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A SPHERICAL CR STRUCTURE ON THE COMPLEMENT OF THE FIGURE EIGHT KNOT WITH DISCRETE HOLONOMY

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Abstract

We describe a general geometrical construction of representations of fundamental groups of 3-manifolds into $PU(2,1)$ and eventually of spherical CR structures defined on those 3-manifolds. We construct branched spherical CR structures on the complement of the figure eight knot and the Whitehead link. They have discrete holonomies contained in $PU(2,1,\mathbb{Z}[\omega])$ and $PU(2,1,\mathbb{Z}[i])$ respectively.

1. Introduction

One of the most important examples of hyperbolic manifolds is the complement of the figure eight knot. It was shown by Riley in [**R**] that the fundamental group of that manifold had a discrete representation in $PSL(2,\mathbb{C})$ with parabolic peripheral holonomy. In fact, he showed that there exists a representation contained in $PSL(2,\mathbb{Z}[\omega])$ where $\mathbb{Z}[\omega]$ is the ring of Eisenstein integers. On the other hand, the construction by Thurston of a real hyperbolic structure on the complement of the figure eight knot is based on gluing of ideal tetrahedra, and that led to general constructions on a large family of 3-manifolds.

The fact that 3-manifolds are all equipped with contact structures and that CR structures are naturally associated to them makes us suspect that spherical CR structures should be fundamental to the study of three manifolds.

A basic question is that of the existence of a representation $\Gamma \to PU(2,1)$ where Γ is the fundamental group of a 3-manifold and $PU(2,1)$ is the group of the homogeneous model for spherical CR geometry, that is $S^3 \subset \mathbb{C}^2$ with the natural CR structure induced from the complex structure of $\mathbb{C}^2$. After that question is solved, one can ask whether the representation obtained comes from a spherical CR structure, as the holonomy map defined by the geometric structure. If there exists such a structure, the holonomy group might be discrete or not, giving rise to a dichotomy (not existant in the real hyperbolic case but analogous

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to that of conformal geometry). If the holonomy is discrete, a further question is whether it is complete, meaning that the manifold is in fact a quotient of the domain of discontinuity $\Omega \subset S^3$ by the holonomy group. Note that it is not excluded that the representation be not injective and the structure be complete.

Very few representations of fundamental groups of 3-manifolds in $PU(2,1)$ are known. The only construction of such a structure on a 3-manifold (which is not a circle bundle) previous to this work is essentially for the Whitehead link and other manifolds obtained from it by Dehn surgery in [**S1**, **S2**] (Schwartz showed they are also complete).

In this paper we address mainly the basic question of existence of representations. We propose a geometrical construction by gluing appropriate tetrahedra adapted to CR geometry. In particular, we prove that the complement of the figure eight knot has a (branched) spherical CR structure with discrete holonomy such that the holonomy of the boundary torus is parabolic and faithful (see Theorem 6.1 and Proposition 6.5). We also prove a rigidity theorem for this representation, namely that it is the only one with faithful purely parabolic torus holonomy (see Theorem 5.7). An interesting related feature of the representation is that its limit set is S^3 (Theorem 6.8). As another example we also construct a representation of the fundamental group of the complement of the Whitehead link with discrete holonomy (Theorem 7.1). It is interesting to observe that we obtain representations of the fundamental groups of those link complements with values in $PU(2,1,\mathbb{Z}[\omega])$ and $PU(2,1,\mathbb{Z}[i])$, that is, the same rings of integers of the complete structures in the case of real hyperbolic geometry.

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2. Complex hyperbolic space

For a complete introduction to complex hyperbolic geometry we refer to Goldman ([**G**]).

2.1. PU$(2,1)$ and the Heisenberg group. Let $\mathbb{C}^{2,1}$ denote the complex vector space equipped with the Hermitian form

$$\langle z, w\rangle = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1.$$

Consider the following subspaces in $\mathbb{C}^{2,1}$:

$$\begin{aligned} V_+ &= \{z \in \mathbb{C}^{2,1} \ : \ \langle z, z\rangle > 0\ \},\\ V_0 &= \{z \in \mathbb{C}^{2,1}\setminus\{0\} \ : \ \langle z, z\rangle = 0\ \},\\ V_- &= \{z \in \mathbb{C}^{2,1} \ : \ \langle z, z\rangle < 0\ \}.\end{aligned}$$

Let $P : \mathbb{C}^{2,1}\setminus\{0\} \to \mathbb{C}P^2$ be the canonical projection onto complex projective space. Then $\mathbf{H}^2_{\mathbb{C}} = P(V_-)$ equipped with the Bergman metric is a complex hyperbolic space. The boundary of complex hyperbolic space is $P(V_0) = \partial \mathbf{H}^2_{\mathbb{C}}$. The isometry group $\widehat{\mathbf{PU}}(2,1)$ of $\mathbf{H}^2_{\mathbb{C}}$ comprises holomorphic transformations in $\mathbf{PU}(2,1)$, the unitary group of $\langle \cdot, \cdot \rangle$, and anti-holomorphic transformations arising elements of $\mathbf{PU}(2,1)$ followed by complex conjugation.

The *Heisenberg group* $\mathfrak{N}$ is the set of pairs $(z,t) \in \mathbb{C} \times \mathbb{R}$ with the product

$$(z,t) \cdot (z',t') = (z + z', t + t' + 2\mathrm{Im}\, z\overline{z}').$$

Using stereographic projection, we can identify $\partial \mathbf{H}^2_{\mathbb{C}}$ with the one-point compactification $\overline{\mathfrak{N}}$ of $\mathfrak{N}$. Heisenberg translations by $(0,t)$ for $t \in \mathbb{R}$ are called *vertical translations*.

Any complex conjugation in hyperbolic space corresponds after a conjugation to the inversion in the x-axis in $\mathbb{C} \subset \mathfrak{N}$ by

$$\iota_x : (z,t) \mapsto (\overline{z}, -t).$$

All these actions extend trivially to the compactification $\overline{\mathfrak{N}}$ of $\mathfrak{N}$ and represent transformations in $\widehat{\mathbf{PU}}(2,1)$ acting on the boundary of complex hyperbolic space.

A point $p = (z,t)$ in the Heisenberg group and the point ∞ are lifted to the following points in $\mathbb{C}^{2,1}$:

$$\hat{p} = \begin{bmatrix} \frac{-|z|^2 + it}{2} \\ z \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{\infty} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Given any three points p_1, p_2, p_3 in $\partial \mathbf{H}^2_{\mathbb{C}}$ we define *Cartan's angular invariant* $\mathbb{A}$ as

$$\mathbb{A}(p_1, p_2, p_3) = \arg(-\langle \hat{p}_1, \hat{p}_2 \rangle \langle \hat{p}_2, \hat{p}_3 \rangle \langle \hat{p}_3, \hat{p}_1 \rangle).$$

In the special case where $p_1 = \infty$, $p_2 = (0,0)$ and $p_3 = (z,t)$ we simply get $\tan(\mathbb{A}) = t/\left|z\right|^2$.

2.2. $\mathbb{R}$-circles, $\mathbb{C}$-circles and $\mathbb{C}$-surfaces. There are two kinds of totally geodesic submanifolds of real dimension 2 in $\mathbf{H}^2_{\mathbb{C}}$: complex lines in $\mathbf{H}^2_{\mathbb{C}}$ are *complex geodesics* (represented by $\mathbf{H}^1_{\mathbb{C}} \subset \mathbf{H}^2_{\mathbb{C}}$) and Lagrangian planes in $\mathbf{H}^2_{\mathbb{C}}$ are *totally real geodesic* 2*-planes* (represented by $\mathbf{H}^2_{\mathbb{R}} \subset \mathbf{H}^2_{\mathbb{C}}$). Each of these totally geodesic submanifolds is a model of the hyperbolic plane.

Consider complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ and its boundary $\partial \mathbf{H}^2_{\mathbb{C}}$. We define $\mathbb{C}$-*circles* in $\partial \mathbf{H}^2_{\mathbb{C}}$ to be the boundaries of complex geodesics in $\mathbf{H}^2_{\mathbb{C}}$. Analogously, we define $\mathbb{R}$-*circles* in $\partial \mathbf{H}^2_{\mathbb{C}}$ to be the boundaries of Lagrangian planes in $H^2_{\mathbb{C}}$.

Proposition 2.1 (see [**G**])**.** *In the Heisenberg model, $\mathbb{C}$-circles are either vertical lines or ellipses, whose projection on the z-plane are circles.*

Finite $\mathbb{C}$-circles are determined by a *centre* $M = (z = a + ib, c)$ and a *radius* R. They may also be described using polar vectors in $P(V_+)$ (see Goldman [**G**, p. 129]).

If we use the Hermitian form $\langle \cdot, \cdot \rangle$, a finite chain with centre $(a+ib, c)$ and radius R has polar vector (that is the orthogonal vector in $\mathbb{C}^{2,1}$ to the plane determined by the chain).

$$\begin{bmatrix} \frac{R^2-a^2-b^2+ic}{2} \\ a+ib \\ 1 \end{bmatrix}.$$

Given two points p_1 and p_2 in Heisenberg space, we write $[p_1, p_2]$ for a choice of one of the two segments of $\mathbb{C}$-circle joining them. The choice is determined by the orientation of the pair as any $\mathbb{C}$-circle is oriented, being a boundary of a complex disc.

Definition 2.2. A $\mathbb{C}$-triangle based on three points $[p_0, p_1, p_2]$ is a triangular surface foliated by segments of $\mathbb{C}$-circles with boundary the segments $[p_0, p_1]$,$[p_0, p_2]$ and $[p_1, p_2]$.

One could in principle work with $\mathbb{R}$-circles instead, but we will restrict ourselves in this paper to objects foliated by $\mathbb{C}$-circles. Each of those triangles could be part of a $\mathbb{C}$-sphere (see [**FZ**]).

A canonical way (which we will need to modify eventually) to fill a triangle between three points is the surface which is the union of segments of $\mathbb{C}$-circles joining p_0 to each point of the segments $\mathbb{C}$-circle $[p_1, p_2]$.

3. CR triangles

We first describe triples of points in the standard spherical CR sphere. They are classified up to $PU(2,1)$ in the following proposition.

Proposition 3.1 ([**C**], see [**G**])**.** *The Cartan invariant classifies ordered triples of points up to* $PU(2,1)$.

3.0.1. Computations. This section can be neglected in a first reading. It describes a practical way to compute an element $g \in U(2,1)$ such that $g(\alpha_1, \alpha_2, \alpha_3) = (\beta_1, \beta_2, \beta_3)$ (when the triples have the same Cartan's invariant). It will be used again and again to obtain the matrices in the following sections without any further comment.

It is convenient to compare each triple to a reference triple which has a fixed Cartan's invariant. We suppose that the three points are not contained in a same $\mathbb{C}$-circle and in that case we might choose the triple $x = (x_1, x_2, x_3)$ with

$$x_1 = \infty, \quad x_2 = (0,0), \quad x_3 = (1,t)$$

whose Cartan's invariant is $\mathbb{A}(x_1, x_2, x_3) = \operatorname{arctg}(t)$. For $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we will determine Φ_α such that $\Phi_\alpha(x) = \alpha$. In the following we will use the same notation for an element in $PU(2,1)$ and a matrix representative in $U(2,1)$. First observe that, from $\Phi_\alpha(x_1) = \alpha_1 = (z_1, t_1)$ and $\Phi_\alpha(x_2) = \alpha_2 = (z_2, t_2)$ and recalling the formula for the lift (if $\alpha_i \neq \infty$)

$$\widehat{\alpha}_i = \begin{bmatrix} \frac{-|z_i|^2+it_i}{2} \\ z_i \\ 1 \end{bmatrix},$$

the matrix Φ_α should be of the form

$$\Phi_\alpha = \begin{bmatrix} \frac{-|z_1|^2+it_1}{2} & b & \lambda\frac{-|z_2|^2+it_1}{2} \\ z_1 & e & \lambda z_2 \\ 1 & h & \lambda \end{bmatrix}.$$

It remains to compute b, e, h, λ.

As a first step, from $\Phi_\alpha \in U(2,1)$, that is, $M = \overline{\Phi}_\alpha^T J_0 \Phi_\alpha = J_0$, we obtain a number of equations. In particular , if $z_1 \neq z_2$ we can solve $M_{12} = M_{32} = 0$ for e, b:

$$e = \frac{h}{2}\frac{|z_1|^2 - |z_2|^2 + i(t_1 - t_2)}{\bar{z}_1 - \bar{z}_2},$$
$$b = -\frac{h}{2}\frac{\bar{z}_2(|z_1|^2 + it_1) - \bar{z}_1(|z_2|^2 + it_2)}{\bar{z}_1 - \bar{z}_2}.$$

On the other hand, if $z_1 = z_2$, $M_{12} = M_{32} = 0$ gives $h = 0$ and $b = -\bar{z}_1 e$.

As a second step, we impose $\Phi_\alpha(x_3) = \alpha_3$. If $z_1 \neq z_2$, that gives two equations in h and λ, the remaining variables. And that can be solved explicitly. Analogously, if $z_1 = z_2$ one can easily solve for the remaining variables b and λ.

The explicit expression for Φ_α, in specific cases, is computed in the following sections according to these two steps and eventually dividing by a cubic root of its determinant in order to obtain a matrix in $SU(2,1)$. Finally, in order to obtain the matrix for g we follow the commutative diagram

$$\begin{tikzcd} \alpha \arrow[rr, "g"] & & \beta \\ & x \arrow[ul, "\Phi_\alpha"] \arrow[ur, "\Phi_\beta"'] & \end{tikzcd}$$

so that $g(\alpha) = \Phi_\beta \circ \Phi_\alpha^{-1}(\alpha) = \beta$.

4. Tetrahedra

Tetrahedra are constructed from configurations of four ordered points in S^3 (the standard CR sphere). The edges of the tedrahedron could be segments of either $\mathbb{R}$-circles or $\mathbb{C}$-circles and the faces should be adapted later to that one skeleton. In this paper we will use $\mathbb{C}$-circles and $\mathbb{C}$-triangles.

4.1. Real hyperbolic tetrahedra. Tetrahedra were used in ([**T**]) to construct three dimensional hyperbolic manifolds after side pairings on their faces. In order to control the quotient manifold an important step is to parametrize in an efficient way the family of tetrahedra.

The simplest tetrahedra to parametrize are the ideal ones, i.e., those having the four vertices at the ideal boundary of hyperbolic space. In that case, using the half-space model, we can normalize the four points to be

$$p_1 = \infty \quad p_2 = 0 \quad p_3 = 1 \quad p_4 = z.$$

The invariant $z = \frac{p_4 - p_2}{p_3 - p_2} \in \mathbb{C}^*$, of the triangle (p_2, p_3, p_4) at p_2, specifies the ideal tetrahedron, and we say that z is the invariant of the edge $[p_1, p_2]$. For a general configuration of points it is given by the cross ratio $\frac{(p_4-p_2)(p_3-p_1)}{(p_3-p_2)(p_4-p_1)}$. One usually considers those tetrahedra with $Imz > 0$. Choosing normalizations based on different edges gives invariants which can be expressed as a function of z. For instance, choosing

$$p_1 = \infty \quad q_1 = 0 \quad p_4 = 1$$

implies

$$p_2 = \frac{1}{1-z}.$$

Each invariant is associated to an edge according to the normalization above, and due to a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry of an ideal tetrahedron, the invariants of opposed edges are equal as shown in Figure 1.

When sewing several tetrahedra along the same edge a necessary condition for the gluing to give rise to a hyperbolic manifold along that edge is that the product of the invariants (associated to the tetrahedra along that edge) be one. We obtain several equations, as many as the number of edges in the quotient manifold.

As a last observation, note that the invariants are well defined for an ordered quadruple of points in $\mathbb{C} \cup \{\infty\}$. They only depend on the 0-skeleton of the tetrahedron. That is the point of view we adopt in the spherical CR setting as, in that case, the edges and especially the 2-skeleton are quite arbitrary.

4.2. CR tetrahedra.

Definition 4.1. A tetrahedron is a configuration of four (ordered) points and a choice of edges, that is, a choice of $\mathbb{C}$-circle segments joining each pair of points.

In order to find a representation it is enough to work with configurations of four points, that is the 0-skeleton. The 1-skeleton is important to define eventually a CR structure on a 3-manifold triangulated by tetrahedra. We could have used $\mathbb{R}$-circles. The advantage of using $\mathbb{C}$-circles is that in the Heisenberg space model, $\mathbb{C}$-circles passing through

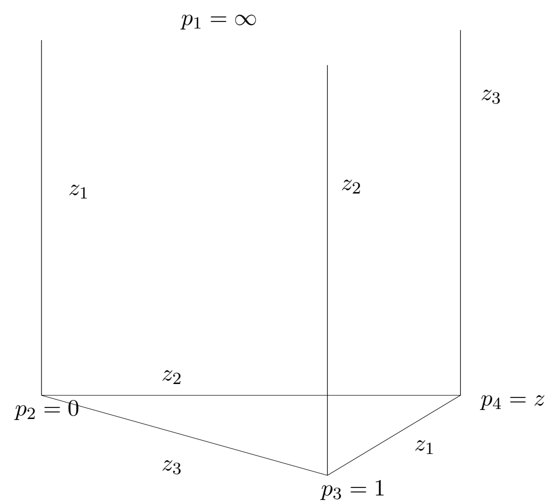

Figure 1. Parameters for a real hyperbolic tetrahedron. Here $z_1 = z$ with $Im(z) > 0$, $z_2 = 1/(1 - z_1)$ and $z_3 = 1 - 1/z_1$.

infinity are vertical lines and this makes the analogy with real hyperbolic ideal tetrahedra more transparent.

For a general configuration of four points we have four Cartan invariants corresponding to each triple of points. But one of them is determined by the others in view of the cocycle condition (see [**G**] p. 219):

$$-A(x_2, x_3, x_4) + A(x_1, x_3, x_4) - A(x_1, x_2, x_4) + A(x_1, x_2, x_3) = 0.$$

A simple consequence is the following.

Proposition 4.2. *If three triples of four points are contained in $\mathbb{R}$-circles ($\mathbb{C}$-circles), the four points are contained in a common $\mathbb{R}$-circle ($\mathbb{C}$-circle).*

Proof. We will prove the result on $\mathbb{R}$-circles, the other case being easier. From the cocycle relation, each triple is contained in an $\mathbb{R}$-circle as $A = 0$ for all triples. Without loss of generality, we can suppose that three of the points are $\infty, [0,0], [1,0]$ in Heisenberg coordinates. The fourth point is in an $\mathbb{R}$-circle containing $\infty, [0,0]$ on one hand, so

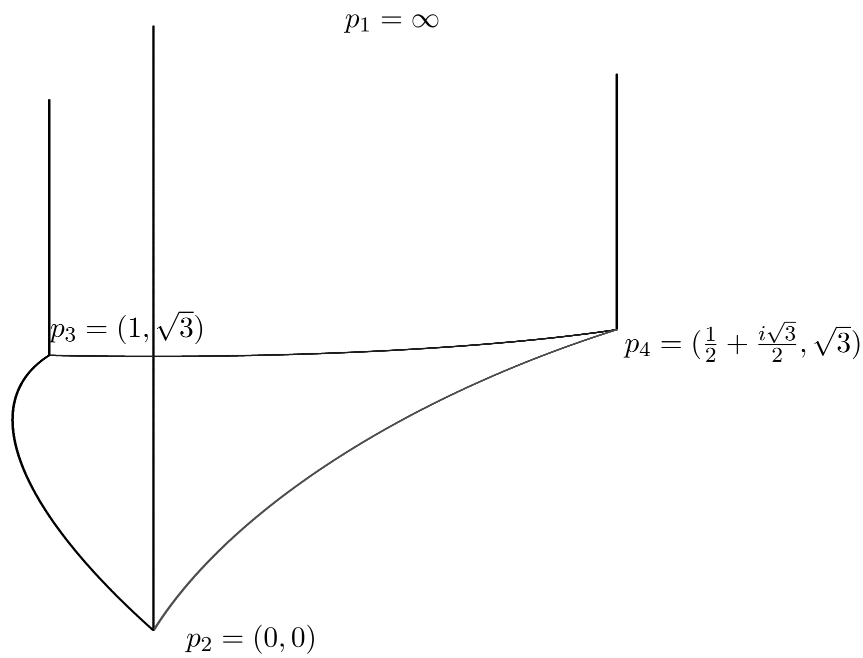

Figure 2. The standard tetrahedron in the Heisenberg group with one vertex at infinity.

it is in the plane $t = 0$. On the other hand, it should be in an $\mathbb{R}$-circle passing through $[1, 0]$ and ∞, that is in the contact plane at $[1, 0]$. The intersection of both planes is precisely the x-axis. q.e.d.

The fundamental example is the following but we will need to change the choice of the edges in later constructions. There are many possible choices for edges depending on a choice of orientation for each edge.

Example: standard tetrahedron. Figure 4.2 shows one tetrahedron with

$$p_1 = \infty \quad p_2 = 0 \quad p_3 = (1, \sqrt{3}) \quad p_4 = \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}, \sqrt{3}\right),$$

in the Heisenbeg group. We choose all $\mathbb{C}$-edges with positive vertical coordinate. The $\mathbb{C}$-edge between p_3 and p_4 is chosen to be the smallest arc between the two points.

We deal first with configuration of ordered points (cf. [**W**] for different normalizations):

Proposition 4.3. *Consider a generic configuration of four ordered points p_1, p_2, p_3, p_4 in S^3 (any three of them not contained in a $\mathbb{C}$-circle) up to overall translation by an element of $PU(2,1)$. Then, there exists a unique representative normalized such that*

$$p_1 = \infty \quad p_2 = 0 \quad p_3 = (1,t) \quad p_4 = (z, s|z|^2)$$

with

$$\bar{z}\frac{i+s}{i+t} \neq 1$$

and

$$z \neq 0, 1.$$

Proof. In fact, we can always suppose $p_1 = \infty \quad p_2 = 0 \quad p_3 = (1,t) \quad p_4 = (z, s|z|^2)$ with $z \neq 0$ because, besides p_2, no other point can lie on the vertical axis. Moreover, the two conditions follow from computing Cartan's invariant. In fact, p_2, p_3, p_4 are in the same $\mathbb{C}$-circle if and only if

$$\mathbb{A}(p_2, p_3, p_4) = \pm\pi/2,$$

therefore if and only if $\operatorname{Re}\{\mathbb{A}(p_2, p_3, p_4)\} = 0$, and a computation shows that

$$\operatorname{Re}\{\mathbb{A}(p_2, p_3, p_4)\} = \frac{|z|^2}{8}|\bar{z}(1-is) - (1-it)|^2$$

so that p_2, p_3, p_4 are not in the same $\mathbb{C}$-circle if and only if $\bar{z}\frac{i+s}{i+t} \neq 1$. Analogously,

$$\operatorname{Re}\{\mathbb{A}(p_1, p_3, p_4)\} = |1-z|^2,$$

so that p_1, p_3, p_4 are not in the same $\mathbb{C}$-circle if and only if $z \neq 1$. q.e.d.

In analogy with the real hyperbolic ideal tetrahedron, the invariant associated to the edge $[p_1, p_2]$ at the vertex p_1 is $z_{12} = z$. Acoordingly, the invariants associated to the edges $[p_1, p_3]$ and $[p_1, p_4]$ at the vertex p_1 are $z_{13} = 1/(1-z)$ and $z_{14} = 1 - 1/z$ respectively.

Remark. The invariants can be defined as cross ratios of four ordered points in a $\mathbb{C}P^1$. In fact, consider the points $p_1 \in S^3$ and the projective space of all complex lines passing through it. There are four distinguished points determined by the three lines passing through p_1 and each of the other three points and a fourth point which is the complex tangent line to S^3 through p_1. Repeating the same procedure for each point we obtain the following invariants.

We consider Figure 3 to describe the parameters of a tetrahedron. Note again that, contrary to the ideal tetrahedron in real hyperbolic geometry, the euclidean invariants at each vertex, although related, are not the same. The following definition puts together the considerations above and makes explicit the invariants once the tetrahedron is given in normalized coordinates.

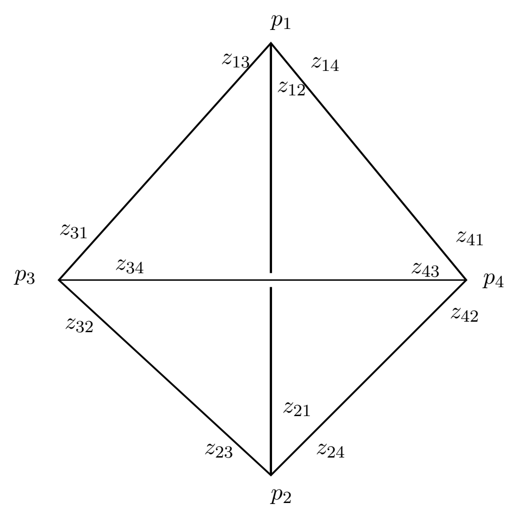

Figure 3. Parameters for a CR tetrahedron.

Definition 4.4. Given an ordered generic quadruple of points

$$p_1 = \infty \quad p_2 = 0 \quad p_3 = (1, t) \quad p_4 = (z, s|z|^2)$$

we associate the following invariants as in Figure 3

$$z_{12} = z, \quad z_{21} = \frac{\bar{z}(i+s)}{i+t}, \quad z_{34} = \frac{z((t+i) - \bar{z}(s+i))}{(z-1)(t-i)},$$

$$z_{43} = \frac{\bar{z}(z-1)(s-i)}{(t+i) - \bar{z}(s+i)}.$$

Also, $z_{13} = \frac{1}{1-z_{12}}$ and $z_{14} = 1 - \frac{1}{z_{12}}$ and similarly for z_{2i}, z_{3i} and z_{4i}.

Remark. Observe that the invariants are associated to a quadruple of points and therefore do not depend on the choice of edges between the vertices. In fact, the compatibility conditions which we will state below are based on the "0-skeleton" of the putative tetrahedra. One will have to verify later that the "1-skeleton" is also well defined.

Proposition 4.5. *Generic configurations (any three points are not contained in a $\mathbb{C}$-circle) of four pairwise distinct points in S^3 are parametrized by the algebraic real variety in $(\mathbb{C} \setminus \{0, 1\})^{12}$ with coordinates*

z_{ij}, $1 \leq i \neq j \leq 4$ *defined by the usual hyperbolic constraints:*

$$z_{ij}z_{ik}z_{il} = -1, \quad z_{13} = \frac{1}{1-z_{12}}, \quad z_{32} = \frac{1}{1-z_{31}},$$
$$z_{24} = \frac{1}{1-z_{21}}, \quad z_{43} = \frac{1}{1-z_{41}}$$

and the six real equations

$$z_{1j}z_{j1} = \overline{z_{kl}z_{lk}}.$$

Remark. In fact, eliminating two variables in each vertex, one can write the six real equations directly in $(\mathbb{C}\setminus\{0,1\})^4$ with variables $z_{12}, z_{21}, z_{34}, z_{43}$. The equations in these variables are:

$$z_{12}z_{21} = \overline{z_{34}z_{43}}$$
$$\frac{1}{1-z_{12}}\frac{1}{1-z_{34}} = \frac{1}{1-\bar{z}_{21}}\frac{1}{1-\bar{z}_{43}}$$
$$\left(1-\frac{1}{z_{12}}\right)\left(1-\frac{1}{z_{43}}\right) = \left(1-\frac{1}{\bar{z}_{34}}\right)\left(1-\frac{1}{\bar{z}_{21}}\right).$$

Proof. The use of the twelve variables makes the equations more symmetric. In order to prove the result we suppose first that $z_{12} \neq \bar{z}_{21}$. Observe that this implies that $z_{34} \neq \bar{z}_{43}$. We solve for z, t, s for a normalized configuration.

From the three equations above we obtain

$$z_{12}z_{21} = \overline{z_{34}z_{43}}$$
$$(1-z_{12})(1-z_{34}) = (1-\bar{z}_{21})(1-\bar{z}_{43})$$
$$(1-z_{12})(1-z_{43})\bar{z}_{21}\bar{z}_{34} = (1-\bar{z}_{21})(1-\bar{z}_{34})z_{12}z_{43}.$$

Dividing the last equation by the second we get

$$|1-z_{12}|^2\bar{z}_{21}\bar{z}_{34} = |1-z_{21}|^2 z_{12}z_{43},$$

and dividing the result by the first one we get

$$\star \qquad |1-z_{12}|^2|z_{34}|^2 = |1-z_{21}|^2|z_{12}|^2.$$

Substituting

$$\bar{z}_{43} = \frac{z_{12}z_{21}}{\bar{z}_{34}}$$

in the second equation, we get

$$(1-z_{12})(1-z_{34}) = (1-\bar{z}_{21})\left(1-\frac{z_{12}z_{21}}{\bar{z}_{34}}\right)$$

and using relation $\star$, we may solve for z_{34}:

$$z_{34} = \bar{z}_{12}\frac{(1-z_{21})(z_{12}-\bar{z}_{21})}{(z_{12}-1)(\bar{z}_{12}-z_{21})},$$

and analogously for z_{43}:

$$z_{43} = \bar{z}_{21}\frac{(1-z_{12})(\bar{z}_{12}-z_{21})}{(z_{21}-1)(z_{12}-\bar{z}_{21})}.$$

The argument is similar for the other cases, that is, when $z_{13} \neq \bar{z}_{31}$ and $z_{14} \neq \bar{z}_{41}$.

If $z_{12} = \bar{z}_{21}$, then $t = s$ for the normalized coordinates. If $z_{14} = \bar{z}_{41}$, then $s\mathrm{Re}\, z + \mathrm{Im}\, z = t$. Therefore, if both equations are satisfied, we conclude that $t = s$ and $t\mathrm{Re}\, z + \mathrm{Im}\, z = t$. That is a regular configuration. The equation $z_{11} = \bar{z}_{31}$ gives a new relation, namely, $\mathrm{Im}\, z = 0$ which determines the configuration, namely $t = s = 0$ and the four points are aligned on an $\mathbb{R}$-circle. q.e.d.

Computations.

This section may be skipped. It justifies the definition of the invariants by an explicit computation using normal coordinates. In order to compute the invariants at p_2 one has to place that vertex at infinity. This can be accomplished using the complex inversion interchanging $p_1 = \infty$ and $p_2 = 0$:

$$I(z,t) = \left(\frac{z}{|z|^2 - it}, \frac{-t}{|z|^4 + t^2}\right).$$

Computing:

$$I(p_1) = (0,0) \quad I(p_2) = \infty \quad I(p_3) = \left(\frac{1}{-1+it}, -\frac{t}{1+t^2}\right)$$

and

$$I(p_4) = \left(\frac{1}{\bar{z}(-1+is)}, -\frac{s}{|z|^2(1+s^2)}\right).$$

The invariant z_{21} defined at the vertex p_2 associated to the edge $[p_1, p_2]$ is the quotient of the z-coordinates of $I(p_3)$ by $I(p_4)$. That is $z_{21} = \frac{\bar{z}(i+s)}{i+t}$.

In the same way, to obtain the invariants at p_3 and p_4 one has to move each of them to ∞. In order to do that we first translate p_3 to the origin by a Heisenberg translation

$$T_1(z', t') = (z' - 1, t' - t - 2\mathrm{Im}\, \bar{z}'),$$

where t is the vertical coordinate of p_3, and then use the complex inversion

$$I(z,t) = \left(\frac{z}{|z|^2 - it}, \frac{-t}{|z|^4 + t^2}\right).$$

Computing:

$$IT_1(p_1) = (0,0) \quad IT_1(p_2) = \left(\frac{i}{t-i}, \frac{t}{1+t^2}\right) \quad IT_1(p_3) = \infty$$

and

$$IT_1(p_4) = \left(\frac{z-1}{|z-1|^2 - (s|z|^2 - t + 2\mathrm{Im}\, z)i}, \right.$$
$$\left. - \frac{s|z|^2 - t + 2\mathrm{Im}\, z}{|z-1|^4 + (s|z|^2 - t + 2\mathrm{Im}\, z)^2} \right).$$

The invariant z_{34} associated to the edge $[p_3, p_4]$ at the vertex p_3 should be the quotient of the z-coordinates of $IT_1(p_2) - IT_1(p_4)$ by the z-coordinates of $IT(p_1) - IT(p_4)$. That is

$$\tilde{z}_1 = \frac{-|z|^2(i+s) + z(t+i)}{(z-1)(t-i)}.$$

Finally, in order to obtain the invariant at p_4 we first translate it to the origin by a Heisenberg translation

$$T_2(z', t') = \left(z' - z, t' - s|z|^2 - 2\mathrm{Im}\, z\bar{z}'\right),$$

where $p_4 = (z, s|z|^2)$, and then use the holomorphic qy inversion I as above to obtain

$$IT_2(z', t') = \left(\frac{z' - z}{|z'-z|^2 - i(t' - s|z|^2 - 2\mathrm{Im}\, z\bar{z}')}, \right.$$
$$\left. - \frac{t' - s|z|^2 - 2\mathrm{Im}\, z\bar{z}'}{|z'-z|^4 + (t' - s|z|^2 - 2\mathrm{Im}\, z\bar{z}')^2} \right).$$

Computing:

$$IT_2(p_1) = (0,0) \quad IT_2(p_2) = \left(-\frac{1}{\bar{z}(1+si)}, \frac{s}{|z|^2(1+s^2)} \right) \quad IT_2(p_4) = \infty$$

and

$$IT_2(p_3) = \left(\frac{1-z}{1 - it - 2\bar{z} + |z|^2(1+si)}, \star \right).$$

The invariant z_{43} associated to the edge $[p_3, p_4]$ at the vertex p_4 should be the quotient of the z-coordinates of $IT_1(p_1) - IT_1(p_3)$ by the z-coordinates of $IT(p_2) - IT(p_3)$. That is

$$\tilde{z}'_1 = \frac{\bar{z}(z-1)(s-i)}{(t+i) - \bar{z}(s+i)}.$$

A simple computation gives the Cartan invariants:

Proposition 4.6. *Given a quadruple of points with coordinates*

$$p_1 = \infty \quad p_2 = 0 \quad p_3 = (1,t) \quad p_4 = (z, s|z|^2),$$

then

$$tg\mathbb{A}(p_1, p_2, p_3) = t$$
$$tg\mathbb{A}(p_1, p_3, p_4) = \frac{|z|^2 s - t + 2\mathrm{Im}\, z}{|z-1|^2}$$
$$tg\mathbb{A}(p_1, p_2, p_4) = s,$$

and

$$tg\mathbb{A}(p_2,p_3,p_4)$$
$$= \frac{2(s-t)\mathrm{Re}\, z + 2(1+ts)\mathrm{Im}\, z + t(1+s^2)|z|^2 - s(1+t^2)}{|(s-i)z+i-t|^2}.$$

4.3. Symmetric Tetrahedra. Important classes of configurations are the ones with a prescribed symmetry. We will consider namely the following class. See [**Wi**] for a complete study of the symmetry group.

Definition 4.7. A symmetric configuration is a configuration of four points with an anti-holomorphic involution which is a product of two transpositions.

Consider a configuration of four points p_1,p_2, p_3 and p_4 and an anti-holomorphic involution φ such that $\varphi(p_1) = p_2$ and $\varphi(p_3) = p_4$. We call $[p_1, p_2]$ or $[p_3, p_4]$ the axis of the involution.

Recall that generic configurations of four points can be normalized as follows in Heisenberg coordinates:

$$p_1 = \infty \quad p_2 = 0 \quad p_3 = (1,t) \quad p_4 = (z, s|z|^2).$$

Lemma 4.8. *The configuration of four points p_1,p_2, p_3 and p_4 has an anti-holomorphic involution φ such that $\varphi(p_1) = p_2$ and $\varphi(p_3) = p_4$ if and only if $t = s$.*

Proof. A simple proof follows writing the general form of an anti-holomorphic transformation permuting ∞ and 0. It is given by conjugating one inversion interchanging both points

$$(z,t) \to \left(-\frac{\bar{z}}{|z|^2+it}, \frac{t}{|z|^4+t^2}\right)$$

by all complex dilations $(z,t) \to (\lambda z, |\lambda|^2 t)$ where $\lambda \in \mathbb{C}^*$. The result is the following general form:

$$f : (z,t) \to \left(-\frac{\lambda^2\bar{z}}{|z|^2+it}, \frac{|\lambda|^4 t}{|z|^4+t^2}\right).$$

Imposing that the points p_3 and p_4 be permuted, that is $f(1,t) = (z, s|z|^2)$, gives the result. q.e.d.

For a generic configuration of four points p_1,p_2, p_3 and p_4 with an anti-holomorphic involution φ such that $\varphi(p_1) = p_2$ and $\varphi(p_3) = p_4$, we obtain, from Proposition 4.6, $\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(p_1, p_2, p_4)$ and $\mathbb{A}(p_1, p_3, p_4) = \mathbb{A}(p_2, p_3, p_4)$. Conversely, $\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(p_1, p_2, p_4)$ implies that the configuration is symmetric. A simple computation shows the following:

Lemma 4.9. *A generic configuration of four points of four pairwise distinct points in S^3 parametrized by z_{ij}, $1 \leq i \neq j \leq 4$ is symmetric with axis $[a, b]$ if and only if*

$$z_{ab} z_{ba} = z_{kl} z_{lk},$$

where $\{a, b, j, k\} = \{1, 2, 3, 4\}$.

Definition 4.10. A regular configuration of four points p_1,p_2, p_3 and p_4 is a configuration satisfying

$$|\mathbb{A}(p_1, p_2, p_3)| = |\mathbb{A}(p_1, p_2, p_4)| = |\mathbb{A}(p_1, p_3, p_4)| = |\mathbb{A}(p_2, p_3, p_4)|.$$

For generic regular configurations, Proposition 4.6 implies that in Heisenberg coordinates $t = \pm s$ and $t = \pm \frac{\operatorname{Im} z}{1 - \operatorname{Re} z}$ (if we have equality of Cartan's invariants above the sign is posivitive). Regular configurations are parametrized by a complex variable.

For symmetric configurations, the computation of invariants is simpler:

Corollary 4.11. *For a symmetric tetrahedron given by*

$$p_1 = \infty \quad p_2 = 0 \quad p_3 = (1, t) \quad p_4 = (z, t|z|^2)$$

with an anti-holomorphic involution satisfying $\varphi(p_1) = p_2$ and $\varphi(p_3) = p_4$, $z_{12} = z$,$z_{21} = \bar{z}$, $z_{34} = \frac{z(1-\bar{z})(t+i)}{(z-1)(t-i)}$ and $z_{43} = \overline{z_{34}}$.

The proposition above shows that the set of symmetric tetrahedra is parametrized by a strictly pseudoconvex CR hypersurface in $\mathbb{C} \setminus \{0, 1\} \times \mathbb{C} \setminus \{0, 1\}$; namely, solving for t, we obtain the equation

$$|z_{12}| = |z_{34}|.$$

We say a symmetric tetrahedron with an anti-holomorphic involution satisfying $\varphi(p_1) = p_2$ and $\varphi(p_3) = p_4$ is special if the complex lines defined by p_1, p_2 and p_3, p_4 are orthogonal. In the coordinates above, that means that $t = s$ (symmetry) and $|z_{12}| = 1$. In that case observe that $z_{34} = \frac{t+i}{t-i}$ and $tg\mathbb{A}(p_1, p_3, p_4) = \frac{\operatorname{Im} z_{12}}{1 - \operatorname{Re} z_{12}}$. Special symmetric tetrahedra have a convenient realization as a finite configuration of points given in the following proposition.

Proposition 4.12. *If the special symmetric tetrahedron with an anti-holomorphic involution satisfying $\varphi(p_1) = p_2$ and $\varphi(p_3) = p_4$ is given by*

$$p_1 = (0, t) \quad p_2 = (0, -t) \quad p_3 = (1, 0) \quad p_4 = (e^{i\theta}, 0),$$

then $z_{12} = e^{i\theta}$ and $z_{34} = \frac{(t+i)^2}{(t-i)^2}$, where, as usual, $z_{13} = \frac{1}{1 - z_{12}}$ and $z_{14} = 1 - \frac{1}{z_{12}}$.

4.3.1. The standard special tetrahedron. We let $t = 2+\sqrt{3}$ in the coordinates above for a special tetrahedra. Using the formulas above we obtain that the tetrahedron defined by $p_1 = (0, 2+\sqrt{3})$, $p_2 = (0, -(2+\sqrt{3}))$, $p_3 = (1, 0)$ and $p_4 = (1, 0)$ has invariant $z_{12} = z_{34} = \frac{1+i\sqrt{3}}{2}$. Moreover, $A(p_3, p_4, p_2) = \frac{\pi}{3}$ and $A(p_1, p_4, p_2) = -\frac{\pi}{3}$.

4.3.2. Another special tetrahedron. We make $p_1 = (0, 1+\sqrt{2})$, $p_2 = (0, -(1+\sqrt{2}))$, $p_3 = (1, 0)$ and $p_4 = (i, 0)$. The tetrahedron defined by p_1, p_2, p_3, p_4 is special symmetric and $z_{12} = i$.

As a last observation, the moduli for a tetrahedron can be expressed using other invariants as the Koranny-Reimann cross-ratios and Cartan's invariants.

4.4. Edges. Given a configuration of four points p_1, p_2, p_3, p_4 we wish to define the tetrahedron, that is, a choice of an edge between each pair of points. For each pair there are two possible segments corresponding to the two halves of the $\mathbb{C}$-circle defined by the pair. That makes 64 possibilities. When gluing tetrahedra one has to chose carefully, for each tetrahedron, edges so that the gluing will be compatible. One practical way to do that is to give an order between each pair. This determines without ambiguity the edge because $\mathbb{C}$-circles, as boundaries of complex discs, are ordered.

4.5. Faces. After the definition of a 1-skeleton the next step to define the tetrahedron is to define the 2-skeleton. Again, one has to be careful to chose the 2-skeleton in a compatible manner when gluing several tetrahedra.

As an example we will consider the case of a special symmetric tetrahedron with edges chosen as in Figure 4.2.

We define the procedure of filling the faces from the one skeleton of the tetrahedra in such a way that the 2-skeleton will be $\mathbb{Z}_2$-invariant:

Definition 4.13. The diverging $\mathbb{C}$-rays procedure is the definition of the 2-skeleton by taking $\mathbb{C}$-segments from p_1 to the edges $[p_3, p_4]$, $[p_4, p_2]$ and $\mathbb{C}$-segments from p_2 to the edges $[p_3, p_4]$, $[p_3, p_1]$.

Observe that the rays start from p_1 or p_2 and not from p_3 or p_4.

Lemma 4.14. *The special symmetric tetrahedron defined by the procedure of diverging $\mathbb{C}$-rays is homeomorphic to a tetrahedron.*

Proof. Figure 4 shows the projection of the 2-faces as will be explained in the proof. Recall that we can use the following coordinates $p_1 = \infty$, $p_2 = 0$, $p_3 = (1, t)$ and $p_4 = (e^{i\theta}, t)$. The edges are the following: $[p_3, p_4] = (e^{i\varphi}, t)$ $0 \le \varphi \le \theta$, $[p_2, p_3]$ is an arc of ellipse which projects on the z-coordinate of the Heisenberg space as an arc of a circle joining p_2 and p_3. It is given by

$$\left(\frac{1+\sqrt{t^2+1}\cos\varphi}{2}, \frac{t+\sqrt{t^2+1}\sin\varphi}{2}, \frac{\sqrt{t^2+1}}{2}(\cos\varphi - \sin\varphi)t\right).$$

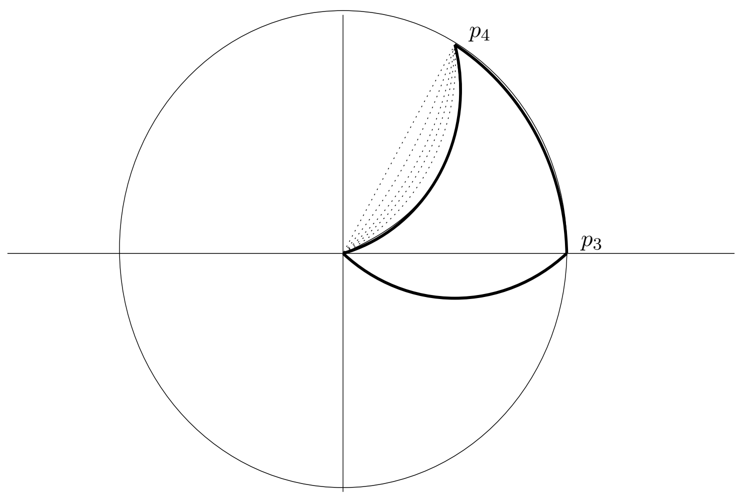

Figure 4. Projection of the 2-faces of a tetrahedron using the diverging rays procedure (here $p_3 = (1, t)$ with $t > 0$).

The projected circle is the circle centered at $(1/2, t/2)$ with radius $\frac{\sqrt{t^2+1}}{2}$. Observe that the projection of a $\mathbb{C}$-segment from p_2 towards $(1, t')$ with $t' > t$ is contained in the region determined by the projected $\mathbb{C}$-segment from p_2 towards $(1, t)$ and the segment $[0, 1]$. Also, $[p_2, p_4]$ is a rotation by $e^{i\theta}$ of the arc of ellipse $[p_2, p_3]$ and analogously for the arcs $[p_2, (e^{i\theta}, t')]$ for $t' > t$. The projected circles are obtained by the same rotation. The other edges are vertical lines going from the finite vertices towards infinity (p_1). By analyzing the projections, the following intersections are obvious:

$$\Delta[p_1, p_2, p_3] \cap \Delta[p_1, p_3, p_4] = [p_2, p_3]$$
$$\Delta[p_1, p_3, p_4] \cap \Delta[p_1, p_3, p_4] = [p_1, p_2]$$
$$\Delta[p_2, p_3, p_4] \cap \Delta[p_1, p_3, p_4] = [p_3, p_4].$$

It remains to examine $\Delta[p_1, p_2, p_3] \cap \Delta[p_1, p_2, p_4]$ and the result will follow by symmetry as the face $\Delta[p_1, p_2, p_4]$ is transformed by the $\mathbb{Z}_2$ symmetry into the face $\Delta[p_1, p_2, p_3]$. To see this, observe that the projections of the rays starting from p_2 and ending in the vertical line over

p_4 gather in a family of arcs which are all contained in the full arc of a circle determined by the projection of $[p_2, p_4]$ and the segment $[0, e^{i\theta}]$; therefore there is no intersection with the projections of the rays in $\Delta[p_1, p_2, p_3]$ which are points on the projection of $[p_2, p_3]$. q.e.d.

In Section 5 we glue tetrahedra to obtain a spherical structure in the complement of the figure eight knot, but the tetrahedra will be defined using a modification of the procedure defined above.

5. Gluing tetrahedra: figure eight knot

5.1. Compatibility equations I: Face equations. A first compatibility condition for a given family of tetrahedra with side pairings is that the pairing of two triples of points are only possible if the corresponding Cartan's invariants are equal.

A computation shows that

$$\tan(\mathbb{A}(p_2, p_3, p_4)) = -i\frac{z_{21}z_{31}z_{41} + 1}{z_{21}z_{31}z_{41} - 1}.$$

This implies that the face gluing conditions between tetrahedra with invariants z_{ij} and $w_{i'j'}$ are given by equations of the form

$$z_{il}z_{jl}z_{kl}w_{i'l'}w_{j'l'}w_{k'l'} = 1,$$

where l and l' correspond to points oposed to the common face. We will impose these relations in the next section. Here we will make computations in normal coordinates.

We refer to the parametrization of a generic configuration using z_i, z_i', $\tilde{z}_i$ and w_i, w_i', $\tilde{w}_i$ as in Figure 5. In this section we obtain the Cartan compatibility equations for gluing two configurations according to the scheme in the figure. The side-pairings g_1, g_2, g_3 and Id will be determined explicitly in a later section for a discrete representation which has purely parabolic peripheral holonomy.

We will use directly Cartan's invariants, leaving the general equations above for a later section. There are four equations corresponding to the four side-pairings. We use again the normalizations

$$p_1 = \infty \quad p_2 = 0 \quad p_3 = (1, t_1) \quad p_4 = (z, s_1|z|^2)$$

and

$$\dot{p}_1 = \infty \quad \dot{p}_2 = 0 \quad \dot{p}_3 = (1, t_2) \quad \dot{p}_4 = (w, s_2|w|^2).$$

Define

$$F(z, t, s) = \frac{2(s-t)\mathrm{Re}\, z + 2(1+ts)\mathrm{Im}\, z + t(1+s^2)|z|^2 - s(1+t^2)}{|(s-i)z + i - t|^2}.$$

Then we can write

$$\mathbb{A}(p_1, p_2, p_4) = \mathbb{A}(\dot{p}_1, \dot{p}_2, \dot{p}_3) \Longrightarrow s_1 = t_2$$

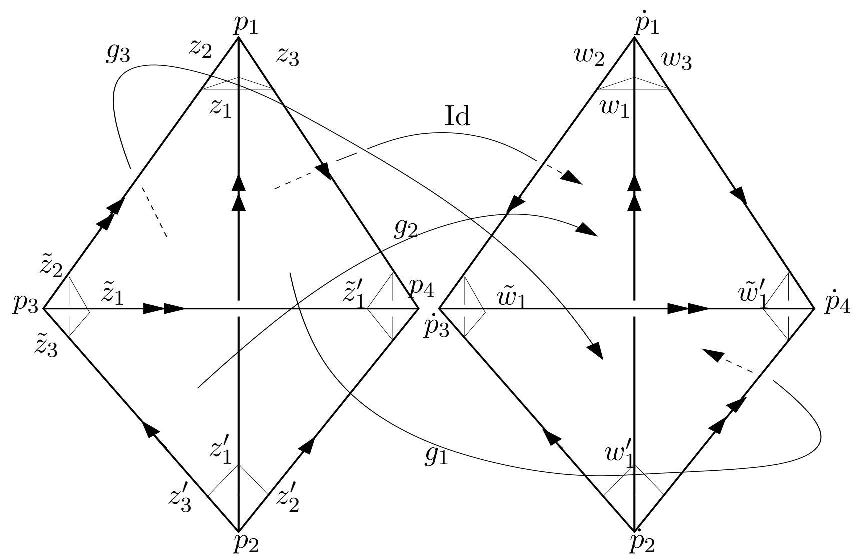

Figure 5. Two tetrahedra with the gluing scheme to obtain the figure eight knot showing the CR-invariants.

$$\mathbb{A}(p_1,p_2,p_3)=\mathbb{A}(\dot{p}_4,\dot{p}_2,\dot{p}_3)\Longrightarrow t_1=F(w,t_2,s_2)$$

$$\mathbb{A}(p_2,p_3,p_4)=\mathbb{A}(\dot{p}_1,\dot{p}_3,\dot{p}_4)\Longrightarrow F(z,t_1,s_1)=\frac{s_2|w|^2-t_2+2\mathrm{Im}\,w}{|w-1|^2}$$

$$\mathbb{A}(p_1,p_3,p_4)=\mathbb{A}(\dot{p}_1,\dot{p}_2,\dot{p}_4)\Longrightarrow \frac{s_1|z|^2-t_1+2\mathrm{Im}\,z}{|z-1|^2}=s_2.$$

There are only three independent equations, the fourth one being a consequence of the cocycle condition for Cartan's invariant. That is, from

$$\mathbb{A}(p_1,p_2,p_4)+\mathbb{A}(p_1,p_4,p_3)+\mathbb{A}(p_1,p_3,p_2)=\mathbb{A}(p_2,p_4,p_3)$$

$\mathbb{A}(p_2,p_3,p_4)=\mathrm{arctg}\,(F(z,t_1,s_1))$ is determined.

Choosing the first tetrahedron arbitrarily one can fix all Cartan's invariants of the second one. In particular, t_2,s_2 being determined and the equation

$$F(z,t_1,s_1)=\frac{s_2|w|^2-t_2+2\mathrm{Im}\,w}{|w-1|^2}$$

is the equation of a quadric in w. That gives a 5 real parameter family of a couple of tetrahedra with compatible Cartan's invariants under the gluing scheme.

5.1.1. Special case (symmetric tetrahedra). If we deal with symmetric tetrahedra, the equations are simplified and we obtain the following special case:

Proposition 5.1. *If two symmetric tetrahedra (with symmetry axis $[p_1, p_2]$) are glued following the scheme above to obtain the complement of the figure eight knot, then they are both regular. In that case, a couple of regular tetrahedra is parametrized by a hypersurface in the variables z and w.*

Proof. For a symmetric tetrahedron $s = t$. From the equations above we obtain that the four triples have the same Cartan's invariant and therefore they are regular. In this case we have

$$t = \frac{\operatorname{Im} z}{1 - \operatorname{Re} z} = \frac{\operatorname{Im} w}{1 - \operatorname{Re} w}.$$

q.e.d.

That is a three dimensional family of couples of tetrahedra parametrized by

$$z = \alpha + t(1-\alpha)i \quad w = \beta + t(1-\beta)i,$$

where t, α, β are reals.

5.1.2. Families of triangulations with compatible faces. It is important to ask whether one can associate to a triangulation of a three-manifold compatible Cartan invariants. A simple answer is given in the following:

Proposition 5.2. *Fix a triangulation (with T tetrahedra) of a three manifold M and let $\{F_i\}$ be the oriented two-faces of the triangulation. Then, to each element $\mathbb{A} \in C^2(M, \mathbb{R})$ (the 2-cocycles of M) such that $-\pi/2 \leq \mathbb{A}(F_i) \leq \pi/2$ one can associate a T-dimensional family of T spherical CR tetrahedra.*

Proof. To each face of a tetrahedron of the triangulation, the cocycle element τ associates one invariant (Cartan). Now, for each tetrahedron in the triangulation, the Cartan invariants being fixed, there exists a one parameter family of configurations of four points with precisely those Cartan invariants. q.e.d.

Of course, we are interested in non-trivial elements of $C^2(M, \mathbb{R})$; otherwise all the tetrahedra would be collapsed in a $\mathbb{R}$-circle.

A special case is obtained if $\mathbb{A}$ is a coboundary $\mathbb{A} = \delta\tau$. Here τ is a 1-cocycle defined in the 1-skeleton. In the case of the figure eight knot there are only two edges, say, $\tau(\rightarrow) = a$ and $\tau(\twoheadrightarrow) = b$. We obtain the following

$$\mathbb{A}(p_1, p_3, p_4) = \delta\tau(p_1, p_3, p_4) = \tau(p_1p_3 + p_3p_4 + p_4p_1) = -b + b - a = -a$$

and analogously,

$$\mathbb{A}(p_1, p_4, p_2) = b \quad \mathbb{A}(p_1, p_2, p_3) = a \quad \mathbb{A}(p_3, p_4, p_2) = b.$$

That shows that the tetrahedron is symmetric with one of the axes of symmetry the edge $[p_3, p_1]$. In that case, the equations defining the tetrahedra are

$$s_1 = t_2 = \frac{s_2|w|^2 - t_2 + 2\mathrm{Im}\, w}{|w-1|^2} \quad s_2 = t_1 = \frac{s_1|z|^2 - t_1 + 2\mathrm{Im}\, z}{|z-1|^2},$$

so that when we fix t_1 and s_1 ($t_1 \neq s_1$) satisfying $t_1(t_1 - 2s_1) \leq 1$ and $s_1(s_1 - 2t_1) \leq 1$, we obtain one circle of solutions for z and another for w. Observe that

$$C^2(\text{quotient of tetrahedra by side pairings}, \mathbb{R})$$

is a 3-dimensional vector space generated by the coboundaries (that is a two-dimensional space corresponding to symmetric tetrahedra with axis $[p_3, p_1]$) and the two-dimensional space of cocycles corresponding to symmetric tetrahedra with axis $[p_1, p_2]$ with

$$\mathbb{A}(p_1, p_3, p_4) = u \quad \mathbb{A}(p_1, p_4, p_2) = -t \quad \mathbb{A}(p_1, p_2, p_3) = t \quad \mathbb{A}(p_3, p_4, p_2) = u.$$

For a general triangulation it is more convenient to start with a coboundary which is easily computed by assigning to each edge a real number and computing the coderivative of that cochain. In the following sections we consider the family of constant cocycles in the case of the figure eight knot complement: those are the coboundaries such that

$$\mathbb{A}(p_1, p_3, p_4) = t \quad \mathbb{A}(p_1, p_4, p_2) = -t \quad \mathbb{A}(p_1, p_2, p_3) = t \quad \mathbb{A}(p_3, p_4, p_2) = t.$$

They are precisely those described in Proposition 5.1.

5.2. Compatibility conditions II: Edge equations. In order to have a coherent gluing of the tetrahedra along the edges, we have to impose some compatibility equations as in the real hyperbolic gluing of ideal tetrahedra. We have four equations (two for each cycle of edges corresponding to the two end points of each cycle):

$$\begin{aligned} z_1 w_1 z_2 \tilde{w}_1' \tilde{z}_1' \tilde{w}_2' &= 1 \\ z_1' w_1' \tilde{z}_2 \tilde{w}_1 \tilde{z}_1 w_2' &= 1 \\ \tilde{z}_2' \tilde{w}_3 \tilde{z}_3 \tilde{w}_2 \tilde{z}_3' \tilde{w}_3' &= 1 \\ z_2' w_3' z_3' w_2 z_3 w_3 &= 1. \end{aligned}$$

Observe that the product of the four equations is 1; that is, one of the equations is obviously dependent on the other three.

5.3. Holonomy. Using the invariants of the tetrahedra introduced above one can compute the rotation part of the peripheral subgroups following essentially the same method as in Thurston's notes. By rotation part of an element of the group of automorphisms of the Heisenberg space we mean the rotation part of its natural projection on the automorphism group of $\mathbb{C}$.

First of all, one has to identify the peripheral subgroups corresponding to the ends of the manifold whose fundamental group is to be embedded in $PSU(2,1)$. In the case of the figure eight complement, Figure 14 shows the link around the unique boundary component. It is a torus triangulated by eight triangles a, b, c, d, e, f, g, h where the side-pairings used to glue each triangle to the next are explicited. The representation of the fundamental group of this torus is obtained following the side-pairings along the triangles which border a loop representing each of the two generators. The rotational part of each generator is precisely the product of each of the invariants at each of the vertices contained in the loop associated to that generator. Calling H_1 and H_2 the two generators of the peripherical group $\mathbb{Z}\oplus\mathbb{Z}$ we obtain for their rotational part $R(H_1)$ and $R(H_2)$ the following formulae:

$$R(H_1) = R(G_1^{-1}G_3G_1^{-1}G_2G_3^{-1}G_1G_3^{-1}) = \tilde{z}_1'\tilde{z}_2'\tilde{w}_3\tilde{z}_3\tilde{z}_1w_2'z_1'z_2'w_3z_3z_1\tilde{w}_2'$$

and

$$R(H_2) = R(G_2^{-1}G_1) = -\tilde{w}_2'\tilde{w}_1'\tilde{z}_1'.$$

Contrary to the real hyperbolic case where, in order to be parabolic, the rotation part of an element should be 1, for an element of the Heisenberg group a necessary condition is that $|R(H_i)| = 1$ (in fact, an element could be ellipto-parabolic).

5.4. A representation with purely parabolic peripheral holonomy. In order to describe all parabolic representations we first observe that they can be described by sewing two tetrahedra.

We consider the following presentation with parabolic generators

$$\langle\ g_1, g_3 \mid [g_3, g_1^{-1}]g_3 = g_1[g_3, g_1^{-1}]\ \rangle.$$

Let p_1 and p_2 be the parabolic fixed points of g_1 and g_3 respectively. Let $p_3 = g_1^{-1}(p_2)$ and $q_3 = g_3^{-1}(p_1)$.

Lemma 5.3. $g_3g_1^{-1}(p_2) = g_1^{-1}g_3(p_1)$.

Proof.

$$\begin{aligned}[g_3, g_1^{-1}]g_3(p_3) &= g_1[g_3, g_1^{-1}](p_3) = g_1g_3g_1^{-1}g_3^{-1}g_1g_1^{-1}(p_2)\\ &= g_1g_3g_1^{-1}(p_2) = g_1(g_3g_1^{-1}(p_2)).\end{aligned}$$

Therefore,

$$g_1^{-1}g_3g_1^{-1}g_3^{-1}g_1(g_3g_1^{-1}(p_2)) = g_3g_1^{-1}(p_2).$$

That means that $p_4 = g_3 g_1^{-1}(p_2)$ is the fixed point of the parabolic element $g_1^{-1} g_3 g_1^{-1} g_3^{-1} g_1$, and that implies that $p_4 = g_1^{-1} g_3(p_1)$. We conclude that

$$g_3 g_1^{-1}(p_2) = g_1^{-1} g_3(p_1).$$

q.e.d.

We obtain therefore two tetrahedra as in the hyperbolic representation by considering the points p_1, p_2, p_3, p_4, q_3 and the same side pairings between the corresponding triples of points as we've been considering until now.

5.4.1. Special case (symmetric tetrahedra). A direct substitution of the symmetric solutions to the face conditions obtained in 5.1.1 into each of the edge equations gives:

Proposition 5.4. *For a couple of symmetric tetrahedra glued according to the scheme above, the edge compatibility equations are equivalent to the equation*

$$\frac{(t^2(1-\alpha)^2+\alpha^2)(t^2(1-\beta)^2+\beta^2)}{(1-\alpha)(1-\beta)(1+t^2)} = 1.$$

That is

$$(\alpha-1)^2(\beta-1)^2t^4 + (2\alpha^2\beta^2 - 2\alpha^2\beta + \beta^2 - \alpha\beta + \alpha^2 + \alpha - 2\alpha\beta^2 - 1 + \beta)t^2$$
$$-1 + \alpha^2\beta^2 + \alpha - \alpha\beta + \beta = 0.$$

That family of couples of tetrahedra defines a two dimensional family of representations of the fundamental group of the complement of the figure eight knot.

A remarkable one dimensional family is obtained imposing $\alpha = \beta$ in the equation of Proposition 5.4, which transforms to:

$$(\alpha-1)^4t^4 + (2\alpha^2-1)(\alpha-1)^2t^2 + (\alpha^2+\alpha-1)(\alpha^2-\alpha+1),$$

which has a solution for $\alpha \leq 5/8$ (positive discriminant) and $1 - 2\alpha^2 + \sqrt{5-8\alpha} \geq 0$ (a solution of the biquadratic equation should be positive). We obtain a solution for $-(1+\sqrt{5})/2 \leq \alpha \leq 5/8$. The solutions are

$$t = \frac{\sqrt{1-2\alpha^2+\sqrt{5-8\alpha}}}{\sqrt{2}(\alpha-1)}$$

if $-(1+\sqrt{5})/2 \leq \alpha \leq 5/8$, and another solution

$$t = \frac{\sqrt{1-2\alpha^2-\sqrt{5-8\alpha}}}{\sqrt{2}(\alpha-1)},$$

which is valid for $(\sqrt{5}-1)/2 \leq \alpha \leq 5/8$.

In the next section, we pick up an element of this family, namely, we make

$$\alpha = \beta = \frac{1}{2},$$

and in that case $t = \sqrt{3}$. Using Proposition 5.4 and after a straightforward calculation one obtains

Lemma 5.5. *When gluing two symmetric tetrahedra as in the scheme above, we obtain the following expressions for the generators of the peripheral group*

$$R(H_1) = \frac{(i+t)^4(t(1-\alpha)-\alpha i)}{(-i+t)^4(t(1-\beta)+\beta i)}$$

$$R(H_2) = -\frac{(t(1-\alpha)-\alpha i)(t(1-\beta)-\beta i)}{(1-it)(\beta-1)}.$$

The necessary condition for parabolicity is $|R(H_i)| = 1$, that is

$$t^2(1-\alpha)^2 + \alpha^2 = t^2(1-\beta)^2 + \beta^2$$

and

$$\frac{(t^2(1-\alpha)^2 + \alpha^2)(t^2(1-\beta)^2 + \beta^2)}{(1-\beta)^2(1+t^2)} = 1.$$

Using again 5.4 and the last equation, we conclude that $\alpha = \beta$.

Proposition 5.6. *There exists a unique representation* (*among representations obtained gluing two symmetric tetrahedra as above*) *of the fundamental group of the complement of the figure eight knot with purely parabolic peripheral holonomy. In that case $R(H_1) = 1$, $\alpha = \beta = 1/2$ and $t = \sqrt{3}$.*

Proof. It follows by solving the equations above. In particular, it suffices to impose that $R(H_2) = 1$. In fact, that equation becomes

$$-\frac{(t(1-\alpha)-\alpha i)^2}{(1-it)(\alpha-1)} = 1,$$

which implies $\alpha = 1/2$, and the proposition follows. q.e.d.

5.4.2. General case of unipotent representations.

Theorem 5.7. *There exists a unique representation of the fundamental group of the complement of the figure eight knot with faithful purely parabolic peripheral holonomy.*

Of course, the representation is the same as the one obtained in the previous proposition with the additional symmetry hypothesis.

Proof. We can assume that the representation is constructed by sewing tetrahedra as the previous lemma shows. The compatibility equations are

Edge equations:

$$\begin{aligned} z_1 w_1 z_2 \tilde{w}'_1 \tilde{z}'_1 \tilde{w}'_2 &= 1 \\ z'_1 w'_1 \tilde{z}_2 \tilde{w}_1 \tilde{z}_1 w'_2 &= 1 \\ \tilde{z}'_2 \tilde{w}_3 \tilde{z}_3 \tilde{w}_2 \tilde{z}'_3 \tilde{w}'_3 &= 1 \\ z'_2 w'_3 z'_3 w_2 z_3 w_3 &= 1. \end{aligned}$$

Face equations:

$$\begin{aligned} z_2 \tilde{z}'_1 z'_3 w_3 \tilde{w}_1 w'_2 &= 1 \\ z_3 \tilde{z}_1 z'_2 w'_1 \tilde{w}'_3 \tilde{w}_2 &= 1 \\ z_1 \tilde{z}'_2 \tilde{z}_3 w_2 \tilde{w}'_1 w'_3 &= 1 \\ z'_1 \tilde{z}_2 \tilde{z}'_3 w_1 \tilde{w}_3 \tilde{w}'_2 &= 1. \end{aligned}$$

Unipotent condition:

$$R(H_1) = \tilde{z}'_1 \tilde{z}'_2 \tilde{w}_3 \tilde{z}_3 \tilde{z}_1 w'_2 z'_1 z'_2 w_3 z_3 z_1 \tilde{w}'_2 = 1$$
$$R(H_2) = -\tilde{w}'_2 \tilde{w}'_1 \tilde{z}'_1 = 1.$$

From equation $R(H_2) = 1$ we obtain $-\tilde{w}'_2 \tilde{w}'_1 \tilde{z}'_1 = 1$, that is $\tilde{z}'_1 = \tilde{w}'_3$. Using the first edge equation we then obtain $z_3 = w_1$. Using the third equation we obtain $\tilde{z}_3 = \tilde{w}_1$ and finally using the second equation we obtain $z'_1 = w'_3$. The face equations therefore are now written:

$$\begin{aligned} w_3 \tilde{w}'_3 w'_2 w_3 \tilde{w}_1 w'_2 &= 1 \\ w_1 \tilde{w}_2 w'_1 w'_1 \tilde{w}'_3 \tilde{w}_2 &= 1 \\ w_2 \tilde{w}'_1 \tilde{w}_1 w_2 \tilde{w}'_1 w'_3 &= 1 \\ w'_3 \tilde{w}_3 \tilde{w}'_2 w_1 \tilde{w}_3 \tilde{w}'_2 &= 1 \end{aligned} \quad \text{that is} \quad \begin{aligned} w_3^2 {w'}_2^2 \tilde{w}'_3 \tilde{w}_1 &= 1 \\ {w'}_1^2 \tilde{w}_2^2 w_1 \tilde{w}'_3 &= 1 \\ w_2^2 \tilde{w}'^2_1 \tilde{w}_1 w'_3 &= 1 \\ \tilde{w}_3^2 \tilde{w}'^2_2 w_1 w'_3 &= 1. \end{aligned}$$

We use the equations

$$\begin{aligned} z_1 z'_1 &= \overline{\tilde{z}_1 \tilde{z}'_1} \\ z_2 \tilde{z}_2 &= \overline{z'_2 \tilde{z}'_2} \\ z_3 \tilde{z}'_3 &= \overline{\tilde{z}_3 z'_3} \end{aligned}$$

and

$$\begin{aligned} w_1 w'_1 &= \overline{\tilde{w}_1 \tilde{w}'_1} \\ w_2 \tilde{w}_2 &= \overline{w'_2 \tilde{w}'_2} \\ w_3 \tilde{w}'_3 &= \overline{\tilde{w}_3 w'_3}. \end{aligned}$$

Writing in terms of normalized coordinates $\infty, (0,0), (1, t_2), (w_1, s_2)$ the configuration parametrized by the coordinates $w_1, w'_1, \tilde{w}_1, \tilde{w}'_1$, we obtain the solutions:

$$w_1 = \beta_1 + i\beta_2 = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \quad t_2 = s_2 = \pm\sqrt{3}$$
$$w_1 = \beta_1 + i\beta_2 = \frac{3}{2} \pm i\frac{\sqrt{7}}{2} \quad t_2 = \mp\sqrt{7} \quad s_2 = 0$$
$$w_1 = \beta_1 + i\beta_2 = \frac{3}{8} \pm i\frac{\sqrt{7}}{8} \quad s_2 = \mp\sqrt{7} \quad t_2 = 0.$$

We remark that the solutions are given by symmetric configurations. The groups obtained from the solutions come in pairs conjugated by an anti-holomorphic reflection. We don't distinguish them up to conjugation in the isometry group.

We will see in section 6.5.1 that the solution

$$w_1 = \frac{3}{8} + i\frac{\sqrt{7}}{8} \quad s_2 = -\sqrt{7} \quad t_2 = 0,$$

so that

$$w_1' = \frac{5}{4} + i\frac{\sqrt{7}}{4} \quad \tilde{w}_1 = \frac{3}{8} - i\frac{\sqrt{7}}{8} \quad \tilde{w}_1' = \frac{5}{4} - \frac{\sqrt{7}}{4}$$

gives rise to a cyclic boundary holonomy. Analogously, the solution

$$w_1 = \beta_1 + i\beta_2 = \frac{3}{2} \pm i\frac{\sqrt{7}}{2} \quad t_2 = \mp\sqrt{7} \quad s_2 = 0,$$

so that

$$w_1' = -\frac{3}{4} - i\frac{\sqrt{7}}{4} \quad \tilde{w}_1 = \frac{3}{2} - i\frac{\sqrt{7}}{2} \quad \tilde{w}_1' = -\frac{1}{4} + \frac{\sqrt{7}}{4}$$

is associated to a cyclic boundary holonomy. The remainder solution will be shown to have a faithful holonomy in the next section. This concludes the proof. q.e.d.

6. Discrete holonomy

Theorem 6.1. *There exists a branched spherical CR-structure on the complement of the figure eight knot with discrete holonomy.*

Proof. We use the same identifications that Thurston used in his construction for a hyperbolic real structure on the figure eight knot. That is, two tetrahedra with the identifications given in Figure 7. We realize the two tetrahedra in the Heisenberg space gluing a pair of sides. The side pairings transformations are shown in Figure 7 where the two tetrahedra are represented with a common side. (Here we introduce the point $q_3 = (\omega, 0)$ where $\omega = \frac{-1+i\sqrt{3}}{2}$.) They are determined by their action on three points and are defined by:

$$\begin{aligned} g_1 &: (p_4, p_3, p_1) \to (q_3, p_2, p_1) \\ g_2 &: (p_2, p_3, p_4) \to (p_1, p_4, q_3) \\ g_3 &: (p_3, p_2, p_1) \to (p_4, p_2, q_3). \end{aligned}$$

We will also use the infinite configuration with

$$p_1 = \infty, p_2 = (0,0), p_3 = (1, \sqrt{3}), p_4 = \left(\frac{1+i\sqrt{3}}{2}, \sqrt{3}\right), q_3 = (\omega, \sqrt{3}).$$

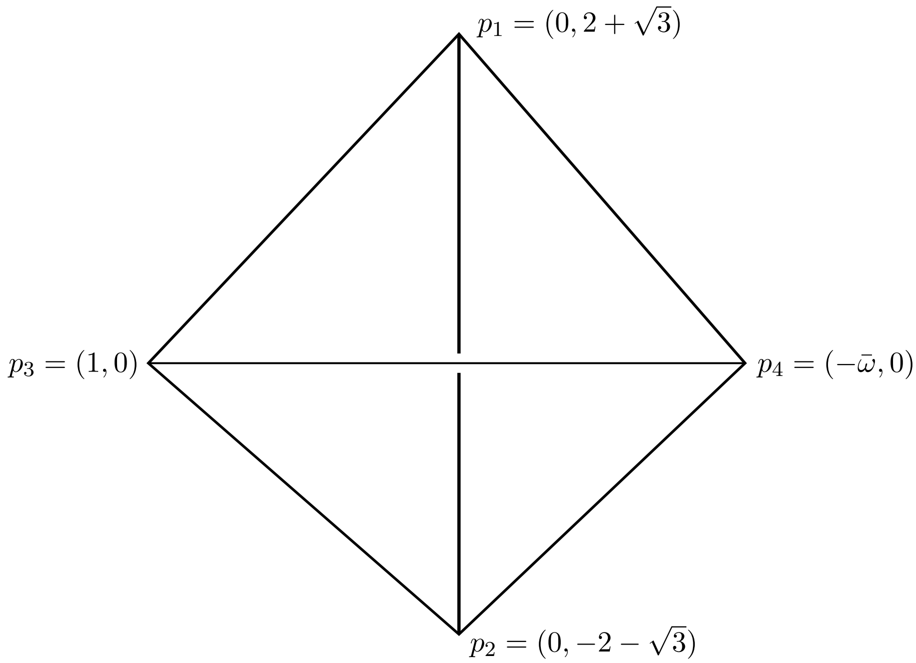

Figure 6. A schematic view of the standard ideal tetrahedron in the Heisenberg group. Here $\omega = \frac{-1+i\sqrt{3}}{2}$.

Using the computations as explained in 3.0.1, we obtain the following generators:

$$G_1 = \begin{bmatrix} 1 & 1 & \bar{\omega} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} \sqrt{3}i & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \bar{\omega} & -1 & 1 \end{bmatrix}.$$

Note that G_1, G_3 are parabolic and G_2 is elliptic of order six. Moreover, the three generators are in $PU(2,1,\mathbb{Z}[\omega])$, proving that the representation (if it exists) is discrete.

6.1. 0-skeleton. The 0-skeleton satisfies the compatibility conditions given by the equations imposing that the invariants multiply to unity.

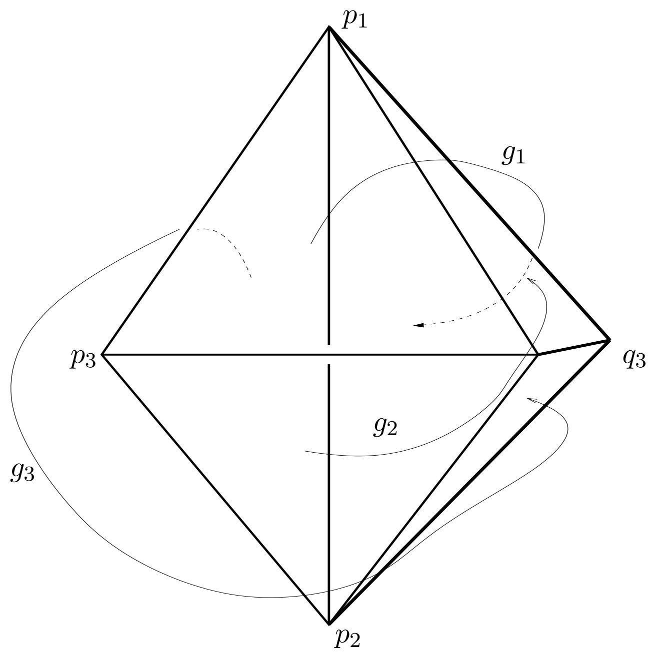

Figure 7. Identifications on the tetrahedra.

That is sufficient to obtain a representation of the fundamental group of the complement of the figure eight knot.

To be explicit, starting with the points

$$p_1 = \infty, p_2 = (0,0), p_3 = (1, \sqrt{3}), p_4 = (-\bar{\omega}, \sqrt{3}), q_3 = (\omega, \sqrt{3}),$$

we can follow the identifications of the triples of points in the following order:

1) $G_1(p_4, p_3, p_1) = (q_3, p_2, p_1)$ and $G_1(p_2) = (-1, -\sqrt{3}) = q_4$.
2) $G_1G_3^{-1}(p_4, p_2, q_3) = (p_2, q_4, p_1)$ and $G_1G_3^{-1}(p_1) = (\omega^2, -\sqrt{3}) = q_5$.
3) $G_1G_3^{-1}G_2(p_2, p_3, p_4) = G_1G_3^{-1}(p_1, p_4, q_3) = (q_5, p_2, p_1)$ and
$$G_1G_3^{-1}G_2(p_1) = (\omega, -\sqrt{3}) = q_6.$$
4) $G_1G_3^{-1}G_2G_1^{-1}(q_3, p_2, p_1) = G_1G_3^{-1}G_2(p_4, p_3, p_1) = (p_1, p_2, q_6)$ and $G_1G_3^{-1}G_2G_1^{-1}(p_4) = p_3$.
5) Finally $G_1G_3^{-1}G_2G_1^{-1}G_3(p_3, p_2, p_1) = G_1G_3^{-1}G_2G_1^{-1}(p_4, p_2, q_3) = (p_3, p_2, p_1)$.

The last equality shows that $G_1G_3^{-1}G_2G_1^{-1}G_3 = Id$ and therefore proves that we have indeed a representation of the fundamental group of the complement of the figure eight.

6.2. 1-skeleton. The one skeleton we choose has to be compatible with the identifications above. Recall that a segment of $\mathbb{C}$-circle is determined by giving the first point and its end point. That follows because the $\mathbb{C}$-circle has an intrinsic orientation. Referring to Figure 13, where the

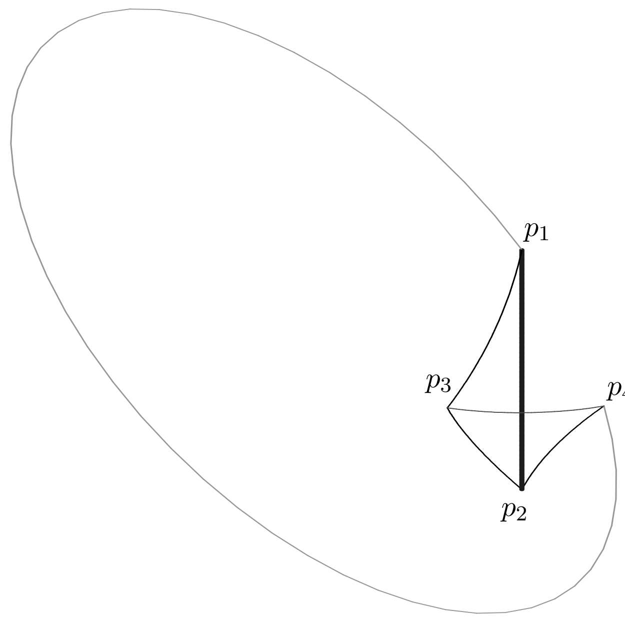

Figure 8. A realistic picture of the 1-skeleton of one of the tetrahedra.

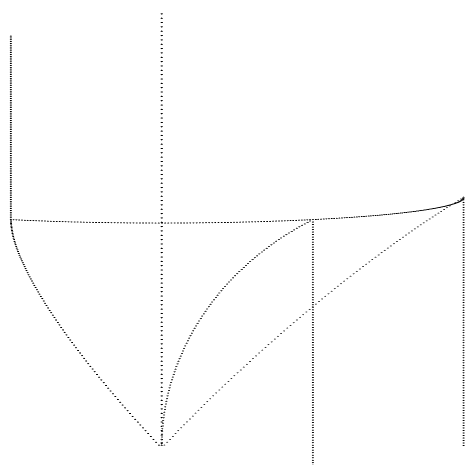

Figure 9. The 1-skeleton of the two tetrahedra with $p_1 = \infty$ in Heisenberg space.

orientations of the edges are explicit, we might choose accordingly the segments of $\mathbb{C}$-circles.

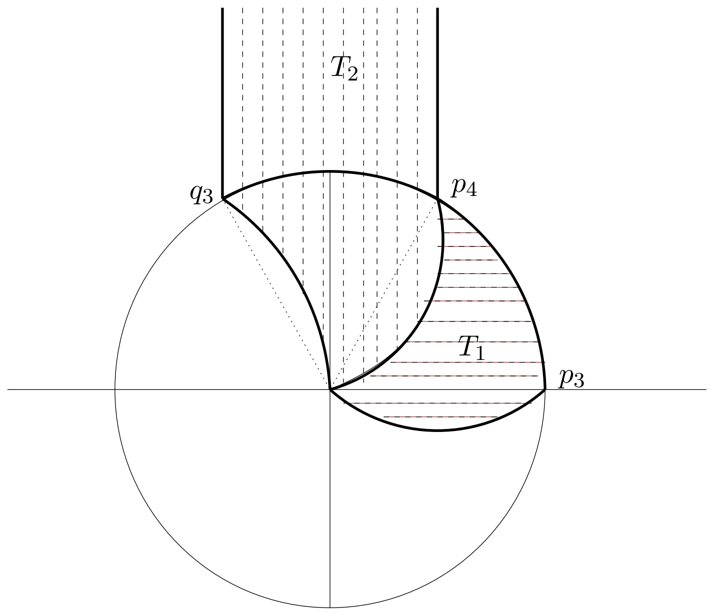

Figure 10. The projection of the two tetrahedra T_1 and T_2 on the z-plane of Heisenberg space.

6.3. 2-skeleton. We use the model with $p_1 = \infty$ in Heisenberg space. The faces are defined in the following way:

6.3.1. T_1**.**

1) $F(p_2, p_3, p_4)$ is defined as the union of ray-segments starting at p_2 and ending at the edge $[p_3, p_4]$. Using standard formulae for the $\mathbb{C}$-circle between two points we obtain that the formula for each of them is

$$(e^{i\varphi}(e^{i\theta} - \bar{\omega}), \sqrt{3}\cos\theta - \sin\theta)$$

for $0 \leq \varphi \leq \pi/3$ and $-2\pi/3 \leq \theta \leq -\pi/3$. They are all obtained from the edge $[p_2, p_3]$ by a rotation of angle φ. Their projection on the z-coordinate of Heisenberg space fills the region defined by the projection of the triangle $[p_2, p_3, p_4]$.

2) All other faces of T_1 are obtained by joining each of the edges to p_1. That gives a cylindrical surface with the base formed by the triangle $[p_2, p_3, p_4]$.

Observe that somewhat strangely there is one edge, namely $[p_1, p_4]$, which is not contained in a 2-face. This edge should be thought of as part of the second tetrahedron. We therefore define T_1 to be the region defined by the 2-faces, neglecting the edge $[p_1, p_4]$ and puting in instead the edge $[p_4, p_1]$.

6.3.2. T_2**.**

1) From $F(p_1, p_4, q_3) = G_2(F(p_2, p_3, p_4))$ we obtain the definition of that face as the union of ray-segments starting at p_1 and ending on the edge $[p_4, q_3]$.
2) From $F(q_3, p_2, p_4) = G_3(F(p_1, p_2, p_3))$ we obtain the definition of that face as the union of ray-segments starting at the edge $[p_2, p_4]$ and ending at q_3. The projection of that surface on the z-coordinate is contained in the region defined by the projection of the edges of the triangle $[p_2, p_{4,3}]$.
3) The face $F(p_1, p_2, p_4)$ is obtained as the union of the ray-segments from the edge $[p_2, p_4]$ to p_1 and all the C-circles passing through p_1 and the half line starting at p_4 parallel to the y-axis. The analogous definition holds for $F(p_1, p_2, q_3)$ as shown in Figure.

The definitions of T_1 and T_2 make clear the following lemma:

Lemma 6.2. *G_1, G_2, G_3 are side pairings of the union $T_1 \cup T_2$.*

Proposition 6.3. *The quotient space of the union of the tetrahedra T_1 and T_2 (excluding the vertices) by the side pairings G_1, G_2, G_3 is the complement of the figure eight knot.*

Proof. The proof follows as in Thurston's, except that we have two extra sides. They don't change the topology of the quotient as the reader can be easily convinced. q.e.d.

6.4. The structure around the edges. The quotient of $T_1 \cup T_2$ by the side pairings inherits a spherical CR-structure except possibly at the edges. There are two of them, represented by $[p_2, p_1]$ and $[p_2, p_4]$. We have to verify that around each there exists a spherical structure. In fact, we show that the neighborhood around those edges is a branched cover of a neighborhood of half of the t-axis in the Heisenberg space. That is what we call a branched spherical CR structure below.

We will make explicit the computations for the edge $[p_2, p_1]$, the other edge being similar. We follow the side pairings as for the 0-skeleton. That is, the neighborhood around $[p_2, p_1]$ should be a union of the neighborhoods contained in (following that order) T_1, T_2, $G_1(T_1)$, $G_1G_3^{-1}(T_2)$, $G_1G_3^{-1}G_2(T_1)$ and $G_1G_3^{-1}G_2G_1^{-1}(T_2)$. We will show that the union of those tetrahedra forms a neighborhood covering three times a standard tubular neighborhood of the edge $[p_2, p_1]$ in Heisenberg space. To show this we need to prove that each tetrahedra, in this series of six, matches the previous one filling around the edge. There might be intersections between two consecutive tetrahedra (besides the matching faces) but those intersections should be *monotone*: that is, following from a point in the face intersection of two consecutive tetrahedra, along a small positively oriented circle around the edge, there exists a positive interval contained in the new tetrahedron and not in the previous one.

1) T_1 and T_2 are well positioned by construction.
2) $G_1(T_1)$ is shown in Figure 11. In fact, G_1 is a Heisenberg translation and therefore the action of G_1 descends to an action on the projection as a complex translation $z \to z - 1$. Its intersection with T_2 is the side pairing face.
3) The match $G_1(T_1)$ with $G_1G_3^{-1}(T_2)$ is analyzed by composing with G_1^{-1}, that is, it corresponds to the match between T_1 and $G_3^{-1}(T_2)$. That is the only time a computation is needed, but it is a simple verification and can be seen in Figure 12. In fact, the image of the projection of the face $\Delta(p_1, p_4, q_3)$ by G_3^{-1} is given by
$$\left(\frac{1}{4}\sqrt{3}t + \frac{3}{4} + \frac{1}{2}\sqrt{t^2+1}\cos\theta, \frac{1}{4}t + \frac{1}{4}\sqrt{3} + \frac{1}{2}\sqrt{t^2+1}\sin\theta\right);$$
those are circles containing the projections of p_3 and p_4 as shown in Figure 12. The projection of $G_3^{-1}(T_2)$ contains the disc centred at p_3 in the figure where a slice of angle $\pi/6$ is deleted.
4) The match $G_1G_3^{-1}(T_2)$ and $G_1G_3^{-1}G_2(T_1)$ corresponds to T_2 and $G_2(T_1)$ which is clear.
5) As we already know that we have a representation of the fundamental group, the last two tetrahedra around the edge are
$$G_1G_3^{-1}G_2(T_1) = G_3^{-1}G_1(T_1)$$
and $G_1G_3^{-1}G_2G_1^{-1}(T_2) = G_3^{-1}(T_2)$. We also know that $G_1(T_1) \cup T_2$ match monotonically, so we conclude that $G_3^{-1}G_1(T_1)$ and $G_3^{-1}(T_2)$ match monotonically.
6) The match between $G_1G_3^{-1}G_2G_1^{-1}(T_2)$ and T_1 can be analyzed again in Figure 12 as it corresponds precisely to the match between $G_3^{-1}(T_2)$ and T_1.

The branched cover $z \to z^n$ shows that a degree n covering of the punctured complex disc is again a punctured disc. In particular, its complex strucure can be extended to the full disc. On the other hand, consider the branched covering $\sigma : D \to B$ where $D = \{\ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^{2n} + |z_2|^2 < 1\ \}$, $B = \{\ (w_1, w_2) \in \mathbb{C}^2 \mid |w_1|^2 + |w_2|^2 < 1\ \}$ is the complex ball and $\sigma(z_1, z_2) = (z_1^n, z_2)$. The boundary of D is a spherical CR-structure except at the singular circle $z_1 = 0$. In fact, the tangent plane $TD \cap JTD$ along that circle is not of contact type.

Definition 6.4. A branched spherical CR-structure on a 3-manifold is a spherical structure defined over that manifold except at a finite number of curves. Each of the curves have a neighborhood which is CR equivalent to a neighborhood of part of the curve $z_1 = 0$ in ∂D as defined above.

6.5. Holonomy of the torus link. Refering to Figure 13 and section 5.3, the holonomy of the torus link at the vertex can be computed

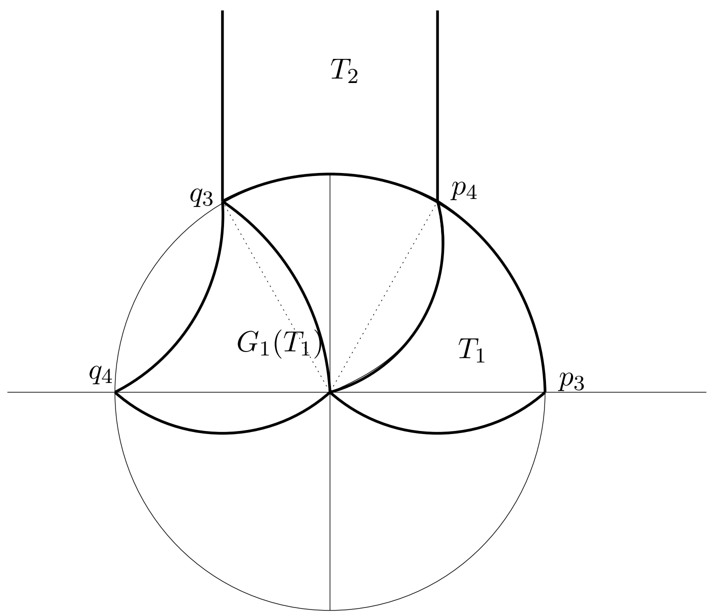

Figure 11. Projected view of T_1 and $G_1(T_1)$.

following the identifications of the triangles forming the link. Starting with the triangle on the right of the first tetrahedron we obtain the generators. Here we conjugated the generators G_i by the element satisfying

$$\gamma : (\infty, 0, [1, -\sqrt{3}]) \to (p_1, p_4, p_3)$$

to simplify the expressions of the generators of the holonomy, so that

$$H_1 = G_1^{-1}G_3G_1^{-1}G_2G_3^{-1}G_1G_3^{-1} = \begin{bmatrix} \frac{-9+i\sqrt{3}}{2} & \frac{3+i5\sqrt{3}}{2} & \frac{5+i\sqrt{3}}{2} \\ \frac{3+i5\sqrt{3}}{2} & 5-i\sqrt{3} & 1-i2\sqrt{3} \\ \frac{5+i\sqrt{3}}{2} & 1-i2\sqrt{3} & -2-i\sqrt{3} \end{bmatrix}$$

$$H_2 = G_2^{-1}G_1 = \frac{-1+i\sqrt{3}}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -\frac{1+i3\sqrt{3}}{2} \end{bmatrix}.$$

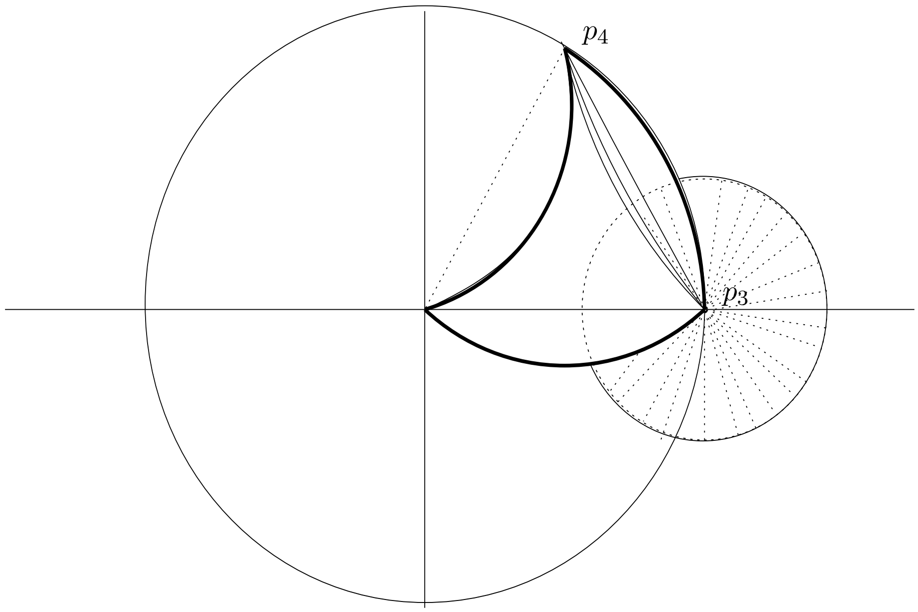

Figure 12. Projected view of T_1, T_2 and $G_3^{-1}(T_2)$.

Writing

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -\omega & \bar{\omega} & 0 \\ \bar{\omega} & \omega & 1 \end{bmatrix} \in U(2,1),$$

we obtain

$$\omega H_1 = A \begin{bmatrix} 1 & -2\bar{\omega} & -1-2\omega \\ 0 & 1 & 2\omega \\ 0 & 0 & 1 \end{bmatrix} A^{-1}$$

and

$$\omega H_2 = A \begin{bmatrix} 1 & \bar{\omega} & \omega \\ 0 & 1 & -\omega \\ 0 & 0 & 1 \end{bmatrix} A^{-1}.$$

Figures 13 and 14 show how to compute those elements. It turns out that they are parabolic and independent:

Proposition 6.5. *The holonomy of the torus link is faithful and parabolic.*

6.5.1. Holonomy of the other representations. We consider first the solution

$$w_1 = \frac{3}{8} + i\frac{\sqrt{7}}{8} \quad s_2 = -\sqrt{7} \quad t_2 = 0.$$

In that case we have

$$z_1 = \frac{5}{4} + i\frac{\sqrt{7}}{4} \quad t_1 = \sqrt{7} \quad s_1 = 0.$$

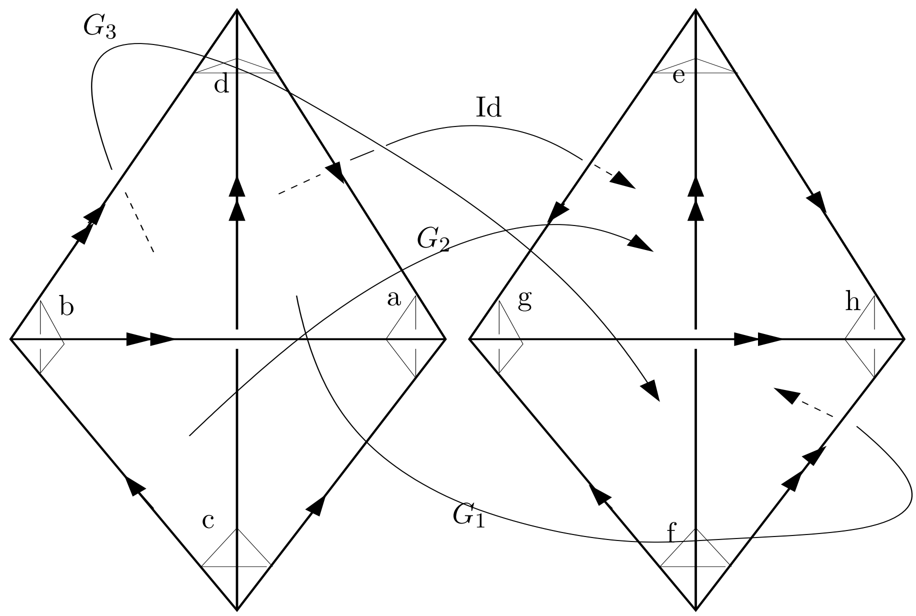

Figure 13. The triangulation of the torus link.

We consider the configuration of points:

$$p_1 = \infty \quad p_2 = (0,0)$$

$$q_1 = (1, \sqrt{7}) \quad q_2 = (z_1, 0) \quad q_3 = (z_1 w_1, s_2|z_1 w_1|^2) = \left(\frac{1}{4} + i\frac{\sqrt{7}}{4}, -\frac{\sqrt{7}}{2}\right),$$

which corresponds to the pairing of the tetrahedra $[p_1, p_2, q_1, q_2]$ and $[p_1, p_2, q_2, q_3]$ along the common face $[p_1, p_2, q_2]$. A simple computation shows that

$$\tan \mathbb{A}(p_1, p_2, q_1) = \sqrt{7} \quad \tan \mathbb{A}(q_1, q_2, p_1) = -\sqrt{7}$$

and the triples of points $[p_1, p_2, q_2]$, $[q_1, q_2, p_2]$ are contained in $\mathbb{R}$-circles. We compute the matrices in $SU(2,1)$ corresponding to the face identifications:

$$G_1 = \begin{bmatrix} 1 & 1 & -\frac{1}{2} - \frac{i\sqrt{7}}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

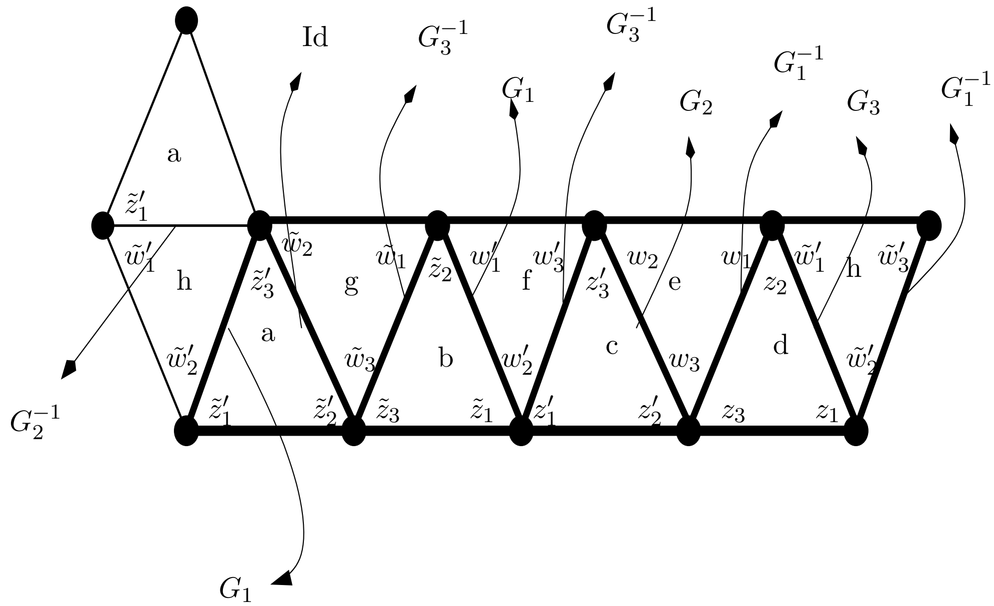

Figure 14. Computation of the holonomy at the vertex.

$$G_2 = \begin{bmatrix} 2 & \frac{3}{2} - \frac{i\sqrt{7}}{2} & -1 \\ -\frac{3}{2} - \frac{i\sqrt{7}}{2} & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{1}{2} + \frac{i\sqrt{7}}{2} & 1 & 1 \end{bmatrix}.$$

The holonomy is

$$H_1 = \begin{bmatrix} 36 & \frac{75}{2} - \frac{15i\sqrt{7}}{2} & -25 \\ -\frac{75}{2} - \frac{15i\sqrt{7}}{2} & -49 & 25 + 5i\sqrt{7} \\ -25 & -25 + 5i\sqrt{7} & 16 \end{bmatrix}$$

and

$$H_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & \frac{5}{2} + \frac{i\sqrt{7}}{2} \\ -1 & -\frac{5}{2} + \frac{i\sqrt{7}}{2} & 4 \end{bmatrix}.$$

We observe that $(H_2)^{-5} = H_1$, and therefore the holonomy is cyclic.

Analogously, the solution

$$w_1 = \beta_1 + i\beta_2 = \frac{3}{2} \pm i\frac{\sqrt{7}}{2} \quad t_2 = \mp\frac{\sqrt{7}}{2} \quad s_2 = 0$$

gives rise to a cyclic holonomy.

6.6. Relation to Eisenstein-Picard group. In [**FP**] we proved that the Eisenstein-Picard Group $PU(2,1,\mathbb{Z}[\omega])$ is generated by

$$P = \begin{bmatrix} 1 & 1 & \omega \\ 0 & \omega & -\omega \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 & \omega \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

In this section we identify the generators of the holonomy in terms of these generators. The information is contained in the following proposition. We first state a lemma whose proof is a simple computation after a guess obtained by identifying the translational part of each parabolic element.

Lemma 6.6. *The holonomy of the torus link is given by*

$$\omega H_1 = A[Q, P^{-1}]^2 P^6 A^{-1}$$

and

$$\omega H_2 = A[P^{-1}, Q]A^{-1},$$

where $A = IQP^{-1}QP^{-1}Q^{-1}PQ^{-1}I$.

From the lemma and a computation we obtain the generators of the group.

Proposition 6.7.

$$G_1 = (QP^{-1})^3 Q^{-1},$$
$$\bar{\omega} G_2 = \bar{\omega} G_1 H_2^{-1} = (QP^{-1})^3 Q^{-1} A[Q, P^{-1}]A^{-1}$$
$$G_3 = IG_1 I = I(QP^{-1})^3 Q^{-1} I.$$

6.7. The limit set. Consider $\Gamma_8 = \langle G_1, G_2, G_3 \rangle \subset G = PU(2,1,\mathbb{Z}[\omega])$. We prove

Theorem 6.8. *The limit set of* Γ_8 *is* S^3.

Proof. First observe that the limit set of a group is the same as of any of its non-elementary normal subgroups. From $G_2 = [G_3, G_1^{-1}]$ we have

$$= \langle G_1, G_3 \rangle \Gamma_8 = \langle G_1, G_2, G_3 \rangle = \langle G_1, G_3 \rangle \subset= \langle G_1, I \rangle.$$

The last inclusion is of index two as $IG_1I = G_3$. Observe that G_1 is unipotent and the element $P^3 = Q^2 = G_3 G_1^{-1} G_3^{-1} G_1 I$ is a unipotent element in the center of the Heisenberg group containing G_1.

We use now the presentation of G obtained in [**FP**]:

$$G = \langle P, Q, I \mid I^2 = (QP^{-1})^6 = PQ^{-1}IQP^{-1}I = P^3Q^{-2} = (IP)^3 \ \rangle.$$

Call $T_1 = [Q, P^{-1}]$, then we claim that

$$\langle G_1, I \rangle \subset \langle G_1, I, T_1 \rangle$$

is a normal inclusion. We compute:

$$T_1 I T_1^{-1} = I G_1^{-1} I G_1 I \quad T_1^{-1} I T_1 = G_1^{-1} I G_1 P^3 I$$

and

$$T_1 G_1 T_1^{-1} = P^{-3} G_1 \quad T_1^{-1} G_1 T_1 = P G_1.$$

This shows the claim.

Observe now that $\langle G_1, I, T_1 \rangle$ has only one cusp. Indeed, the parabolic subgroup generated by $\langle G_1, P^3, T_1 \rangle$ is cocompact in the Heisenberg group and a fundamental domain (in the Heisenberg group) is contained in the isometric sphere for I (cf [**FP**]). The subgroup $\langle G_1, I, T_1 \rangle$ is therefore a subgroup of $PU(2,1,\mathbb{Z}[\omega])$ of finite index. Its limit set is therefore S^3. q.e.d.

Using the following lemma we conclude that $\langle G_1, I, T_1 \rangle$ coincides with the normalizer of Γ_8 in $PU(2,1,\mathbb{Z}[\omega])$.

Lemma 6.9. *The normal subgroup of* $PU(2,1,\mathbb{Z}[\omega])$ *generated by* Γ_8 (*the normalizer of* Γ_8) *is of index* 6.

Proof. We use again the presentation

$$G = \langle P, Q, I \mid I^2 = (QP^{-1})^6 = PQ^{-1}IQP^{-1}I = P^3Q^{-2} = (IP)^3 \ \rangle.$$

The quotient $H = G/N(\Gamma_8)$ has the following presentation:

$$H = \langle \ \ P, Q, I \mid I^2 = (QP^{-1})^6 = PQ^{-1}IQP^{-1}I = P^3Q^{-2} = (IP)^3 = G_1 = G_2 = G_3 \ \ \rangle.$$

$$H = \langle \ \ P, Q, I \mid I^2 = (QP^{-1})^6 = PQ^{-1}IQP^{-1}I = P^3Q^{-2} = (IP)^3 = (QP^{-1})^3Q^{-1} = (QP^{-1})^3Q^{-1}A[Q, P^{-1}]A^{-1} = I(QP^{-1})^3Q^{-1}I \ \ \rangle$$

with $A = IQP^{-1}QP^{-1}Q^{-1}PQ^{-1}I$. We may clearly substitute the last three relations by

$$(QP^{-1})^3Q^{-1} = [Q, P^{-1}] = Id.$$

On the other hand the relation $[Q, P^{-1}] = Id$ plus the relation $P^3Q^{-2} = Id$ implies $(QP^{-1})^3Q^{-1} = Id$. The presentation above is therefore the same as

$$H = \langle \ \ P, Q, I \mid I^2 = (QP^{-1})^6 = PQ^{-1}IQP^{-1}I = P^3Q^{-2} = (IP)^3 = [Q, P^{-1}] \ \rangle.$$

1) From $P^3Q^{-2} = Id$ we obtain $P^2 = Q^2P^{-1}$ and $P^2Q^{-1} = P^{-1}Q$, therefore

$$Id = [P,Q] = PQP^{-1}Q^{-1} = PQ^{-1}Q^2P^{-1}Q^{-1} =$$

$$PQ^{-1}P^2Q^{-1} = PQ^{-1}P^{-1}Q = [P,Q^{-1}].$$

2) It follows, from $(QP^{-1})^6 = Q^6P^{-6} = Id$ and $P^3Q^{-2} = Id$, that $Q^2 = Id$.
3) The representation becomes:

$$H = \langle\ P,Q,R \mid I^2 = Q^2 = P^3 = PQ^{-1}IQP^{-1}I = (IP)^3 = [P,Q]\ \rangle$$

or

$$H = \langle\ P,Q,R \mid I^2 = Q^2 = P^3 = [PQ,I] = (IP)^3 = [P,Q]\ \rangle.$$

4) From $(IP)^3 = Id$ we have $PIP = IP^{-1}I = IP^2I$. Therefore

$$Id = [PQ,I] = PQIQP^2I = QPIPQPI = QIP^2IQPI = QIP^2I^2QP$$

where we used in the last equality $[PQ,I] = Id$, and therefore

$$Id = QIP^2QP.$$

This implies that $I = QP^2QP = Id$. We identify then the group

$$H = \langle\ P,Q \mid Q^2 = P^3 = [P,Q]\ \rangle$$

as the cyclic group of order six.

q.e.d.

7. Gluing tetrahedra: The Whitehead Link

Using the other special tetrahedra defined in 4.3.2, we obtain a representation of the fundamental group of the complement of the Whitehead link. It suffices to observe that we can glue four tetrahedra as in Thurston, forming an octahedra with dihedral angles equal to $\pi/2$. We make $p_1 = (0, 1+\sqrt{2})$, $p_2 = (0, -(1+\sqrt{2}))$, $p_3 = (1,0)$ and $p_4 = (i,0)$. We have $z_1 = \tilde{z}_1 = i$ with $\mathbb{A}(p_1,p_3,p_4) = \pi/4$. We want to show completeness. Define $q_3 = (-1,0)$ and $q_4 = (-i,0)$.

The generators of the group are given by

$$g_A : [p_1,p_3,p_4] \to [p_4,q_3,p_2]$$

$$g_B : [p_1,p_4,q_3] \to [q_4,p_2,q_3]$$

$$g_C : [p_1,q_3,q_4] \to [q_4,p_3,p_2]$$

$$g_D : [p_1,q_4,p_3] \to [p_4,p_2,p_3].$$

Conjugating the generators above with the mapping

$$[p_1,p_3,p_4] \to [\infty,0,(1,1)],$$

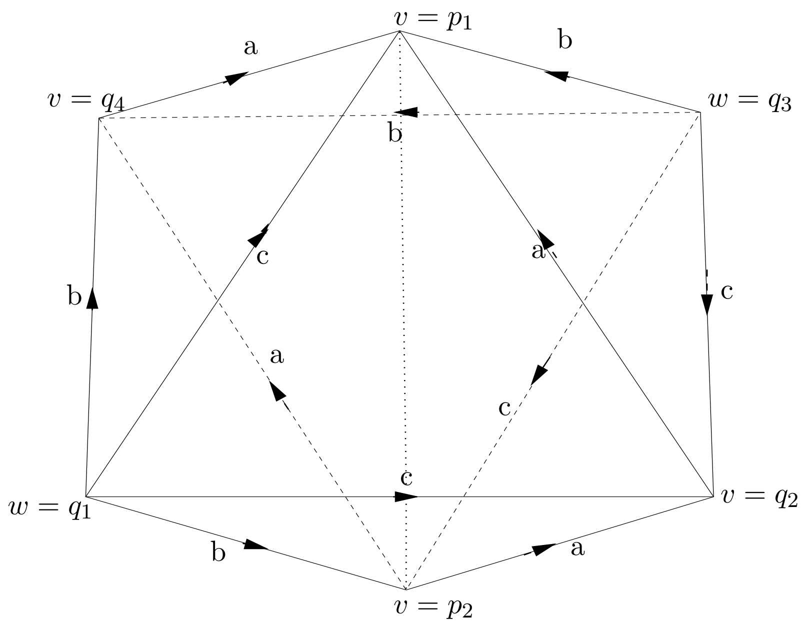

Figure 15. The Whitehead link complement.

we obtain the following matrices in $SU(2,1)$ representing the generators:

$$G_1 = \begin{bmatrix} 1 & 0 & -i \\ -1-i & 1 & -1+i \\ -1-i & 1-i & i \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 1 & 1-i & -1+i \\ -1-i & -1 & 1-i \\ -1+i & 1+i & -1-2i \end{bmatrix}$$

$$G_3 = \begin{bmatrix} i & 1+i & -i \\ 1-i & -1-2i & 2i \\ -1-i & -3+i & 3+2i \end{bmatrix}$$

$$G_4 = \begin{bmatrix} -i & 0 & 0 \\ -1+i & -1 & 0 \\ -1+i & -1+i & -i \end{bmatrix}.$$

G_1 and G_3 have trace $2+i$ and therefore are loxodromic, G_2 and G_4 have trace $-1-2i$ and are elliptic of order four.

We obtained the following:

Theorem 7.1. *The representation of the fundamental group of the Whitehead link complement generated by* G_1, G_2, G_3, G_4 *is in* $PU(2, 1, \mathbb{Z}[i])$ *and is therefore discrete.*

7.1. Holonomy. There are two tori. We use the notation as in [**Ra**]. We compute their holonomy as in the case of the figure eight knot. The first torus has holonomy generated by

$$H_1 = G_3^{-1} G_1^{-1} = \begin{bmatrix} -1-6i & -6-4i & 2+4i \\ -4+6i & 1+8i & 2-4i \\ 2+4i & 4+2i & -1-2i \end{bmatrix} \quad \textit{and} \quad H_2 = G_2.$$

Observe that H_1 is parabolic but H_2 is elliptic. The other torus has holonomy generated by

$$H_1' = G_3 G_1^{-2} G_3 = \begin{bmatrix} 5 & 2-6i & -4 \\ -8-4i & -7+8i & 6+2i \\ -8+8i & 8+12i & 5-8i \end{bmatrix} \quad \textit{and} \quad H_2' = Id.$$

Here H_1' is parabolic. Note that the holonomy of that torus is not faithful.

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