm}t}dt = \delta_{mk}, \quad \int_0^T \Psi_{mn}^{2,\mp}(t)e^{-i\lambda_{kn}^{\pm}t}dt = 0, \qquad \forall m, k \in \mathbb{N}^*.$$

Note that $(\Psi_{mn}^{1,\pm})_{n\geq 1}$ and $(\Psi_{mn}^{2,\pm})_{m\geq 1}$ are biorthogonals to the families $\left(e^{-i\lambda_{ml}^{\pm}t}\right)_{l\geq 1}$ and $\left(e^{-i\lambda_{kn}^{\pm}t}\right)_{k\geq 1}$ respectively. The numbers 1 and 2 in the biorthogonal's notation show which of the indices is kept fixed, the first or the second one.

Once the biorthogonals $(\Psi_{mn}^{1,\pm})_{n\geq 1}$ and $(\Psi_{mn}^{2,\pm})_{m\geq 1}$ are determined for each $m$ and $n$, controls may be constructed for problems (P1) and (P2). Indeed, a control $v(t,y) \in L^2((0,T)\times(0,b))$ for problem (P1), acting on the edge $\{(a,y) : y \in (0,b)\}$, is given by

$$v(t,y) = \sum_{m\geq 1}\sum_{n\geq 1} \beta_{mn}^{\pm} \sin\left(\frac{n\pi}{b}y\right)\Psi_{mn}^{1,\pm}(t). \tag{3}$$

A pair of controls $(v^1(t,y), v^2(t,x)) \in L^2((0,T)\times(0,a)) \times L^2((0,T)\times(0,b))$ for problem (P2), acting on the edges $\{(a,y) : y \in (0,b)\}$ and $\{(x,b) : x \in (0,a)\}$ respectively, is given by

$$v^1(t,y) = \sum_{m\geq 1}\sum_{n\geq 1} \beta_{mn}^{\pm} \sin\left(\frac{m\pi}{b}y\right)\Psi_{mn}^{1,\pm}(t), \quad v^2(t,x) = \sum_{n\geq 1}\sum_{m\geq 1} \beta_{mn}^{\pm} \sin\left(\frac{n\pi}{a}x\right)\Psi_{mn}^{2,\pm}(t). \tag{4}$$

The coefficients $(\beta_{mn}^{\pm})$ in (4) are proportional to the Fourier coefficients of the initial data we want to control and are explicitly given in (65). Hence, once the biorthogonals $(\Psi_{mn}^{1,\pm})_{n\geq 1}$ and $(\Psi_{mn}^{2,\pm})_{m\geq 1}$ are found, the problem is reduced to see for which coefficients $(\beta_{mn}^{\pm})$ the series (3) and (4) converge in $L^2$. An answer to this question may be obtained by studying the behavior of the norms of the biorthogonals $(\Psi_{mn}^{1,\pm})_{n\geq 1}$ and $(\Psi_{mn}^{2,\pm})_{m\geq 1}$. Roughly speaking, greater the norms of the biorthogonals are, smaller the coefficients $(\beta_{mn}^{\pm})$ (and, consequently, the space of controllable initial data) are.

We recall that $\sin\left(\frac{m\pi}{b}y\right)\Psi_{mn}^{1,\pm}(t)$ and $\sin\left(\frac{n\pi}{b}x\right)\Psi_{mn}^{2,\pm}(t)$ are controls for the $(m,n)-$th eigenmode, corresponding to the eigenvalue $\lambda_{mn}^{\pm}$ and acting on the edge $\{(a,y) : y \in (0,b)\}$ or $\{(x,b) : x \in (0,a)\}$ respectively.

In [7] it is shown that, for any $k \geq 1$ there exists a constant $C = C(k)$ such that

$$(5)\qquad \begin{aligned} ||\Psi^{1,\pm}_{mn}||_{L^2(0,T)} &\leq C(m), \quad \forall n \in \mathbb{N}^*, \\ ||\Psi^{2,\pm}_{mn}||_{L^2(0,T)} &\leq C(n), \quad \forall m \in \mathbb{N}^*. \end{aligned}$$

Moreover, the constant $C$ increases exponentially. More precisely, there exist four constants $M_1$, $M_2$, $\omega_1$ and $\omega_2$ such that

$$M_1 \exp(\omega_1 k) \leq C(k) \leq M_2 \exp(\omega_2 k), \quad \forall k \geq 1.$$

These estimates allow to deduce that a special space of initial data (very regular in one direction) may be controlled from one edge.

In [1] a deeper analysis shows that a longer control time $T$ increases the space of controllable data. Moreover, it was noted that better constant $C$ may be found in (5). In [14] the problem of evaluating the norms of $\Psi^{1,\pm}_{mn}$ and $\Psi^{2,\pm}_{mn}$ is reconsidered and it is proved that

$$(6)\qquad \begin{aligned} ||\Psi^{1,\pm}_{mn}||_{L^2(0,T)} &\leq \begin{cases} C(m) & \frac{n}{b} \leq I\left(\frac{m}{a}\right), \\ C' & \frac{n}{b} > I\left(\frac{m}{a}\right) \end{cases} \\ ||\Psi^{2,\pm}_{mn}||_{L^2(0,T)} &\leq \begin{cases} C(n) & \frac{m}{a} \leq I\left(\frac{n}{b}\right), \\ C' & \frac{m}{a} > I\left(\frac{n}{b}\right) \end{cases} \end{aligned}$$

where $C(k)$ is like above, $C'$ is a constant independent of $m$ and $n$ and $I(k) \sim \exp(k)$.

Note that (6) implies that the biorthogonals $(\Psi^{1,\pm}_{mn})_{n\geq 1}$ are uniformly bounded in the range $R_1 = \left\{(m,n) \,:\, \frac{n}{b} \geq I\left(\frac{m}{a}\right)\right\}$ and the biorthogonals $(\Psi^{2,\pm}_{mn})_{m\geq 1}$ are uniformly bounded in the range $R_2 = \left\{(m,n) \,:\, \frac{m}{a} \geq I\left(\frac{n}{b}\right)\right\}$. This proves that even a larger space of initial data may be controlled from one edge. However, since $R_1 \cup R_2 \neq \mathbb{N}^* \times \mathbb{N}^*$, we cannot say that any eigenmode may be controlled with uniformly bounded controls from one edge or the other.

We point out that techniques like, for instance, multipliers allow to deduce exact controllability in the space $L^2 \times H^{-1}$ when the control acts on two adjacent edges but do not tell how the control depends on the frequencies and if we can use only one edge for some frequencies and the other for the rest of them. Moreover, no information may be obtained for the space of controllable initial data from one edge and how large the controls are.

The biorthogonal technique is used in this paper to show that in (6) we may take $I(k) = k$ if $T$ is large enough. This result has several interesting consequences:

- Related to problem (P1), since the index $I$ is smaller than before, a larger class of initial data is shown to be controllable from one edge. For example, any initial data such that

  $$(7)\qquad \sum_{n\geq 1} \sum_{\frac{m}{a} \geq \frac{n}{b}} \left|\beta^{\pm}_{mn}\right|^2 + \sum_{n\geq 1} e^{2\frac{a}{b}n} \sum_{\frac{m}{a} < \frac{n}{b}} \left|\beta^{\pm}_{mn}\right|^2 < \infty$$

  is controllable with an $L^2-$ control acting only on the edge $\{(0,y) \,:\, y \in (0,b)\}$.

- Related to problem (P2), since now $R_1 \cup R_2 = \mathbb{N}^* \times \mathbb{N}^*$, uniformly bounded controls on one edge or another may be alternatively chosen to control the whole range of frequencies. Indeed, we may choose a pair of controls $(v^1(t,y), v^2(t,x)) \in L^2((0,T)\times(0,a))\times L^2((0,T)\times(0,b))$ acting on the edges $\{(0,y)\,:\, y\in(0,b)\}$ and $\{(x,0)\,:\, x\in(0,a)\}$ respectively not like in (4), where each frequency is controlled simultaneously from both edges, but as follows

$$
\begin{aligned}
v^1(t,y) &= \sum_{m\geq 1}\sum_{\frac{n}{b}\geq\frac{m}{a}} \beta_{mn}^{\pm} \sin\left(\frac{m\pi}{b}y\right)\Psi_{mn}^{1,\pm}(t),\\
v^2(t,x) &= \sum_{n\geq 1}\sum_{\frac{m}{a}>\frac{n}{b}} \beta_{mn}^{\pm} \sin\left(\frac{n\pi}{a}x\right)\Psi_{mn}^{2,\pm}(t).
\end{aligned}
\tag{8}
$$

  Now, the controls $v^1$ and $v^2$ acts only on the frequencies $R_1$ and $R_2$ respectively. This choice shows that any initial data with

$$
\sum_{m\geq 1}\sum_{n\geq 1}\left|\beta_{mn}^{\pm}\right|^2 < \infty. \tag{9}
$$

  is controllable with an $L^2-$ pair of controls acting alternatively on two adjacent edges. This is equivalent to the controllability of any initial data in $L^2(\Omega)\times H^{-1}(\Omega)$.

An interesting spectral method is used in [32] to show the controllability of any initial data from $L^2 \times H^{-1}$. It consists of reducing the problem to a Hautus test for the eigenvalues of the Laplace operator (see also [26, 33, 34]). To our knowledge this is the first study for the controllability problem of the wave equation in 2-D which uses Fourier methods. Published after the submission of our paper, [22] gives a direct Ingham type proof for the boundary observability of N-dimensional wave equation. Our approach, based on Fourier expansion and biorthogonal sequences, is different and offers more detailed information about the norm of the controls needed to control each eigenmode from one or two edges. However, our method has a drawback due to some technical aspects: it does not give the optimal controllability time (see Remark 10 at the end of the paper).

The remaining part of the article is organized in the following way. Section 2 introduces the controllability problems and it gives equivalent formulations in terms of some moments problems. The biorthogonal sequences are introduced and evaluated in section 3 and the controllability results are deduced in section 4.

## 2 The controllability problem

Let us consider $\Omega = (0,a)\times(0,b)\subset\mathbb{R}^2$ and divide its boundary in two parts $\partial\Omega = \overline{\Gamma_0}\cup\overline{\Gamma_1}$ such that $\Gamma_0\cap\Gamma_1=\emptyset$.

We are concerned with the following boundary exact controllability property of the wave equation in $\Omega$: given $T > 0$ and $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control function $v(t, y) \in L^2((0, T) \times \Gamma_0)$ such that the solution of the equation

$$\left\{ \begin{array}{ll} u'' - \Delta u = 0 & \text{for } (x, y) \in \Omega, \;\; t > 0, \\ u(t, x, y) = v(t, x, y) & \text{for } (x, y) \in \Gamma_0, \;\; t > 0, \\ u(t, x, y) = 0 & \text{for } (x, y) \in \Gamma_1, \;\; t > 0, \\ u(0, x, y) = u^0(x, y), & \text{for } (x, y) \in \Omega \\ u'(0, x, y) = u^1(x, y), & \text{for } (x, y) \in \Omega \end{array} \right. \tag{10}$$

satisfies

$$u(T, \cdot) = u'(T, \cdot) = 0. \tag{11}$$

By $'$ we denote the time derivative.

Observe that the reversibility of (10) allows to show that (11) is achieved if and only if for every target state $(u_T^0, u_T^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists $v$ such that $u(T) = u_T^0, u'(T) = u_T^1$. This do not hold in other contexts as the non linear framework or the heat equation.

The exact controllability property of (10) may be characterized by the following immediate property (see, for instance, [25]).

THEOREM 2.1 *Let $T > 0$ and $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. The following two properties are equivalent:*

*(i) There exists a control function $v(t, y) \in L^2((0, T) \times \Gamma_0)$ such that the solution of the equation (10) verifies (11).*

*(ii) The following equality holds*

$$\begin{array}{c} \displaystyle\int_0^T \int_{\Gamma_0} v(t, x, y) \frac{\partial \overline{\varphi}}{\partial \nu}(t, x, y) dx dy dt = \\ \displaystyle < u^1, \varphi(0) >_{-1,1} - \int_\Omega u^0(x, y) \overline{\varphi}'(0, x, y) dx dy \end{array} \tag{12}$$

*for any $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $(\varphi, \varphi')$ being the solution of the homogeneous adjoint equation*

$$\left\{ \begin{array}{ll} \varphi'' - \Delta \varphi = 0, & \textit{for } (x, y) \in \Omega, \;\; t > 0, \\ \varphi(t, x, y) = 0, & \textit{for } (x, y) \in \partial\Omega, \;\; t > 0, \\ \varphi(T, x, y) = \varphi^0(x, y), & \textit{for } (x, y) \in \Omega, \\ \varphi'(T, x, y) = \varphi^1(x, y), & \textit{for } (x, y) \in \Omega. \end{array} \right. \tag{13}$$

In (12), $<\cdot,\cdot>_{-1,1}$ denotes the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Moreover, we shall make the following notation

$$\langle (w^0,w^1),(\psi^0,\psi^1)\rangle_D = < w^1,\psi^0 >_{-1,1} - \int_\Omega w^0(x,y)\overline{\psi}^1(x,y)dxdy \tag{14}$$

for any $(w^0,w^1)\in L^2(\Omega)\times H^{-1}(\Omega)$ and $(\psi^0,\psi^1)\in H_0^1(\Omega)\times L^2(\Omega)$.

Our aim is to give sharp estimates for the control's norm when $\Gamma_0$ consists of one edge or two adjacent edges. As we have said before, our analysis is based on the Fourier expansion of solutions. Therefore, let us now introduce the eigenvalues of the wave operator,

$$\lambda_{mn}^{\pm} = \pm\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}},\quad (m,n)\in\mathbb{N}^*\times\mathbb{N}^* \tag{15}$$

and the corresponding eigenfunctions

$$\Phi_{mn}^{\pm}(x,y) = \sqrt{\frac{2}{ab}}\begin{pmatrix}\frac{1}{i\lambda_{mn}^{\pm}}\\ -1\end{pmatrix}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),\quad (m,n)\in\mathbb{N}^*\times\mathbb{N}^*. \tag{16}$$

We denote by $\Phi_{mn}^{1\pm}$ and $\Phi_{mn}^{2\pm}$ the two components of $\Phi_{mn}^{\pm}$.

The sequence $(\Phi_{mn}^{\pm})_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*}$ forms an orthonormal basis in $H_0^1(\Omega)\times L^2(\Omega)$. Moreover,

$$||\Phi_{mn}^{\pm}||_{L^2(\Omega)\times H^{-1}(\Omega)} = \frac{1}{|\lambda_{mn}^{\pm}|}.$$

**Remark 1** *Note that it is sufficient to show that (12) is verified by $(\varphi^0,\varphi^1)=\Phi_{mn}^{\pm}$ for all $m,n\in\mathbb{N}^*$. Indeed, the continuity of the linear form $\Lambda: H_0^1(\Omega)\times L^2(\Omega)\to\mathbb{C}$, defined by*

$$\Lambda(\varphi^0,\varphi^1) = \int_0^T\int_{\Gamma_0} v(t,x,y)\frac{\partial\overline{\varphi}}{\partial\nu}(t,x,y)dxdydt - \langle (u^0,u^1),(\varphi(0),\varphi'(0))\rangle_D \tag{17}$$

*implies that (12) holds for any $(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)$ if and only if it is verified on a basis of the space $H_0^1(\Omega)\times L^2(\Omega)$.*

□

By considering $(\varphi^0,\varphi^1)=\Phi_{mn}^{\pm}$ in (12), we obtain the following result.

Theorem 2.2 *The control $v$ drives to zero the initial data*

$$(u^0,u^1) = \sum_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*}\alpha_{mn}^{\pm}\Phi_{mn}^{\pm} \tag{18}$$

*of (10) if and only if, for all $(k,l)\in\mathbb{N}^*\times\mathbb{N}^*$,*

$$\int_0^T e^{-i\lambda_{kl}^{\pm}t}\left(\int_{\Gamma_0} v(t,x,y)\frac{\partial\overline{\Phi_{kl}^{1\pm}}}{\partial\nu}(x,y)dxdy\right)dt = \frac{4}{iab\lambda_{kl}^{\pm}}\alpha_{kl}^{\pm}. \tag{19}$$

In (19) and elsewhere we shall use the summation rule, $\alpha^{\pm}_{mn}\Phi^{\pm}_{mn} = \alpha^{+}\Phi^{+}_{mn} + \alpha^{-}\Phi^{-}_{mn}$.

The following two corollaries are direct consequences of the previous Theorem.

**Corollary 1** *If $\Gamma_0 = \{(a,y) \,:\, y \in (0,b)\}$ then the control $v \in L^2((0,T)\times(0,b))$ drives to zero the initial data data (18) if and only if, for all $(k,l) \in \mathbb{N}^* \times \mathbb{N}^*$,*

$$\int_0^T e^{-i\lambda^{\pm}_{kl}t}\left(\int_0^b v(t,y)\sin\left(\frac{l\pi y}{b}\right)dy\right)dt = (-1)^{k+1}\frac{2\sqrt{2}}{\sqrt{ab}}\frac{a}{\pi k}\alpha^{\pm}_{kl}. \tag{20}$$

**Corollary 2** *If $\Gamma_0 = \{(a,y) \,:\, y \in (0,b)\} \cup \{(x,b) \,:\, x \in (0,b)\}$ then the control $(v^1, v^2) \in L^2((0,T)\times(0,b)) \times L^2((0,T)\times(0,a))$ drives to zero the initial data (18) if and only if, for all $(k,l) \in \mathbb{N}^* \times \mathbb{N}^*$,*

$$\begin{aligned}\int_0^T e^{-i\lambda^{\pm}_{kl}t}\left((-1)^{k+1}\frac{k}{a}\int_0^b v^1(t,y)\sin\left(\frac{l\pi y}{b}\right)dy + (-1)^{l+1}\frac{l}{b}\int_0^a v^2(t,x)\sin\left(\frac{k\pi x}{a}\right)dx\right)dt = \\ = \frac{2\sqrt{2}}{\sqrt{ab}\pi}\alpha^{\pm}_{kl}.\end{aligned} \tag{21}$$

## 3 Biothogonal sequences

The controls $v$ and $(v^1, v^2)$ from Corollaries 1 and 2 are obtained from an explicitly given biorthogonal sequence. Let us consider the family of complex exponentials $\Lambda = \left(e^{i\lambda^{\pm}_{mn}t}\right)_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*}$.

**Definition 3.1** *Let $m \in \mathbb{N}^*$ be fixed. The sequence $(\Theta^{1,\pm}_{mn})_{n\in\mathbb{N}^*} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ is $(1,m)-$***biorthogonal** *to the family $\Lambda$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ if*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta^{1,\pm}_{mn}(t)e^{-i\lambda^{\pm}_{ml}t}dt = \delta_{nl}, \quad \int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta^{1,\mp}_{mn}(t)e^{-i\lambda^{\pm}_{ml}t}dt = 0, \qquad \forall n,l \in \mathbb{N}^*. \tag{22}$$

**Definition 3.2** *Let $n \in \mathbb{N}^*$ be fixed. The sequence $(\Theta^{2,\pm}_{mn})_{m\in\mathbb{N}^*} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ is $(2,n)-$***biorthogonal** *to the family $\Lambda$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ if*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta^{2,\pm}_{mn}(t)e^{-i\lambda^{\pm}_{kn}t}dt = \delta_{mk}, \quad \int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta^{2,\mp}_{mn}(t)e^{-i\lambda^{\pm}_{kn}t}dt = 0, \qquad \forall m,k \in \mathbb{N}^*. \tag{23}$$

The notations $(1,m)$ and $(2,n)$ in the above definitions mark the position and name of the fixed index.

Note that, once the biorthogonal sequences $(\Theta^{1,\pm}_{mn})_{n\in\mathbb{N}^*}$ and $(\Theta^{2,\pm}_{mn})_{m\in\mathbb{N}^*}$ are available, controls $v$ acting on one edge or $(v_1, v_2)$ acting on two adjacent edges may be obtained as linear combinations of the biorthogonals. However, some conditions have to be imposed to the Fourier coefficients $\alpha_{mn}$ of the initial data $(u^0, u^1)$ to ensure the convergence of these combinations in $L^2$. The conditions are directly related with the magnitude of the norms of the biorthogonal elements. Therefore, the first thing to do is to construct and evaluate the norm of the biorthogonal sequences $(\Theta^{1,\pm}_{mn})_{n\in\mathbb{N}^*}$ and $(\Theta^{2,\pm}_{mn})_{m\in\mathbb{N}^*}$ for any $m$ and $n$ respectively. This is our aim in the present section. In the following one we use this information to give the wanted answers to the controllability problems.

Let us define

$$\xi^{1+}_{mn}(z) = \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}} \left( \frac{1 - \frac{z}{i\lambda^+_{mk}}}{1 - \frac{\lambda^+_{mn}}{\lambda^+_{mk}}} \right) \prod_{k\in\mathbb{N}^*} \left( \frac{1 - \frac{z}{i\lambda^-_{mk}}}{1 - \frac{\lambda^+_{mn}}{\lambda^-_{mk}}} \right). \tag{24}$$

LEMMA 3.1 *Let $\xi^{1+}_{mn}(z)$ be defined as above. Then,*

$$\xi^{1+}_{mn}(i\lambda^+_{ml}) = \begin{cases} 0 & \text{if } l \neq n, \\ 1 & \text{if } l = n \end{cases}, \qquad \xi^{1+}_{mn}(i\lambda^-_{ml}) = 0 \tag{25}$$

*Moreover, $\xi^{1+}_{mn}$ is an entire function of exponential type, i.e. there exist two constants $A_{mn}$, depending on $m$ and $n$, and $B$, independent of $m$ and $n$, such that*

$$|\xi^{1+}_{mn}(z)| \leq A_{mn} e^{B|z|}, \quad \forall z \in \mathbb{C}. \tag{26}$$

*Proof:* We have that $|\xi^{1+}_{mn}(z)| = Q_{mn}\, P_{mn}(z)$ where

$$P_{mn}(z) = \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}} \left| 1 - \frac{z}{i\lambda^+_{mk}} \right| \prod_{k\in\mathbb{N}^*} \left| 1 - \frac{z}{i\lambda^-_{mk}} \right|, \quad Q_{mn} = \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}} \left| \frac{1}{1 - \frac{\lambda^+_{mn}}{\lambda^+_{mk}}} \right| \prod_{k\in\mathbb{N}^*} \left| \frac{1}{1 - \frac{\lambda^+_{mn}}{\lambda^-_{mk}}} \right|.$$

$$Q_{mn} = \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}} \left| \frac{1}{1 - \frac{\lambda^+_{mn}}{\lambda^+_{mk}}} \right| \prod_{k\in\mathbb{N}^*} \left| \frac{1}{1 - \frac{\lambda^+_{mn}}{\lambda^-_{mk}}} \right| = \frac{1}{2} \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}} \frac{(\lambda^+_{mk})^2}{|(\lambda^+_{mk})^2 - (\lambda^+_{mn})^2|} =$$
$$= \frac{1}{2} \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}} \frac{\frac{b^2}{a^2} m^2 + k^2}{|k^2 - n^2|} = \frac{n^2}{p^2 + n^2} \prod_{k\in\mathbb{N}^*} \frac{p^2 + k^2}{k^2}$$

where $p = \frac{b}{a} m$. Consequently,

$$Q_{mn} = \frac{n^2}{p^2 + n^2} \prod_{k\in\mathbb{N}^*} \frac{p^2 + k^2}{k^2}. \tag{27}$$

By taking into account the Euler formula

$$\frac{\sin(\pi z)}{\pi z} = \prod_{k\in\mathbb{N}^*}\left(1-\frac{z^2}{k^2}\right) \tag{28}$$

we deduce that

$$Q_{mn} = \frac{n^2}{p^2+n^2}\frac{\sin i\pi p}{i\pi p}. \tag{29}$$

Also, from (29), the following estimate may be deduced

$$Q_{mn} \le \frac{2n^2\pi^2}{b^2|\lambda^{\pm}_{mn}|^2}\exp\left(\frac{\pi b}{a}m\right). \tag{30}$$

On the other hand,

$$P_{mn}(z) = \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}}\left|1-\frac{z}{i\lambda^+_{mk}}\right|\prod_{k\in\mathbb{N}^*}\left|1-\frac{z}{i\lambda^-_{mk}}\right| = \left|1-\frac{z}{i\lambda^-_{mn}}\right|\prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}}\left|1+\frac{z^2}{(\lambda^+_{mk})^2}\right| \le$$

$$\le \left(1+\frac{b|z|}{\pi n}\right)\prod_{k\in\mathbb{N}^*}\left(1+\frac{b^2|z|^2}{\pi^2k^2}\right) \le \left(1+\frac{b|z|}{\pi n}\right)\exp\left(\int_0^\infty \ln\left(1+\frac{b^2|z|^2}{\pi^2t^2}\right)dt\right) =$$

$$= \left(1+\frac{b|z|}{\pi n}\right)\exp\left(\frac{b|z|}{\pi}\int_0^\infty \ln\left(1+\frac{1}{s^2}\right)ds\right) = \left(1+\frac{b|z|}{\pi n}\right)\exp\left(b|z|\right).$$

The proof ends by considering $A_{mn} = \frac{2\,\pi^2\,n^2}{b^2\,|\lambda^{\pm}_{mn}|^2}\exp\left(\frac{\pi\,b}{a}\,m\right)$ and $B$ any number strictly larger that $b$.

■

The most important element of our construction is the following evaluation of $\xi^{1+}_{mn}$ on the imaginary axis.

Lemma 3.2 *The following estimate holds for the function $\xi^{1+}_{mn}$ on the imaginary axis*

$$\left|\xi^{1+}_{mn}(x\,i)\right| \le C\times\begin{cases}\exp\left(b\sqrt{\dfrac{m^2\pi^2}{a^2}-x^2}\right) & |x|\le\frac{m\pi}{a}\\ 1 & \frac{m\pi}{a}<|x|\le\frac{m\pi}{a}\sqrt{1+\frac{a^2n^2}{4b^2m^2}}\\ \dfrac{b}{n\pi}\lambda^+_{mn} & \frac{m\pi}{a}\sqrt{1+\frac{a^2n^2}{4b^2m^2}}<|x|\end{cases} \tag{31}$$

*where $C$ is a positive constant independent of $m$ and $n$.*

*Proof:* We have that, for $x \in I\!R$,

$$\xi^{1+}_{mn}(ix) = \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}} \left( \frac{1-\frac{x}{\lambda^+_{mk}}}{1-\frac{\lambda^+_{mn}}{\lambda^+_{mk}}} \right) \prod_{k\in\mathbb{N}^*} \left( \frac{1-\frac{x}{\lambda^-_{mk}}}{1-\frac{\lambda^+_{mn}}{\lambda^-_{mk}}} \right) =$$

$$= \frac{1}{2}\left(\frac{\lambda^+_{mn}+x}{\lambda^+_{mn}}\right) \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}} \frac{(\lambda^+_{mk})^2}{(\lambda^+_{mk})^2-(\lambda^+_{mn})^2} \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}} \frac{(\lambda^+_{mk})^2-x^2}{(\lambda^+_{mk})^2}.$$

The first product is evaluated by (27). Let us now evaluate the second product

$$P_{mn}(u) = \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}} \frac{(\lambda^+_{mk})^2-x^2}{(\lambda^+_{mk})^2} = \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}} \frac{p^2+k^2-u^2}{p^2+k^2}$$

where $p=\frac{b}{a}m$ and $\frac{bx}{\pi}=u$. We analyze separately the cases $0\le u\le p$ and $p>u$.

*Case 1:* $0\le u\le p$. Let us denote $v=\sqrt{p^2-u^2}$. We have that

$$|P_{mn}(u)| = \frac{p^2+n^2}{p^2+n^2-u^2} \prod_{k\in\mathbb{N}^*} \frac{p^2+k^2-u^2}{p^2+k^2} = \frac{p^2+n^2}{n^2+v^2} \prod_{k\in\mathbb{N}^*} \frac{k^2+v^2}{k^2+p^2} =$$

$$= \frac{p^2+n^2}{n^2+v^2} \prod_{k\in\mathbb{N}^*} \frac{k^2}{k^2+p^2} \prod_{k\in\mathbb{N}^*} \frac{k^2+v^2}{k^2} = \frac{p^2+n^2}{n^2+v^2} \prod_{k\in\mathbb{N}^*} \frac{k^2}{k^2+p^2} \frac{\sin i\pi v}{i\pi v}.$$

By taking into account (27), it follows that,

$$|\xi^{1+}_{mn}(ix)| = \frac{1}{2}\left|\frac{\lambda^+_{mn}+x}{\lambda^+_{mn}}\right| \frac{\frac{n^2\pi^2}{b^2}}{(\lambda^+_{mn})^2-x^2} \frac{\sin\left(i\,b\sqrt{\frac{m^2\pi^2}{a^2}-x^2}\right)}{i\,b\sqrt{\frac{m^2\pi^2}{a^2}-x^2}}, \quad \text{if } |x|\le \frac{m\pi}{a}. \tag{32}$$

Since in this case $|x|\le\lambda^+_{mn}$ and $(\lambda^+_{mn})^2-x^2\ge\frac{n^2\pi^2}{b^2}$, we obtain that

$$|\xi^{1+}_{mn}(ix)| \le \exp\left(b\sqrt{\frac{m^2\pi^2}{a^2}-x^2}\right), \quad \text{if } |x|\le\frac{m\pi}{a}. \tag{33}$$

*Case 2:* $p<u$. Let us denote $v=\sqrt{u^2-p^2}$ and remark that, if $v=[v]+\alpha$, with $\alpha\in[0,1)$,

$$|P_{mn}(u)| = \prod_{\substack{k\in\mathbb{N}^*\\ k\neq n}} \frac{|k^2-v^2|}{k^2+p^2} = \frac{n^2+p^2}{|n^2-v^2|} \prod_{k\in\mathbb{N}^*} \frac{1}{k^2+p^2} \prod_{k\in\mathbb{N}^*} (v+k) \prod_{1\le k\le[v]} (v-k) \prod_{k>[v]} (k-v) =$$

$$= \frac{n^2+p^2}{|n^2-v^2|} \prod_{k\in\mathbb{N}^*} \frac{1}{k^2+p^2} \prod_{[v]+1\le j} (j+\alpha) \prod_{0\le j\le [v]-1} (j+\alpha) \prod_{j\in\mathbb{N}^*} (j-\alpha) =$$

$$= \frac{n^2+p^2}{|n^2-v^2|} \frac{\alpha}{[v]+\alpha} \prod_{k\in\mathbb{N}^*} \frac{1}{k^2+p^2} \prod_{j\in\mathbb{N}^*} (j^2-\alpha^2) = \frac{n^2+p^2}{|n^2-v^2|} \frac{\alpha}{[v]+\alpha} \prod_{k\in\mathbb{N}^*} \frac{k^2-\alpha^2}{k^2+p^2} =$$

$$= \frac{n^2+p^2}{|n^2-v^2|} \frac{\alpha}{[v]+\alpha} \prod_{k\in\mathbb{N}^*} \frac{k^2}{k^2+p^2} \prod_{k\in\mathbb{N}^*} \frac{k^2-\alpha^2}{k^2} = \frac{n^2+p^2}{|n^2-v^2|} \frac{\sin(\pi\alpha)}{\pi v} \prod_{k\in\mathbb{N}^*} \frac{k^2}{k^2+p^2}.$$

Now, from (27), it follows that

$$|\xi^{1+}_{mn}(ix)| = \frac{1}{2}\left|\frac{\lambda^+_{mn}+x}{\lambda^+_{mn}}\right| \frac{\frac{n^2\pi^2}{b^2}}{\left|(\lambda^+_{mn})^2-x^2\right|} \frac{\sin\left(\pi\left(\frac{b}{\pi}\sqrt{x^2-\frac{m^2\pi^2}{a^2}}-\left[\frac{b}{\pi}\sqrt{x^2-\frac{m^2\pi^2}{a^2}}\right]\right)\right)}{b\sqrt{x^2-\frac{m^2\pi^2}{a^2}}}.$$

In order to evaluate $|\xi^{1+}_{mn}(ix)|$ we analyze the following cases:

- If $\dfrac{m\pi}{a} < x \le \dfrac{m\pi}{a}\sqrt{1+\dfrac{a^2n^2}{4b^2m^2}}$ then $0<\sqrt{x^2-\frac{m^2\pi^2}{a^2}}<\frac{n\pi}{2b}$ and consequently

$$|\xi^{1+}_{mn}(ix)| = \frac{\frac{n^2\pi^2}{b^2}}{2\lambda^+_{mn}} \frac{x+\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}{\left|\frac{n^2\pi^2}{b^2}-\left(x^2-\frac{m^2\pi^2}{a^2}\right)\right|} \frac{\left|\sin\left(b\sqrt{x^2-\frac{m^2\pi^2}{a^2}}\right)\right|}{b\sqrt{x^2-\frac{m^2\pi^2}{a^2}}} \le \frac{\frac{n^2\pi^2}{b^2}}{2\lambda^+_{mn}} \frac{2\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}{\frac{n^2\pi^2}{b^2}-\frac{n^2\pi^2}{4b^2}} = \frac{4}{3}.$$

- If $\dfrac{m\pi}{a}\sqrt{1+\dfrac{a^2n^2}{4b^2m^2}} < x$ then

$$|\xi^{1+}_{mn}(ix)| = \frac{\frac{n^2\pi^2}{b^2}}{2\lambda^+_{mn}} \frac{x+\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}{\left(\frac{n\pi}{b}+\sqrt{x^2-\frac{m^2\pi^2}{a^2}}\right)\sqrt{x^2-\frac{m^2\pi^2}{a^2}}} \frac{\left|\sin\left(n\pi-b\sqrt{x^2-\frac{m^2\pi^2}{a^2}}\right)\right|}{\left|n\pi-b\sqrt{x^2-\frac{m^2\pi^2}{a^2}}\right|} \le$$

$$\le \frac{\frac{n^2\pi^2}{b^2}}{2\lambda^+_{mn}} \frac{x+\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}{x^2-\frac{m^2\pi^2}{a^2}} \le \frac{n\pi}{2b} \frac{x+\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}{\left(x+\frac{m\pi}{a}\right)\left(x-\frac{m\pi}{a}\right)}.$$

From $\dfrac{m\pi}{a}\sqrt{1+\dfrac{a^2n^2}{4b^2m^2}} < x$ we deduce that $\pi\sqrt{\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}} < 2x$. Moreover, since

$$x-\frac{m\pi}{a} > \frac{m\pi}{a}\sqrt{1+\frac{a^2n^2}{4b^2m^2}} - \frac{m\pi}{a} = \frac{\frac{\pi a n^2}{4b^2m}}{\sqrt{1+\frac{a^2n^2}{4b^2m^2}}+1} \ge \frac{\frac{\pi^2n^2}{4b^2}}{2\lambda^+_{mn}}$$

we deduce that

$$|\xi^{1+}_{mn}(ix)| \le \frac{12\,b}{n\,\pi}\lambda^+_{mn}.$$

The proof of the lemma is now complete.

■

LEMMA 3.3 *Let $m, n \in \mathbb{N}^*$ and $\varphi_{mn} : \mathbb{R} \to \mathbb{R}$ be the continuous function defined by*

$$\varphi_{mn}(x) = \begin{cases} 0, & x \le \lambda^+_{mn} - \frac{m\pi}{a}, \\ b\sqrt{\frac{m^2\pi^2}{a^2} - (\lambda^+_{mn} - x)^2}, & x \in \left(\lambda^+_{mn} - \frac{m\pi}{a}, \lambda^+_{mn}\right) \\ b\frac{m\pi}{a}, & x \ge \lambda^+_{mn}. \end{cases} \tag{34}$$

*There exists an entire function $G_{mn}$ of exponential type less or equal than $\dfrac{e\,b}{\sqrt{1 + \frac{a^2}{b^2}\frac{n^2}{m^2}} - 1}$ such that*

$$|G_{mn}(x)| \le e\,\exp\left(-\varphi_{mn}(x)\right), \quad \forall x \in \mathbb{R}, \tag{35}$$

$$|G_{mn}(0)| = 1. \tag{36}$$

*Proof:* We use an idea of A. E. Ingham [13], generalized by R. M. Redheffer [35]. We introduce the function

$$A(u) = [\varphi_{mn}(eu)], \quad u \ge 0 \tag{37}$$

and we define the product

$$G_{mn}(z) = \prod_{k=1}^{\left[\frac{bm\pi}{a}\right]} \frac{\sin(\rho_k z)}{\rho_k z} \tag{38}$$

where $\frac{1}{\rho_k}$ denotes the $k-$th discontinuity point of the function $A$. In (37) and in the sequel $[a]$ denotes the integer part of $a$.

The sequence $(\rho_k)_k$ is decreasing and

$$\sum_{k=1}^{\left[\frac{bm\pi}{a}\right]} \rho_n = \sum_{k=1}^{\left[\frac{bm\pi}{a}\right]} \frac{e}{\lambda^+_{mn} - \sqrt{\frac{m^2\pi^2}{a^2} - \frac{k^2}{b^2}}} \le$$

$$\le e\int_0^{\frac{bm\pi}{a}} \frac{1}{\lambda_{mn}^{+}-\sqrt{\frac{m^2\pi^2}{a^2}-\frac{t^2}{b^2}}}dt = e\,b\int_0^1 \frac{1}{\sqrt{1+\frac{a^2}{b^2}\frac{n^2}{m^2}}-\sqrt{1-s^2}}ds \le \frac{e\,b}{\sqrt{1+\frac{a^2}{b^2}\frac{n^2}{m^2}}-1}.$$

Since

$$\left|\frac{\sin(\rho_k z)}{\rho_k z}\right| = \left|\sum_{i\ge 0}(-1)^i\frac{(\rho_k z)^{2i}}{(2i+1)!}\right| \le \sum_{i\ge 0}\frac{|\rho_k z|^{2i}}{(2i)!} \le \exp(\rho_k |z|)$$

we have that

$$|G_{mn}(z)| = \prod_{k=1}^{\left[\frac{bm\pi}{a}\right]}\left|\frac{\sin(\rho_k z)}{\rho_k z}\right| \le \exp\left(|z|\sum_{k=1}^{\left[\frac{bm\pi}{a}\right]}\rho_k\right)$$

and $G_{mn}$ is an entire function of exponential type less or equal than $\frac{e\,b}{\sqrt{1+\frac{a^2}{b^2}\frac{n^2}{m^2}}-1}$ .

Moreover, $G_{mn}(0)=1$.

We pass to evaluate $G_{mn}(x)$. Evidently, $|G_{mn}(x)|\le 1$ for any $x\in\mathbb{R}$. For $x\ge \lambda_{mn}^{+}-\frac{m\pi}{a}$,

$$|G_{mn}(x)| \le \prod_{\rho_k\,x\ge 1}\frac{|\sin(\rho_k\,x)|}{|\rho_k\,x|} \le \prod_{\rho_k\,x\ge 1}\frac{1}{\rho_k\,|x|} = \exp\left(\sum_{\rho_k\,x\ge 1}\ln\left(\frac{1}{\rho_k\,x}\right)\right).$$

Since

$$\sum_{\rho_k\,x\ge 1}\ln\left(\frac{1}{\rho_k\,x}\right) = -\int_0^x\frac{A(u)}{u}du \le -\int_{\frac{x}{e}}^x\frac{\varphi_{mn}(eu)-1}{u}du \le -\int_{\frac{x}{e}}^x\frac{\varphi_{mn}(x)-1}{u}du \le 1-\varphi_{mn}(x)$$

we deduce that $|G_{mn}(x)|\le e\exp(-\varphi_{mn}(x))$ and the proof ends.

■

We define now

$$M_{mn}(z) = G_{mn}(\lambda_{mn}^{+}-z). \tag{39}$$

Evidently, $M_{mn}$ is an entire function of exponential type less or equal than $\dfrac{e\,b}{\sqrt{1+\frac{a^2}{b^2}\frac{n^2}{m^2}}-1}$ such that $M_{mn}(\lambda_{mn}^{+})=1$. Moreover,

$$|M_{mn}(x)| = |G_{mn}(\lambda_{mn}^{+}-x)| \le e\times\begin{cases} 1 & x\ge\frac{m\pi}{a}\\ \exp\left(-b\sqrt{\dfrac{m^2\pi^2}{a^2}-x^2}\right) & 0\le x\le\frac{m\pi}{a}\\ \exp\left(-\dfrac{bm\pi}{a}\right) & x\le 0\end{cases} \tag{40}$$

We are now ready to prove the desired result on the existence of a biorthogonal sequence:

Theorem 3.3 *Let $T > 2b\left(1 + e\,(\sqrt{2}+1)\right)$. For any $m \in \mathbb{N}^*$ there exists a $(1,m)-$biorthogonal $(\Theta^{1,\pm}_{mn})_{n\in\mathbb{N}^*} \subset L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ to the family $\Lambda$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ such that*

$$||\widehat{\Theta}^{1,\pm}_{mn}||_{L^\infty(\mathbb{R})} \leq C, \quad \forall\, n \geq \frac{b}{a}m \tag{41}$$

$$||\widehat{\Theta}^{1,\pm}_{mn}||_{L^\infty(\mathbb{R})} \leq C \exp\left(\frac{b}{a}m\pi\right), \quad \forall\, n < \frac{b}{a}m \tag{42}$$

*where $C$ is a positive constant independent of $m$ and $n$.*

**Remark 2** *In Theorem 3.3 and in the sequel $\widehat{f}$ denotes the Fourier transform of the function $f \in L^2(-B,B)$ and it is defined by*

$$\widehat{f}(z) = \frac{1}{\sqrt{2\pi}}\int_{-B}^{B} f(t)e^{-izt}dt. \tag{43}$$

*We recall that, according to the inversion theorem,*

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{f}(x)e^{itx}dx, \tag{44}$$

*and from Plancherel's Theorem*

$$||f||_{L^2(-B,B)} = ||\widehat{f}||_{L^2(\mathbb{R})}. \tag{45}$$

*Finally, the convolution rule says that*

$$\widehat{f*g} = \sqrt{2\pi}\widehat{f}\cdot\widehat{g}, \quad \widehat{f\cdot g} = \frac{1}{\sqrt{2\pi}}\widehat{f}*\widehat{g}. \tag{46}$$

$\square$

*Proof:* For $n \geq \frac{b}{a}m$, we define

$$\Xi^+_{mn}(z) = \xi^{1+}_{mn}(i\,z)M_{mn}(z)\frac{\sin(\delta(z-\lambda^+_{mn}))}{\delta(z-\lambda^+_{mn})}, \tag{47}$$

where $\xi^{1+}_{mn}$ and $M_{mn}$ are given by (24) and (39) respectively and $\delta > 0$ is arbitrary.

We have that

- $\Xi^+_{mn}(\lambda^+_{ml}) = \delta_{nl}, \quad \Xi^+_{mn}(\lambda^-_{ml}) = 0, \qquad n \geq \frac{b}{a}m, \quad l \in \mathbb{N}^*.$

- $\Xi^+_{mn}$ is an entire function of exponential type less or equal than

(48)
$$B = b + \frac{e\,b}{\sqrt{1+\frac{a^2}{b^2}\frac{n^2}{m^2}}-1} + \delta \le b\left(1+e\,(\sqrt{2}+1)\right)+\delta$$

where, in the last inequality, we have taken into account that $n \ge \frac{b}{a}m$.

- $\Xi^+_{mn}(x) \in L^2(\mathbb{R})$.

Let $\Theta^{1,+}_{mn}$ be defined as follows from the Fourier transform of $\Xi^+_{mn}$,

(49)
$$\Theta^{1,+}_{mn}(z) = \frac{1}{\sqrt{2\pi}}\,\widehat{\Xi}^+_{mn}(z).$$

From the properties of $\Xi^+_{mn}$, by using Paley-Wiener Theorem, it follows that $\Theta^{1,+}_{mn}(t)$ has support included in $[-B,B]$, it belongs to $L^2(-B,B)$ and

$$\int_{-B}^{B}\Theta^{1,+}_{mn}(t)e^{-i\lambda^+_{ml}t}dt = \Xi^+_{mn}(\lambda^+_{ml}) = \delta_{nl},\ \int_{-B}^{B}\Theta^{1,+}_{mn}(t)e^{-i\lambda^-_{ml}t}dt = \Xi^+_{mn}(\lambda^-_{ml}) = 0,\ n \ge \frac{b}{a}m,\ l \in \mathbb{N}^*.$$

Moreover, from the decay property of $M_{mn}$ (40) and estimates (31) of $\xi^+_{mn}$ on the imaginary axis we obtain immediately that (41) is verified.

The elements $\Theta^{1,-}_{mn}$ are defined and evaluated as before, being the Fourier transforms of

(50)
$$\Xi^-_{mn}(z) = \xi^{1+}_{mn}(-i\,z)M_{mn}(-z)\frac{\sin(\delta(z-\lambda^-_{mn}))}{\delta(z-\lambda^-_{mn})}.$$

For $n < \frac{b}{a}m$, we define

(51)
$$\Xi^\pm_{mn}(z) = \xi^{1+}_{mn}(\pm i\,z)\frac{\sin(\delta(z-\lambda^\pm_{mn}))}{\delta(z-\lambda^\pm_{mn})},$$

and we introduce $\Theta^{1,\pm}_{mn}$ as in (49)

From estimate (31) of $\xi^{1,+}_{mn}$ on the imaginary axis we deduce that (42) holds.

It follows that $(\Theta^{1,\pm}_{mn})_{n\in\mathbb{N}^*}$ is a biorthogonal sequence to $\left(e^{\lambda^\pm_{mn}t}\right)_{n\in\mathbb{N}^*}$ in $L^2(-B,B)$ which verifies (41)-(42). The proof ends.

■

**Remark 3** *We may evaluate the $L^2-$norm of the biorthogonal too. If $n \ge \frac{b}{a}m$, we have that*

$$\int_{-\infty}^{\infty}|\Xi^\pm_{mn}(x)|^2dx \le e^2C^2\int_{-\infty}^{\infty}\left|\frac{\sin(\delta(x-\lambda^\pm_{mn}))}{\delta(x-\lambda^\pm_{mn})}\right|^2dx \le \frac{e^2\,C^2}{\delta}\int_{-\infty}^{\infty}\left|\frac{\sin(t)}{t}\right|^2dt = \frac{e^2\,C^2\,\pi}{\delta}.$$

*From Plancherel's Theorem we deduce that*

$$\| \Theta^{1,\pm}_{mn} \|_{L^2(-B,B)} = \| \Xi^{\pm}_{mn} \|_{L^2(\mathbb{R})} \leq \frac{e^2\, C^2\, \pi}{\delta}. \tag{52}$$

*Hence, the* $L^2-$*norm of* $(\Theta^{1,\pm}_{mn})_{n\geq \frac{b}{a}m}$ *is uniformly bounded in* $m$*. We shall construct a new biorthogonal with even better norm properties in Theorem 3.4.*

□

**Remark 4** *Note that, for any* $\sigma > 0$*, there exists* $T = T(\sigma)$ *with the property that there exists a* $(1, m)-$*biorthogonal* $(\Theta^{1,\pm}_{mn})_{n\in\mathbb{N}^*} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ *to the family* $\Lambda$ *such that*

$$||\widehat{\Theta}^{1,\pm}_{mn}||_{L^\infty(\mathbb{R})} \leq C, \quad \forall n \in \mathbb{N}^*, \quad n \geq \sigma m \tag{53}$$

*where* $C$ *is a positive constant independent of* $m$ *and* $n$*. Indeed, the exponential type* $B$ *of* $\Xi^{\pm}_{mn}$ *is, for* $n \geq \sigma m$*,*

$$B = b + \frac{e\, b}{\sqrt{1 + \frac{a^2}{b^2}\frac{n^2}{m^2}} - 1} + \delta \leq b + \frac{e\, b}{\sqrt{1 + \frac{a^2}{b^2}\sigma^2} - 1} + \delta.$$

*If we take*

$$T > b\left(1 + \frac{e}{\sqrt{1 + \frac{a^2}{b^2}\sigma^2} - 1}\right)$$

*the same proof as in Theorem 3.3 gives (53).*

*Hence, a larger set biorthogonals is uniformly bounded if we consider a larger time* $T$*. Note also that* $\lim_{\sigma\to 0} T = \infty$ *and consequently not all the elements of the biorthogonal sequence may be uniformly bounded in a given finite time* $T$*.*

*This type of results were already obtained in [1], where it is proved that, the space of controllable initial data from one part of the boundary increases if the time is larger.*

□

For any $\varepsilon > 0$, let

$$K_\varepsilon = \frac{\sqrt{2\pi}}{\epsilon^2}(\chi_\varepsilon * \chi_\varepsilon)$$

where $\chi_\varepsilon$ is the characteristic function of the interval $\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$. The following properties of $K_\varepsilon$ are almost immediate:

- $\mathrm{supp}(K_\varepsilon) \subseteq [-\varepsilon, \varepsilon]$

- $\widehat{K}_\varepsilon(\xi) = \frac{4}{\varepsilon^2}\frac{\sin^2\left(\frac{\varepsilon}{2}\xi\right)}{\xi^2}, \ \forall \xi \in \mathbb{R}\setminus\{0\},$

- $\widehat{K}_\varepsilon(0) = 1$.

We define $\rho^{\pm}_{mn}(x) = e^{i\lambda^{\pm}_{mn}x}K_\varepsilon(x)$ and we note that supp $(\rho^{\pm}_{mn}) \subseteq [-\varepsilon, \varepsilon]$.

The following useful result is a consequence of Theorem 3.3 and gives a new biorthogonal to the family $\Lambda$ in a slightly larger time interval but with better norm properties. It is inspired from [16] (see also [19] and [36]).

THEOREM 3.4 *Let $T > 2b\left(1 + e\left(\sqrt{2}+1\right)\right)$ and $(\Theta^{1,\pm}_{mn})_{n\in\mathbb{N}^*}$ be the $(1,m)-$biorthogonal given by Theorem 3.3. For any $\varepsilon > 0$, the family $(\Psi^{1,\pm}_{mn})_{n\in\mathbb{N}^*}$ given by*

$$\Psi^{1,\pm}_{mn} = \frac{1}{2\pi}\Theta^{1,\pm}_{mn} * \rho^{\pm}_{mn} \tag{54}$$

*is an $(1,m)-$biorthogonal to the family $\Lambda$ in $L^2\left(-\frac{T}{2}-\varepsilon, \frac{T}{2}+\varepsilon\right)$ and there exists a positive constant $C$ independent of $m$ and $n$ such that*

$$\int_{-\frac{T}{2}-\varepsilon}^{\frac{T}{2}+\varepsilon} \left|\sum_{n\in\mathbb{N}^*} c^{\pm}_n \Psi^{1,\pm}_{mn}(t)\right|^2 dt \leq C\left[e^{\frac{2b}{a}m\pi}\sum_{n<\frac{b}{a}m}|c^{\pm}_n|^2 + \sum_{n\geq\frac{b}{a}m}|c^{\pm}_n|^2\right] \tag{55}$$

*for any finite sequence $(c^{\pm}_n)_{n\in\mathbb{N}^*}$.*

*Proof:* Evidently, $\operatorname{supp}(\Psi^{1,\pm}_{mn}) \subset \left(-\frac{T}{2}-\varepsilon, \frac{T}{2}+\varepsilon\right)$ and $(\Psi^{1,\pm}_{mn})_{n\in\mathbb{N}^*} \subset L^2\left(-\frac{T}{2}-\varepsilon, \frac{T}{2}+\varepsilon\right)$. Moreover,

$$\int_{-\frac{T}{2}-\varepsilon}^{\frac{T}{2}+\varepsilon} \Psi^{1,\pm}_{mn}(t)e^{-i\lambda^{\pm}_{ml}t}dt = \sqrt{2\pi}\widehat{\Psi}^{1,\pm}_{mn}(\lambda^{\pm}_{ml}) = \widehat{\Theta}^{1,\pm}_{mn}(\lambda^{\pm}_{ml})\widehat{\rho}^{\pm}_{mn}(\lambda^{\pm}_{ml}) = \delta_{nl}.$$

On the other hand,we have that

$$\int_{-\frac{T}{2}-\varepsilon}^{\frac{T}{2}+\varepsilon} \left|\sum_{n\in\mathbb{N}^*} c^{\pm}_n \Psi^{1,\pm}_{mn}(t)\right|^2 dt = \frac{1}{2\pi}\int_{\mathbb{R}}\left|\sum_{n\in\mathbb{N}^*} c^{\pm}_n \widehat{\Theta}^{1,\pm}_{mn}(x)\widehat{\rho}^{\pm}_{mn}(x)\right|^2 dx \leq$$

$$\leq \frac{1}{2\pi}\int_{\mathbb{R}}\left(\sum_{n\in\mathbb{N}^*}|c^{\pm}_n|\left|\left|\widehat{\Theta}^{1,\pm}_{mn}\right|\right|_{L^\infty(\mathbb{R})}|\widehat{K}_\varepsilon(x-\lambda^{\pm}_{mn})|\right)^2 dx =$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\left(K_\varepsilon(t)\sum_{n\in\mathbb{N}^*}|c^{\pm}_n|\left|\left|\widehat{\Theta}^{1,\pm}_{mn}\right|\right|_{L^\infty(\mathbb{R})}e^{i\lambda^{\pm}_{mn}t}\right)^2 dx \leq$$

$$\leq \frac{2}{\varepsilon^2}\int_{-\varepsilon}^{\varepsilon}\left|\sum_{n<\frac{b}{a}m}|c^{\pm}_n|\left|\left|\widehat{\Theta}^{1,\pm}_{mn}\right|\right|_{L^\infty(\mathbb{R})}e^{i\lambda^{\pm}_{mn}t}\right|^2 dt + \frac{2}{\varepsilon^2}\int_{-\varepsilon}^{\varepsilon}\left|\sum_{n\geq\frac{b}{a}m}|c^{\pm}_n|\left|\left|\widehat{\Theta}^{1,\pm}_{mn}\right|\right|_{L^\infty(\mathbb{R})}e^{i\lambda^{\pm}_{mn}t}\right|^2 dt.$$

Now, we note that $|\lambda^{\pm}_{mn+1} - \lambda^{\pm}_{mn}| \geq \frac{\pi}{2\sqrt{2}b}$ if $n \geq \frac{b}{a}m$. Since the family $(\lambda^{\pm}_{mn})_{n\geq \frac{b}{a}m}$ has uniform gap we use Ingham's inequality [12] and obtain that

$$\int_{-\varepsilon}^{\varepsilon} \left| \sum_{n\geq \frac{b}{a}m} |c^{\pm}_n| \left|\left|\widehat{\Theta}^{1,\pm}_{mn}\right|\right|_{L^\infty(\mathbb{R})} e^{i\lambda^{\pm}_{mn}t} \right|^2 dt \leq C \sum_{n\geq \frac{b}{a}m} |c^{\pm}_n|^2 \left|\left|\widehat{\Theta}^{1,\pm}_{mn}\right|\right|^2_{L^\infty(\mathbb{R})}$$

with $C$ a positive constant independent of $n$ and $m$.

On the other hand, for the family $(\lambda^{\pm}_{mn})_{n< \frac{b}{a}m}$ we have that

$$\int_{-\varepsilon}^{\varepsilon} \left| \sum_{n< \frac{b}{a}m} |c^{\pm}_n| \left|\left|\widehat{\Theta}^{1,\pm}_{mn}\right|\right|_{L^\infty(\mathbb{R})} e^{i\lambda^{\pm}_{mn}t} \right|^2 dt \leq 2\varepsilon\frac{b^2}{a^2}m^2 \sum_{n< \frac{b}{a}m} |c^{\pm}_n|^2 \left|\left|\widehat{\Theta}^{1,\pm}_{mn}\right|\right|^2_{L^\infty(\mathbb{R})}.$$

Estimate (55) follows from the last two inequalities and Theorem 3.3.

■

Similar results hold for the other type of biorthogonal sequence. We state them without proofs.

THEOREM 3.5 *Let $T > 2a\left(1 + e\left(\sqrt{2}+1\right)\right)$. For any $n \in \mathbb{N}^*$ there exists a $(2,n)-$biorthogonal $(\Theta^{2,\pm}_{mn})_{m\in\mathbb{N}^*} \subset L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ to the family $\Lambda$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ such that*

$$||\widehat{\Theta}^{1,\pm}_{mn}||_{L^\infty(\mathbb{R})} \leq C, \quad \forall\, m \geq \frac{a}{b}n \tag{56}$$

$$||\widehat{\Theta}^{1,\pm}_{mn}||_{L^\infty(\mathbb{R})} \leq C\exp\left(\frac{a}{b}n\pi\right), \quad \forall\, m < \frac{a}{b}n \tag{57}$$

*where $C$ is a positive constant independent of $m$ and $n$.*

THEOREM 3.6 *Let $T > 2a\left(1 + e\left(\sqrt{2}+1\right)\right)$ and $(\Theta^{2,\pm}_{mn})_{m\in\mathbb{N}^*}$ be the $(2,n)-$biorthogonal given by Theorem 3.5. For any $\varepsilon > 0$, the family $(\Psi^{2,\pm}_{mn})_{m\in\mathbb{N}^*}$ given by*

$$\Psi^{2,\pm}_{mn} = \frac{1}{2\pi}\Theta^{2,\pm}_{mn} * \rho^{\pm}_{mn} \tag{58}$$

*is an $(2,n)-$biorthogonal to the family $\Lambda$ in $L^2\left(-\frac{T}{2}-\varepsilon,\frac{T}{2}+\varepsilon\right)$ and there exists a positive constant $C$ independent of $m$ and $n$ such that*

$$\int_{-\frac{T}{2}-\varepsilon}^{\frac{T}{2}+\varepsilon} \left| \sum_{m\in\mathbb{N}^*} c^{\pm}_m \Psi^{2,\pm}_{mn}(t)\right|^2 dt \leq C\left[e^{\frac{2a}{b}n\pi} \sum_{m<\frac{a}{b}n} |c^{\pm}_m|^2 + \sum_{m\geq\frac{a}{b}n} |c^{\pm}_m|^2\right] \tag{59}$$

*for any finite sequence $(c^{\pm}_m)_{m\in\mathbb{N}^*}$.*

# 4 Controllability results

We pass to study the controllability problem (10)-(11). Let the initial data be

$$(u^0, u^1) = \sum_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*} \alpha_{mn}^{\pm}\Phi_{mn}^{\pm}. \tag{60}$$

In the sequel $(\Psi_{mn}^{1,\pm})_{n\in\mathbb{N}^*}$ and $(\Psi_{mn}^{2,\pm})_{m\in\mathbb{N}^*}$ are the biorthogonal sequences in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ given by Theorems 3.4 and 3.6 respectively.

THEOREM 4.1 *Let $T > 2\,b\left(1+e\,(\sqrt{2}+1)\right)$. Then, any initial data (60) such that*

$$\sum_{n\in\mathbb{N}^*}\left(e^{2\frac{a}{b}n}\sum_{m<\frac{a}{b}n}|\alpha_{mn}^{\pm}|^2+\sum_{m\geq\frac{a}{b}n}\frac{|\alpha_{mn}^{\pm}|^2}{m^2+n^2}\right)<\infty \tag{61}$$

*may be driven to zero by a control $v\in L^2((0,T)\times(0,b))$. Moreover, $v$ may be taken as follows*

$$v(t,y)=\sum_{n\in\mathbb{N}^*}\sin\left(\frac{n\pi y}{b}\right)\left(\sum_{m\in\mathbb{N}^*}\frac{(-1)^{m+1}}{m\pi}\frac{4\sqrt{2}}{b\sqrt{ab}}\alpha_{mn}^{\pm}e^{-i\lambda_{mn}^{\pm}\frac{T}{2}}\Psi_{mn}^{2,\pm}\left(t-\frac{T}{2}\right)\right). \tag{62}$$

*Proof:* It is easy to see that, from the properties of the biorthogonal sequence $(\Psi_{mn}^{2,\pm})_{m\in\mathbb{N}^*}$, $v$ given by (62) verifies (20). Hence, we only have to prove that the series from (62) converges in $L^2((0,T)\times(0,b))$. Indeed, this follows immediately from the estimate

$$\int_0^T\left\|\sum_{n\in\mathbb{N}^*}\sin\left(\frac{n\pi y}{b}\right)\left(\sum_{m\in\mathbb{N}^*}\frac{(-1)^{m+1}}{m\pi}\frac{4\sqrt{2}}{b\sqrt{ab}}\alpha_{mn}^{\pm}e^{-i\lambda_{mn}^{\pm}\frac{T}{2}}\Psi_{mn}^{2,\pm}\left(t-\frac{T}{2}\right)\right)\right\|_{L^2(0,b)}^2=$$

$$=\frac{b}{2}\int_0^T\sum_{n\in\mathbb{N}^*}\left|\sum_{m\in\mathbb{N}^*}\frac{(-1)^{m+1}}{m\pi}\frac{4\sqrt{2}}{b\sqrt{ab}}\alpha_{mn}^{\pm}e^{-i\lambda_{mn}^{\pm}\frac{T}{2}}\Psi_{mn}^{2,\pm}\left(t-\frac{T}{2}\right)\right|^2\leq$$

$$\leq C\sum_{n\in\mathbb{N}^*}\left(e^{2\frac{a}{b}n\pi}\sum_{m<\frac{a}{b}n}|\alpha_{mn}^{\pm}|^2+\sum_{m\geq\frac{a}{b}n}\frac{|\alpha_{mn}^{\pm}|^2}{m^2+n^2}\right)$$

where in the last inequality we have used (59) from Theorem 3.6.

■

**Remark 5** *Theorem 4.1 shows which part of the spectrum may be uniformly controlled from the edge $\{(a,y): y\in(0,b)\}$: the eigenmodes $(m,n)$ such that $n\geq\frac{b}{a}m$. This result improves the one obtained in [14] where the condition $n\geq\exp(m)$ was obtained.*

□

**Remark 6** *In [7] the equivalent moment problem and biorthogonal estimates are used to study the controllability properties of the linear wave equation in a parallelepiped from one face. It is proved that any initial data* $(u^0, u^1)$ *like in (60) such that*

$$\sum_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*} |\alpha_{mn}^{\pm}|^2 e^{2\frac{a}{b}n} = \tau(u^0,u^1) < \infty$$

*are controllable from the edge* $\{(a,y)\,:\,y\in(0,b)\}$ *with a control* $v\in L^2((0,T)\times(0,b))$ *such that*

$$||v||_{L^2((0,T)\times(0,b))} \le C\tau(u^0,u^1).$$

*Note that our space of controllable initial data given by Theorem 4.1 is larger, since on the range* $m \ge \frac{a}{b}n$ *the conditions on the Fourier coefficients are less restrictive.*

*Let us also mention that, in [7], it is proved that there exists infinitely differentiable initial data which are not controllable from one face. This is obtained by proving an estimate from below for the biorthogonal sequence.*

□

Let us now pass to the problem with a control acting on two adjacent edges.

THEOREM 4.2 *Let* $T > 2\max\left\{a + e\,(\sqrt{2}+1)a,\; b + e\,(\sqrt{2}+1)b\right\}$. *Then, any initial data (60) such that*

$$\sum_{m\in\mathbb{N}^*}\sum_{n\in\mathbb{N}^*} \frac{|a_{mn}^{\pm}|^2}{m^2+n^2} < \infty \tag{63}$$

*may be driven to zero by a control* $(v^1,v^2)\in L^2((0,T)\times(0,b))\times L^2((0,T)\times(0,a))$. *Moreover,* $(v^1,v^2)$ *may be taken as follows*

$$\begin{aligned} v^1(t,y) &= \sum_{n\in\mathbb{N}^*} \sin\left(\frac{n\pi y}{b}\right)\left(\sum_{m\in\mathbb{N}^*,\,\frac{m}{a}>\frac{n}{b}} \frac{(-1)^{m+1}}{m\pi}\,\frac{4\sqrt{2}}{b\sqrt{ab}}\alpha_{mn}^{\pm} e^{-i\lambda_{mn}^{\pm}\frac{T}{2}}\Psi_{mn}^{2,\pm}\left(t-\frac{T}{2}\right)\right)\\ v^2(t,x) &= \sum_{m\in\mathbb{N}^*} \sin\left(\frac{m\pi x}{a}\right)\left(\sum_{n\in\mathbb{N}^*,\,\frac{n}{b}\ge\frac{m}{a}} \frac{(-1)^{n+1}}{n\pi}\,\frac{4\sqrt{2}}{a\sqrt{ab}}\alpha_{mn}^{\pm} e^{-i\lambda_{mn}^{\pm}\frac{T}{2}}\Psi_{mn}^{1,\pm}\left(t-\frac{T}{2}\right)\right). \end{aligned} \tag{64}$$

*Proof:* It is easy to see that

$$(-1)^{k+1}\frac{k\pi}{a}\int_0^T e^{-i\lambda_{kl}^{\pm}t}\int_0^b v^1(t,y)\sin\left(\frac{l\pi y}{b}\right)dydt = \begin{cases} 0 & \text{if } \frac{l}{b}\ge\frac{k}{a}\\ \dfrac{2\sqrt{2}}{\sqrt{ab}}\alpha_{kl}^{\pm} & \text{if } \frac{l}{b}<\frac{k}{a}, \end{cases}$$

$$(-1)^{l+1}\frac{l\pi}{b}\int_0^T e^{-i\lambda_{kl}^{\pm}t}\int_0^a v^2(t,x)\sin\left(\frac{k\pi x}{a}\right)dxdt = \begin{cases} 0 & \text{if } \frac{l}{b}<\frac{k}{a} \\ \frac{2\sqrt{2}}{\sqrt{ab}}\alpha_{kl}^{\pm} & \text{if } \frac{l}{b}\geq\frac{k}{a}. \end{cases}$$

It follows that (21) is verified by the controls (64). Moreover, inequalities (55)-(59) from Theorems 3.4 and 3.6 and condition (63) ensure the convergence of the series in (64) in $L^2((0,T)\times(0,b))$ and $L^2((0,T)\times(0,a))$ respectively.

■

**Remark 7** *Condition (63) is equivalent to $(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega)$, which is known to be the optimal space of controllable initial data.*

□

**Remark 8** *From estimate (59) on the $L^2-$norm of the biorthogonal sequence $(\Psi_{mn}^{2,\pm})_m$ we may deduce information about the dependence of the controls' norms with respect to the rage of frequencies of the initial data $(u^0,u^1)$. For instance, if we want to control from the edge $\{(a,y): y\in(0,b)\}$ an initial data of the form $(u^0,u^1)=\Phi_{mn}^{\pm}$ we may use the control*

$$v_{mn}(t,y)=\frac{(-1)^{m+1}}{m\pi}\frac{4\sqrt{2}}{b\sqrt{ab}}\sin\left(\frac{n\pi y}{b}\right)e^{-i\lambda_{mn}^{\pm}\frac{T}{2}}\Psi_{mn}^{2,\pm}\left(t-\frac{T}{2}\right).$$

*From (59) we deduce that*

$$||v_{mn}||_{L^2((0,T)\times(0,b))}\leq C\times\begin{cases}\frac{1}{m} & \textit{if } m\geq\frac{a}{b}n \\ \frac{1}{m}\exp\left(\frac{a}{b}n\pi\right) & \textit{if } m<\frac{a}{b}n\end{cases}$$

*where $C$ is a constant independent of $m$ and $n$.*

*Hence, the norm of the control $v_{mn}$ is bounded in the range $m\geq\frac{a}{b}n$ but may be exponentially large for $m<\frac{a}{b}n$. This gives a quantitative expression of the fact that it may be very costly to control, from a vertical edge, very oscillatory modes in the $x-$direction. In [7] estimates from below for the norms $||v_{mn}||$ are given, showing that these are indeed exponentially large for $n$ large enough.*

□

**Remark 9** *Note that the pair of controls given by (64) has the following alternating property: $v_1$ controls the eigenmodes $(m,n)$ such that $\frac{m}{a}>\frac{n}{b}$ and leaves unchanged the other ones. On the contrary, $v_2$ controls the eigenmodes $(m,n)$ such that $\frac{n}{b}\geq\frac{m}{a}$ and leaves unchanged the first ones.*

*Of course, other choices are possible for the controls. For instance,*

$$
(65)\qquad
\begin{aligned}
v^1(t,y) &= \frac{1}{2}\sum_{n\in\mathbb{N}^*}\sin\left(\frac{n\pi y}{2}\right)\left(\sum_{m\in\mathbb{N}^*}\frac{(-1)^{m+1}}{m\pi}\frac{4\sqrt{2}}{b\sqrt{ab}}\alpha_{mn}^{\pm}e^{-i\lambda_{mn}^{\pm}\frac{T}{2}}\Psi_{mn}^{2,\pm}\left(t-\frac{T}{2}\right)\right)\\
v^2(t,x) &= \frac{1}{2}\sum_{m\in\mathbb{N}^*}\sin\left(\frac{m\pi x}{2}\right)\left(\sum_{n\in\mathbb{N}^*}\frac{(-1)^{n+1}}{n\pi}\frac{4\sqrt{2}}{a\sqrt{ab}}\alpha_{mn}^{\pm}e^{-i\lambda_{mn}^{\pm}\frac{T}{2}}\Psi_{mn}^{1,\pm}\left(t-\frac{T}{2}\right)\right).
\end{aligned}
$$

*However, in this case, the space of controllable initial data is much smaller, since the presence of the biorthogonals $\Psi_{mn}^{1,\pm}$ with $\frac{n}{b}\geq\frac{m}{a}$ and $\Psi_{mn}^{2,\pm}$ with $\frac{m}{a}>\frac{n}{b}$, which have greater norms, imposes additional restrictive conditions on the corresponding coefficients $\alpha_{mn}$.*

□

**Remark 10** *The controllability time given by Theorem 4.2 is*

$$T > 2\max\left\{a + e\,(\sqrt{2}+1)a,\ b + e\,(\sqrt{2}+1)b\right\}$$

*which is greater than the known optimal one, $2\sqrt{a^2+b^2}$. Further investigations on the multiplier function $G_{mn}$ from Lemma 3.3 are needed to obtain this optimal time. More precisely, we should deduce the existence of a function $G_{mn}$ with the same properties as in Lemma 3.3 but with exponential type equal to $\sqrt{a^2+b^2}-b$. This remains an open problem.*

□

**Acknowledgement**: The authors are grateful to Enrique Zuazua and Marius Tucsnak for several interesting suggestions and discussions. Part of this work was done during the first author's visit at Universidad Nacional Autónoma de México.

## References

[1] B. Allibert, Analytic controllability of the wave equation over cylinder, *ESAIM: COCV*, **4** (1999), 177-207.

[2] S. A. Avdonin and S. A. Ivanov, *Families of exponentials. The method of moments in controllability problems for distributed parameter systems*, Cambridge University Press, 1995.

[3] C. Baiocchi, V. Komornik and P. Loreti, Ingham-Beurling type theorems with weakened gap condition, *Acta Math. Hungar.*, **97**, (2002), 55-95.

[4] J. Ball and M. Slemrod, Nonharmonic Fourier series and the stabilization of distributed semi-linear control systems, *Comm. Pure Appl. Math.*, XXXII (1979), 555-587.

[5] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, *SIAM J. Control Optim.*, **30** (1992), 1024-1065.

[6] R. P. Boas, *Entire functions*, Academic Press, New York, 1954.

[7] H. O. Fattorini, Estimates for sequences biorthogonal to certain exponentials and boundary control of the wave equation, *Lecture Notes in Control and Inform. Sci.*, 111-124.

[8] H. O. Fattorini and D.L. Russell, Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations, *Quarterly J. Appl. Math.*, **32** (1974), 45-69.

[9] H. O. Fattorini and D. L. Russell, Exact controllability theorems for linear parabolic equation in one space dimension, *Arch. Rat. Mech. Anal.*, **43** (1971), 272-292.

[10] S. Hansen, Bounds on functions biorthogonal to a set of complex expoenentials: Control of damped elastic system, *J. Math. Anal. Appl.*, **158** (1991), 487-508.

[11] A. Haraux, Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire , *J. Math. Pures et Appl.*, **68** (1989), 457-465.

[12] A. E. Ingham, Some trigonometrical inequalities with applications to the theory of series, *Math. Z.*, **41**(1936), 367-369.

[13] A. E. Ingham, A note on Fourier transform, *J. London Math. Soc.*, **9** (1934), pp. 29-32.

[14] S. Jaffard and S. Micu, Estimates of the constants in generalized Ingham's inequality and applications to the control of the wave equation, *Asymptotic Analysis*, **28** (2001), 181-214.

[15] S. Jaffard, M. Tucsnak and E. Zuazua, On a theorem of Ingham, *J. Fourier Anal. Appl.*, **3** (1997), 577-582.

[16] J. P. Kahane, Pseudo-Périodicité et Séries de Fourier Lacunaires, *Ann. Scient. Ec. Norm. Sup.*, **37** (1962), 93-95.

[17] V. Komornik, *Exact controllability and stabilization. The multiplier method.*, RAM: Research in Applied Mathematics, Masson, Paris, 1994.

[18] V. Komornik and P. Loreti, Ingham Type Theorems for Vector-Valued Functions and Observability of Coupled Linear Systems, *SIAM J. Control Optim.*, **37** (1998), 49-74.

[19] V. Komornik and P. Loreti, *Fourier Series in Control Theory*, Springer-Verlag, New York, 2005.

[20] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués, Tome 1: Contrôlabilité exacte*, Masson, RMA, Paris, 1988.

[21] A. López, X. Zhang and E. Zuazua, Null controllability of the heat equation as singular limit of the exact controllability of dissipative wave equations, *J. Math. Pures Appl.* **79**, 8 (2000) 741-808.

[22] M. Mehrenberger, An Ingham type proof for the boundary observ- ability of a N-d wave equation, *C.R. Acad. Sci. Paris, Série I*, **347** (2009), 63-68.

[23] S. Micu, Uniform boundary controllability of a semi-discrete 1-D wave equation, *Numer. Math.*, **91** (2002), 723-728.

[24] S. Micu and E. Zuazua, Boundary controllability of a linear hybrid system arising in the control of noise, *SIAM J. Control Optim.*, **35** (1997), 1614-1637.

[25] S. Micu and E. Zuazua, An introduction to the controllability of partial differential equations, Chapter in *Q*uelques questions de thorie du contrle, Ed. Tewfik Sari, Collection Travaux en Cours, Editions Hermann 2005, 69-157.

[26] L. Miller, Controllability cost of conservative systems: resolvent condition and transmutation, *J. Funct. Anal.*, **218** (2005), 425–444.

[27] M. Negreanu and E. Zuazua, Convergence of a multi-grid method for the controllability of a 1-d wave equation, *C.R. Acad. Sci. Paris, Série I*, **338**(5) (2004), 413-418.

[28] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in Complex Domains*, AMS Colloq. Publ., 19, 1934.

[29] D. L. Russell, A unified boundary controllability theory for hyperbolic and parabolic partial differential equations, *Studies in Appl. Math.*, **52** (1973), 189-221.

[30] D. L. Russell, Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions, *SIAM Rev.*, **20** (1978), 639-739.

[31] L. Schwartz, *Etude des sommes d'exponentielles*, Herman, Paris, 1959.

[32] K. Ramdani, T. Takahashi, G. Tenenbaum and M. Tucsnak, A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator, *J. Funct. Anal.*, **226** (2005), 193-229.

[33] K. Ramdani, T. Takahashi and M. Tucsnak, Internal stabilization of the plate equation in a square: the continuous and the semi-discretized problems, *J. Math. Pures Appl.*, **85** (2006), 17-37.

[34] K. Ramdani, T. Takahashi and M. Tucsnak, Uniformly Exponentially Stable Approximations for a Class of Second Order Evolution Equations, *ESAIM: COCV*, **13** (2007), 503-527.

[35] R. M. Redheffer, Completeness of sets of complex exponentials, *Advances in Mathematics*, **24** (1977), 1-62.

[36] M. Tucsnak and G. Weiss, *Observability and Controllability of Infinite Dimensional Systems*, to appear.

[37] R. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, 1980.

[38] J. Zabczyk, *Mathematical Control Theory: An Introduction*, Birkhäuser, Basel, 1992.

[39] E. Zuazua, Propagation, Observation, Control and Numerical Approximation of Waves approximated by finite difference methods, *SIAM Review*, **47** (2) (2005), 197-243.