Mon. Not. R. astr. Soc. (1979) **189**, 189–201

A spherically symmetric self-similar universe

C. C. Dyer *Scarborough College and David Dunlap Observatory, University of Toronto, West Hill, Ontario, Canada M1C 1A4*

Received 1979 March 30; in original form 1979 February 22

Summary. A spherically symmetric self-similar dust-filled universe is considered as a simple model of a hierarchical universe. Observable differences between the model in parabolic expansion and the corresponding homogeneous Einstein–de Sitter model are considered in detail. It is found that an observer at the centre of the distribution has a maximum observable redshift and can in principle see arbitrarily large blueshifts. It is found to yield an observed density–distance law different from that suggested by the observations of de Vaucouleurs. The use of these solutions as central objects for Swiss-cheese vacuoles is discussed.

1 Introduction

This paper considers some of the observable properties of spherically-symmetric self-similar spacetimes with particular emphasis on them as models of a portion of a universe which is hierarchical over some scales. This class of solutions of the Einstein field equations has been considered generally by Cahill & Taub (1971), and applied to astrophysical problems by some authors. Carr & Hawking (1974), Lin, Carr & Fall (1976), Henriksen & Wesson (1978) and Bicknell & Henriksen (1978) have used such solutions in attempting to treat the formation of primordial black holes in the early universe.

de Vaucouleurs (1970) has proposed, on the basis of observational data, that the density of matter in the universe is not uniform on a cosmological scale, finding instead that the density goes roughly as $d^{-1.7}$ for large distance, d. This is suggestive of the hierarchical cosmological models proposed by Charlier (1908), and revived recently in a Newtonian treatment by Wertz (1971). In these models, the universe appears the same to all standard observers, provided they average their results over a sufficiently large region of the universe. Though it may at first seem that the observer occupies an undeservedly special position in these models, the very existence of a neighbourhood in which the observer could have evolved depends upon the fact that the average density in that neighbourhood, say a galaxy, significantly exceeds the density averaged over much larger regions.

Bonnor (1972) has more recently considered a specific Bondi–Tolman dust solution in an attempt to model the density law proposed by de Vaucouleurs. His solution is chosen specifically so that the density on constant t slices goes as $r^{-1.5}$ for large r, similar to the proposal of de Vaucouleurs. Unfortunately the optical equations are not simple in this solution so that comparison with observations is difficult.

In this paper we follow the lead of Bonnor, but instead of assuming a particular density law, we assume self-similarity. Though this leads to a density law similar to that of de Vaucouleurs on constant t slices, it has the advantage that optical calculations are easily performed and thus observational consequences are easily considered. One result of particular interest is that along the past null cone, the density law differs drastically from the desired one, even though the density fall-off on constant t slices exceeds the density fall-off proposed by de Vaucouleurs. Though it becomes clear that this type of solution cannot be used as a model for the universe on the largest scale, due to the existence of a maximum redshift, among other reasons, limited portions of such self-similar dust spheres could be of use in constructing more realistic models of an inhomogeneous universe. The use of such spheres as the central objects in Swiss-cheese vacuoles is briefly discussed.

2 The self-similar solution

We consider a dust-filled universe for which the energy–momentum tensor is

$$T_{\mu\nu} = \rho u_\mu u_\nu , \tag{1}$$

where ρ is the matter density and u^μ is its 4-velocity. We shall assume spherical symmetry, and write the line element in the form:

$$ds^2 = dt^2 - \exp(2\lambda)\, dr^2 - R^2\, d\Omega^2 , \tag{2}$$

where t is the proper time of the dust and r is a comoving radial coordinate. $d\Omega^2$ is the line element on the unit sphere and λ and R are functions of t and r only. Following the procedure of Tolman (1934) and Bondi (1947), and taking the cosmological constant to be zero, the Einstein equations yield the first integral:

$$1 + (\partial R/\partial t)^2 - \exp(-2\lambda)\, (\partial R/\partial r)^2 = \frac{2m}{R} . \tag{3}$$

Since the pressure is zero, m is a function of r only, given by:

$$m(r) = 4\pi \int_0^r \rho R^2\, dR, \tag{4}$$

and this can be interpreted as the mass within the comoving sphere of radius r. Since zero pressure implies that each shell follows a geodesic motion, we have the additional integral of the field equations:

$$E(r) = \frac{1}{2}\left(\frac{\partial R}{\partial t}\right)^2 - \frac{m(r)}{R} . \tag{5}$$

This can be interpreted as the conservation of energy per unit mass of an expanding or contracting shell. Expanding shells with $E(r) \geqslant 0$ will expand forever, while $E(r) < 0$ implies an eventual re-collapse. Constant t slices are flat for $E = 0$. From equations (3) and (5) we have:

$$\exp(2\lambda) = \left(\frac{\partial R}{\partial t}\right)^2 \{1 + 2E(r)\}^{-1} . \tag{6}$$

We now specialize the above solution to the more restricted case of a self-similarity solution, as discussed by Cahill & Taub (1971). Such a self-similar solution will admit a homothetic Killing vector, ξ^a, satisfying:

$$\mathscr{L}_\xi g_{\mu\nu} = \xi_{\mu\|\nu} + \xi_{\nu\|\mu} = 2g_{\mu\nu}, \tag{7}$$

where $\mathscr{L}$ is the usual Lie derivative, $g_{\mu\nu}$ is the metric tensor, and $_{\|\nu}$ indicates covariant differentiation. Hence if one goes in the direction ξ^μ, all lengths scale up or down at a uniform rate. Following Cahill & Taub, and taking ξ^μ to have t and r components, one has $\xi^a = (t, r, 0, 0)$ for a suitable choice of the r coordinate. Then under the scaling transformation $t \to \alpha t$ and $r \to \alpha r$, the metric is invariant. If this is to be the case, then $R(t, r)$ must scale as $t^\beta r^{1-\beta}$, and otherwise be a function only of the self-similarity variable, $s = t/r$. Hence imposing self-similarity leaves us with the line element:

$$ds^2 = dt^2 - \exp(2\lambda)\, dr^2 - r^2 S^2\, d\Omega^2, \tag{8}$$

where λ and S are functions of s only.

As a consequence of self-similarity, dimensionless functions of t and r must be expressible as functions of t and r only in the combination $s = t/r$. Two quantities of particular interest are $E(r)$ and $m(r)/r$, both dimensionless (we are using geometrized units) and hence functions of s only. Since both are known to be independent of t, both must in fact be constants. In the usual Friedmann universe we have strict spatial homogeneity, while in this case we have the 'next best' case, i.e. each shell of dust expands with the same constant energy per unit mass.

Under the change of independent variables from (t, r) to (s, r), we have:

$$\left(\frac{\partial}{\partial t}\right)_r = \frac{1}{r}\left(\frac{\partial}{\partial s}\right)_r \quad \text{and} \quad \left(\frac{\partial}{\partial r}\right)_t = \left(\frac{\partial}{\partial r}\right)_s - \frac{s}{r}\left(\frac{\partial}{\partial s}\right)_r, \tag{9}$$

and equation (5) becomes:

$$S'^2 = 2E + 2/S, \tag{10}$$

where $' \equiv d/ds$ and we have chosen to scale r such that $r = m(r)$. This equation has the form of the Friedmann equation, and has the usual solutions:

$$s + a = \pm \frac{1}{\sqrt{2}} E^{3/2} \left(\sqrt{ES}\sqrt{1 + ES} - \sinh^{-1}\sqrt{ES}\right) \text{ for } E > 0,$$

$$s + a = \pm \frac{\sqrt{2}}{3} S^{3/2} \text{ for } E = 0, \tag{11}$$

$$s + a = \pm \frac{1}{\sqrt{2}} (-E)^{-3/2} \left(\sqrt{-ES}\sqrt{1 + ES} - \sin^{-1}\sqrt{-ES}\right) \text{ for } E < 0.$$

where a is a dimensionless constant related to the degree of inhomogeneity.

It follows from equation (4) that ρR^2 is a dimensionless function, and hence a function of s only. The conservation equations $T^{\mu\nu}{}_{\|\nu} = 0$ imply that $\rho R^2 \exp(\lambda)$ is independent of t, and it then follows that $\rho R^2 \exp(\lambda)$ is constant. Writing equation (6) in terms of s, we have:

$$\exp(\lambda) = (S - sS')/\sqrt{1 + 2E}, \tag{12}$$

so that the density is given by:

$$\rho \propto 1/r^2 S^2 (S - sS'). \tag{13}$$

By a straightforward calculation, the Kretschmann curvature invariant is found to be:

$$R^{abcd}R_{abcd}=\frac{1}{r^4S^6}\left\{6+\frac{(S-2sS')^2+2(sS')^2}{(S-sS')^2}\right\}. \tag{14}$$

These two equations yield singularities in the metric and the density distribution, and it follows that these singularities coincide. A spacelike singularity occurs at $S-sS'=0$, corresponding to the vanishing of g_{11}, while a timelike singularity occurs at $rS=0$, corresponding to the vanishing of g_{22} and g_{33}. The apparent singularity at $r=0$ for $t\neq 0$ arises due to the choice of radial coordinate r which results in $m(r)\propto r$. Transformation to a coordinate such that $m(\omega)\propto\omega^3$ for $r\simeq 0$ eliminates this problem.

Equations (10) and (12) allow us to write:

$$\rho t^2\frac{S^{3/2}}{s}\left(\frac{S^{3/2}}{s}-\sqrt{2}\sqrt{1+ES}\right)=\text{const.}, \tag{15}$$

and for $E=0$, we have the simple form:

$$\rho=\rho_0 t_0^2/(t+ar)(t+3ar), \tag{16}$$

where ρ_0 is the density at the centre of the distribution at time t_0. When $t\gg ar$, ρ becomes roughly proportional to t^{-2} so that at very late times or near the centre of the distribution, the density decreases with time in the same way it does in the corresponding homogeneous universe, i.e. the Einstein–de Sitter universe. At the other extreme, where $ar\gg t$, we have ρ going roughly as r^{-2}, so that there is a strong central concentration. For $E=0$, the spacelike singularity occurs along $s=-a$, and the timelike singularity occurs along $s=-3a$.

To consider further any correspondence of these solutions with the homogeneous Friedmann solutions, it is useful to consider the Weyl conformal curvature tensor. A straightforward calculation shows that the vanishing of all components of the Weyl tensor requires the vanishing of $2S-3sS'$. If this is to be true throughout the spacetime, then we must have $s\propto S^{3/2}$. From equations (11), this cannot be the case for $E\neq 0$, for any choice of the constant a, while for $E=0$ we do have $s\propto S^{3/2}$ if $a=0$. Hence these solutions only reduce to Friedmann homogeneous solutions in the particular case of $E=0=a$, yielding the Einstein–de Sitter universe.

3 Optics

We now consider the propagation of light in these solutions. It is well known that if ξ^μ is a Killing vector, i.e. $\xi_{(\mu\|\nu)}=0$, then $k^\mu\xi_\mu$ is constant along the geodesic with k^μ as tangent vector. This is useful in determining geodesics in situations of sufficient symmetry. In the case of spherical symmetry, one has the three Killing vectors $(0,0,\cos\phi,-\sin\phi\cot\theta)$, $(0,0,\sin\phi,\cos\phi\cot\theta)$, and $(0,0,0,1)$, from which it follows that the geodesic lies in a plane containing the centre of the sphere, and that after a suitable orientation of coordinates we can take $k^2=0$ and $k^3\propto 1/R^2$. To proceed further by this method would require the existence of a Killing vector with an r or t component so that k^0 and k^1 can be determined, as in the case of stationary or homogeneous spacetimes.

Suppose that the spacetime admits ξ^μ as a conformal Killing vector, i.e. $\mathscr{L}_\xi g_{\mu\nu}=2\phi g_{\mu\nu}$, where $\phi=\tfrac{1}{4}\xi^a_{\|a}$. It then follows that $k^\mu\xi_\mu$ is constant along a null geodesic to which k^μ is a tangent vector. For a non-null geodesic $k^\mu\xi_\mu$ is not constant, but changes as $\int\phi\, d\tau$ along the geodesic, where τ is the affine parameter along the geodesic. In our case $\xi^\mu=(t,r,0,0)$ is a homothetic Killing vector ($\phi\equiv 1$), and with the three Killing vectors of spherical symmetry allows us to solve algebraically for k^a, a tangent vector to a null geodesic, yielding:

$$k^a = \left(\frac{s + \epsilon H \exp(\lambda)}{r\,[s^2 - \exp(2\lambda)]},\ \frac{\exp(\lambda) + \epsilon s H}{r \exp(\lambda)\ [s^2 - \exp(2\lambda)]},\ 0,\ \frac{h}{R^2} \right) \tag{17}$$

up to some constant factor. The function $H(s)$ is given by:

$$H(s) = \sqrt{1 - h^2\,[s^2 - \exp(2\lambda)]\,/S^2}, \tag{18}$$

and h is an impact parameter. The indicator ϵ is -1 for incoming portions of a light ray and $+1$ for outgoing portions, changing sign at the point where $H(s)$ vanishes, at which point $ds/d\tau = 0$ while neither $dr/d\tau$ nor $dR/d\tau$ vanishes here. The innermost (or outermost) dust shell through which a photon passes is such that $k^1 = 0$, yielding $h^2 s^2 = S^2$ at the minimum (maximum) of the comoving radial coordinate.

Consider now a comoving observer at (t_0, r_0) and, for simplicity limit consideration to photons received from the direction of the centre of the distribution and the opposite direction, and so take $h = 0$ for the central ray. Regardless of the value of h, the relation $r(s)$ along the null geodesic Γ is, from equation (17):

$$(ds/dr)_\Gamma = [(k^0/k^1) - s]/r, \tag{19}$$

and since $k^0/k^1 = \epsilon \exp(\lambda)$ for $h = 0$, we have:

$$r/r_0 = \exp \int_s^{s_0} \frac{ds}{s - \epsilon \exp(\lambda)}. \tag{20}$$

It is sometimes more convenient to write this in the alternate form:

$$\frac{r}{r_0} = \frac{[s - \epsilon \exp(\lambda)]_0}{s - \epsilon \exp(\lambda)} \exp[-I(s_0, s)], \tag{21}$$

where we define:

$$I(s_0, s) = \int_{s_0}^{s} \frac{\epsilon\, d\,[\exp(\lambda)]}{s - \epsilon \exp(\lambda)}. \tag{22}$$

Since an observer with 4-velocity u^a measures a frequency proportional to $u^a k_a$, we have for the redshift factor from a source at (t, r) to an observer at (t_0, r_0):

$$1 + z = \frac{r_0\,[s - \epsilon \exp(\lambda)]_0}{r\,[s - \epsilon \exp(\lambda)]}, \tag{23}$$

and using equation (21), we finally have for the redshift factor:

$$1 + z = \exp[I(s_0, s)]. \tag{24}$$

If we consider an observer at the centre of the distribution at time t_0, the angular size distance, $D_<$, to some distant comoving object is just $R = rS$ for this observer. Noting that $r_0\,[s + \exp(\lambda)]_0 \to 0$ as $r_0 \to 0$, equation (21) then yields:

$$\mathscr{D}_< = S/[s + \exp(\lambda)]\ (1 + z), \tag{25}$$

where $\mathscr{D}_< = D_</t_0$ is the angular size distance in units of t_0. With equations (22) and (24) we can then construct the $\mathscr{D}_<(z)$ relation for these models. Using the Etherington (1933) relation between $\mathscr{D}_<$ and the luminosity distance $\mathscr{D}_L$ (in units of t_0), we also have:

$$\mathscr{D}_L = S(1+z)/s + \exp(\lambda). \tag{26}$$

The distances obtained above have assumed a strictly smooth density distribution, while it is quite obvious that the observed universe is somewhat lumpy. Hence it is useful to suppose, for simplicity, that all the matter is in lumps such as galaxies, and that the typical line of sight to distant (observationally selected) objects passes between these lumps. Then the average density within the beam is vanishingly small relative to the average local density in the universe. It is useful to also assume that the line of sight passes sufficiently far from any lumps that we may ignore Weyl focusing due to the individual lumps, as in the 'zero shear' relations obtained by Dyer & Roeder (1972, 1973) for models homogeneous on the large scale. At worst this gives a lower limit for the apparent luminosity of a distant source in such a lumpy universe, From the optical scalar equations (Sachs 1961), it then follows that $\tilde{\mathscr{D}}_<$, the angular size distance in this lumpy universe, is just τ, the affine parameter distance, up to some multiplicative constant, where we take $\tau = 0$ at the observer. Hence we need the $\tau(z)$ relation, and from this we find:

$$\tilde{\mathscr{D}}_< = \int_s^\infty \frac{\exp(\lambda - 2I)}{[s + \exp(\lambda)]^2}\, ds. \tag{27}$$

It is of interest to determine the location of the maximum of $\mathscr{D}_<$ (no such maximum exists for $\tilde{\mathscr{D}}_<$ since there is no matter within the beam), or equivalently, to find the point where a radially directed beam of light has zero expansion, as discussed by Carr & Hawking (1974). The expansion, $\theta \equiv \tfrac{1}{2}k^a\|a$, can be calculated directly with the result that θ_ϵ vanishes for:

$$\eta\sqrt{2/S + 2E} + \epsilon\sqrt{1+2E} = 0, \tag{28}$$

where $\eta = \pm 1$ is the sign of $(\partial R/\partial t)_r$ and $\eta = +1$ indicates expansion of the dust distribution. Thus θ vanishes only for $S = 2$, where $2m/R = 1$, and for expansion $(\eta = +1)$ it is the incoming $(\epsilon = -1)$ beam that has $\theta = 0$ and vice versa for contraction. When $S = 2$, the quantity $s + \exp(\lambda)$ becomes $2(1+2E)^{-1/2}$, so that the maximum angular diameter distance for this central observer is

$$\mathscr{D}_{<\max} = \sqrt{1+2E}/(1+z_2), \tag{29}$$

where $1 + z_2$ is the redshift factor corresponding to $S = 2$.

From equation (23), we have for the variation of the redshift factor along the ray:

$$\frac{d\log(1+z)}{d\tau} = -k^0 s/rS^2(S - sS'), \tag{30}$$

so that $1+z$ is extremal at $s = 0$, or equivalently, at $t = 0$. In fact, an observer with $t_0 > 0$ observes a maximum redshift for comoving sources with $t = 0$, and this maximum observable redshift is just $\exp I(s_0, s)$. Hence if these solutions are to be used to describe the whole universe, it is going to appear somewhat different from the conventional Robertson–Walker situation.

In the foregoing discussion, it is evident that the hypersurface Σ_0 where $s^2 - \exp(2\lambda)$ vanishes is of fundamental interest. It is on this hypersurface that the null cone becomes tangent to an $s = \text{const.}$ hypersurface so that the homothetic Killing trajectories become null. These hypersurfaces are the same as those with $V = 1$ in the work of Carr & Hawking

(1974). In fact these hypersurfaces are in reality where the homothetic Killing vector, ξ^μ, becomes null, and represent a homothetic static limit surface, yielding an infinite frequency shift. Since ξ^μ is irrotational, these hypersurfaces are in fact null geodesic, as discussed by Dyer & Honig (1979a, b), and are prime candidates for horizons in these solutions. They represent a special case of the more general conformal stationary limit surface defined by the nullity of a conformal Killing vector, which also becomes a null geodesic hypersurface if the vector is irrotational (see Dyer & Honig 1979a, b).

4 Solutions with $E = 0$

The solutions with $a = 0$ and E small and negative have been considered by Carr & Hawking (1974) with reference to the formation of black holes in the early universe. Here, we shall now restrict our consideration to the solutions with $E = 0$ while $a \neq 0$ and investigate the observational consequences for a central observer.

When $E = 0$, it is useful to define a new function, $u(s)$, such that $S' = 2u$, and from equation (10), we then have $S = 1/2u^2$. Since $(\partial R/\partial t)_r$ is just S', $u > 0$ implies expansion of the dust while $u < 0$ implies contraction. Equation (10) also implies $S'' = -1/S^2$, so that we have:

$$s = (1 - Cu^3)/6u^3, \tag{31}$$

where $C = 6a$ is a constant, and also:

$$\exp(\lambda) = (1 + 2Cu^3)/6u^2. \tag{32}$$

To ensure that the line element is not singular at the centre of the distribution, we can now introduce a new comoving radial coordinate, ω, defined by $r = 2\omega^3/9t_0^2$. Using equations (31) and (32), the line element can now be written in the form:

$$ds^2 = dt^2 - \left(\frac{t + \gamma\omega^3}{t_0}\right)^{4/3} \left\{\left(\frac{t + 3\gamma\omega^3}{t + \gamma\omega^3}\right)^2 d\omega^2 + \omega^2\, d\Omega^2\right\}, \tag{33}$$

where $\gamma = C/27t_0^2$ is a constant. When $\omega \to 0$, this line element approaches the Einstein–de Sitter line element, and it remains non-singular on $\omega = 0$. In terms of this new radial coordinate, the homothetic Killing vector is $(t, \omega/3, 0, 0)$.

The null geodesic hypersurfaces where ξ^a is null, i.e. Σ_0, are defined by solutions of the equations:

$$s = m \exp(\lambda) \tag{34}$$

where $m = \pm 1$. From equations (31) and (32), each of these yields a quartic equation for u_H, where the subscript H implies the value on a Σ_0. It is convenient to define the parameters $x = mu_H$, $\Gamma = mC$, and $\sigma_H = ms_H$, so that equations (31), (32), and (34) yield the following parametric equations for the function $\sigma_H(\Gamma)$:

$$\Gamma = \frac{1 - x}{x^3(1 + 2x)} \quad \text{and} \quad \sigma_H = \frac{1}{2x^2(1 + 2x)}. \tag{35}$$

Fig. 1 is a schematic representation of $\sigma_H(\Gamma)$, and also shows the location of the timelike and spacelike singularities. The extremal values of Γ are $\Gamma_\pm = -2(26 \pm 15\sqrt{3})$ where $\sigma_\pm = 7 \pm 4\sqrt{3}$, while σ has an extremal value of 27/2 at $\Gamma = -108$.

Using Fig. 1, it is then straightforward to determine fundamental properties of the spacetime diagram as a function of the constant C. Taking $C > 0$, one obtains the three diagrams shown in Fig. 2 for C in each of the three ranges:

I: $0 < C < 2(26 - 15\sqrt{3})$

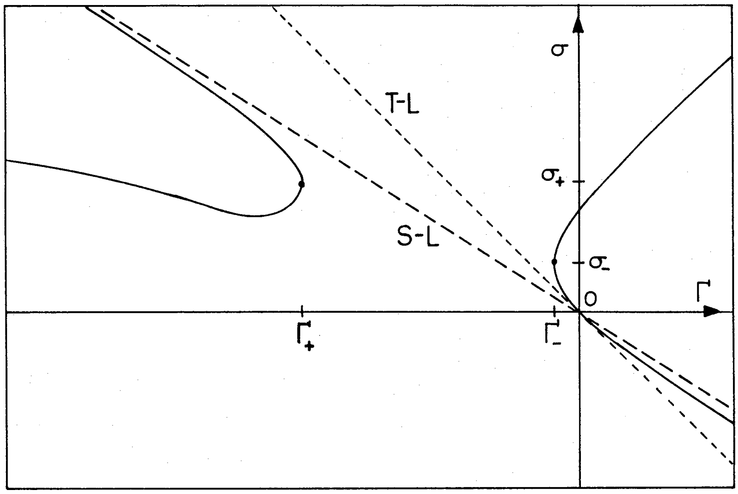

Figure 1. The broken lines show the location of singularities, and the solid curves show the location of null geodesic hypersurfaces defined by $\xi^a \xi_a = 0$, as functions of Γ.

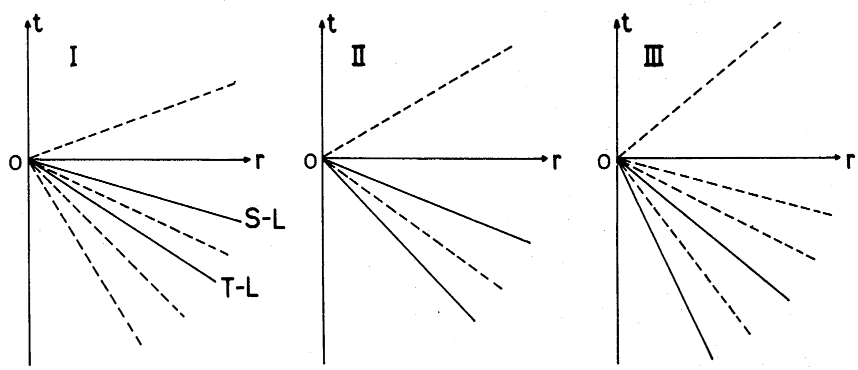

Figure 2. The three types of spacetime diagram when $E = 0$.

II: $2(26 - 15\sqrt{3}) < C < 2(26 + 15\sqrt{3})$

III: $C > 2(26 + 15\sqrt{3})$,

and the Σ_0 hypersurfaces are indicated by broken lines and the singularities by solid lines. When $C \to 0$, the two roots near $\sigma = 1/6$ both converge to $s_H = \pm 1/6$, while the two roots near $\sigma = 0$ both converge to $s_H = 0$, but in so doing blanket the timelike singularity. The spacetime diagram thus degenerates into that of the Einstein–de Sitter universe with its spacelike singularity and conformal flatness for $C = 0$. For $C < 0$, the given spacetime diagrams are just inverted, or, equivalently, the direction of the time flow is reversed.

Considering again a central observer, the redshift factor he observes for a source at $s(u)$ is just $1 + z = \exp(I)$, where I can now be written as:

$$I = 2\int_0^u \frac{(1 - Cu^3)\, du}{1 + u - Cu^3(1 - 2u)}. \tag{36}$$

Since we can obtain the zeroes of the denominator of the integrand (they are the zeroes of $s^2 - \exp(2\lambda)$), this integral is not difficult, but somewhat messy, and a numerical integration is more useful. It is also useful to obtain the lowest order in C correction to the homogeneous model, which we obtain by expanding the integral:

$$I = 2\{\log(1 + u) - 3Cf(u) + O(C^2)\}, \tag{37}$$

where:

$$f(u)=u\left\{\frac{u^2}{3}-u+3+\frac{1}{1+u}\right\}-4\log(1+u). \tag{38}$$

We then have the redshift factor:

$$1+z=(1+u)^2\{1-6Cf(u)+O(C^2)\} \tag{39}$$

and since $f>0$ for $u>0$, increasing C from zero implies smaller observed redshifts, which is reasonable since the incoming photons are falling into a deeper potential well due to the central concentration of mass. Equation (25) then yields for $\mathscr{D}_<$:

$$\mathscr{D}_<=3u/(1+z)\{1+u-Cu^3(1-2u)\}, \tag{40}$$

and taking $z \ll 1$, so $u \ll 1$ yields $\mathscr{D}_< \simeq 3Z/2$. Comparing this with the usual Hubble relation for small z, we have $t_0=2/3H_0$ as the age of the universe at the centre, where H_0 is the usual Hubble parameter. Of course, this is the age one would calculate in the homogeneous Einstein–de Sitter universe. Expanding equation (40), we obtain the lowest order corrections for $\mathscr{D}_<$ and $\mathscr{D}_L$ so that:

$$\mathscr{D}_<=\frac{3u}{(1+u)^3}\{1+Cp_+(u)+O(C^2)\}, \tag{41}$$

and

$$\mathscr{D}_L=3u(1+u)\{1+Cp_-(u)+O(C^2)\}, \tag{42}$$

where:

$$p_\pm(u)=u^3(1-2u)/(1+u)\pm 6f(u), \tag{43}$$

and both p_+ and p_- are positive for small $u>0$. Hence equations (39), (41) and (42) imply that for a given redshift $z \ll 1$, the observable distances, $\mathscr{D}_<$ and $\mathscr{D}_L$, are larger for $C>0$ than they would be in the corresponding homogeneous universe.

From equation (16), it is clear that spacelike slices having t constant show a decrease in the density for increasing r, but it is the variation of the density along the past light cone that is of observational interest. Along the past light cone of the central observer, we have, using equations (9) and (17), the derivative:

$$\left(\frac{d}{dr}\right)_\Gamma=\left(\frac{\partial}{\partial r}\right)_s-\frac{s+\exp(\lambda)}{r}\left(\frac{\partial}{\partial s}\right)_r, \tag{44}$$

and using earlier results for dz/du, $(dr/ds)_\Gamma$, and ds/du, we have

$$\left(\frac{d\log\rho}{d\log z}\right)_\Gamma=\frac{3(1-2Cu^2+3Cu^3-C^2u^5+2C^2u^6)}{(1+z)(1-Cu^3)(1+2Cu^3)} \tag{45}$$

When $z\to 0$, this implies that $\rho/\rho_0 \to (1+z)^3$, so it is of interest to consider ρ/ρ_{R-W} for small z. If we consider the smallest integer n such that $d\log(\rho/\rho_{R-W})/d(z^n)$ not vanish at $z=0$, then $n=3$, so upon expanding in powers of z^3, we have:

$$\rho/\rho_{R-W}=1-\tfrac{1}{2}Cz^3+O(z^6). \tag{46}$$

Thus for small redshifts, this model predicts a deficit in the observed density, relative to the homogeneous model, which is proportional to the constant C. This fall off in density is far

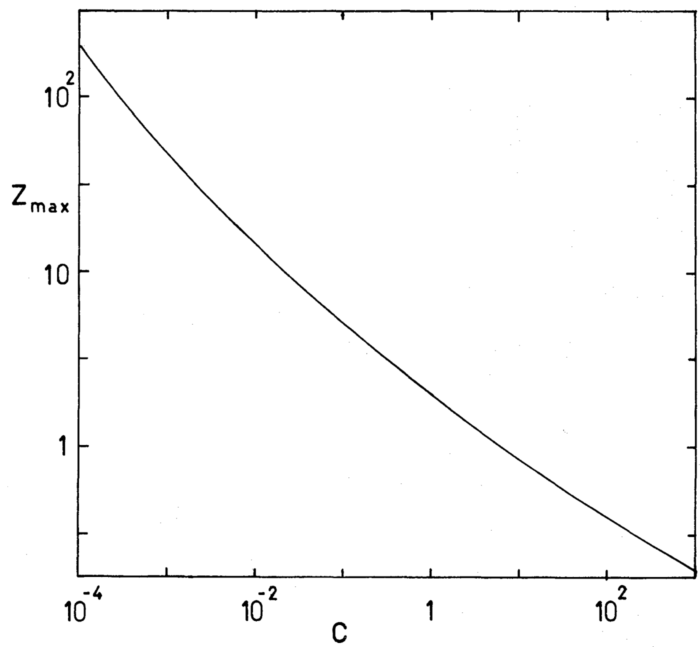

Figure 3. The maximum observable redshift as a function of C.

less than one would find along constant t slices, where one can have the density going roughly as $r^{-\beta}$ where β can be as large as 2, which would be required to agree with the observations of de Vaucouleurs (1970).

Each observer with $t > 0$ observes a maximum redshift, z_{max}, which from equation (30) occurs at $s = 0 = t$, which itself occurs at $u = C^{-1/3}$ for $E = 0$. Numerical integration of equation (36) then yields Fig. 3 which shows z_{max} as a function of the constant C. It is clear that the observed redshifts of galaxies and quasars put a rather stringent upper limit on the value of C.

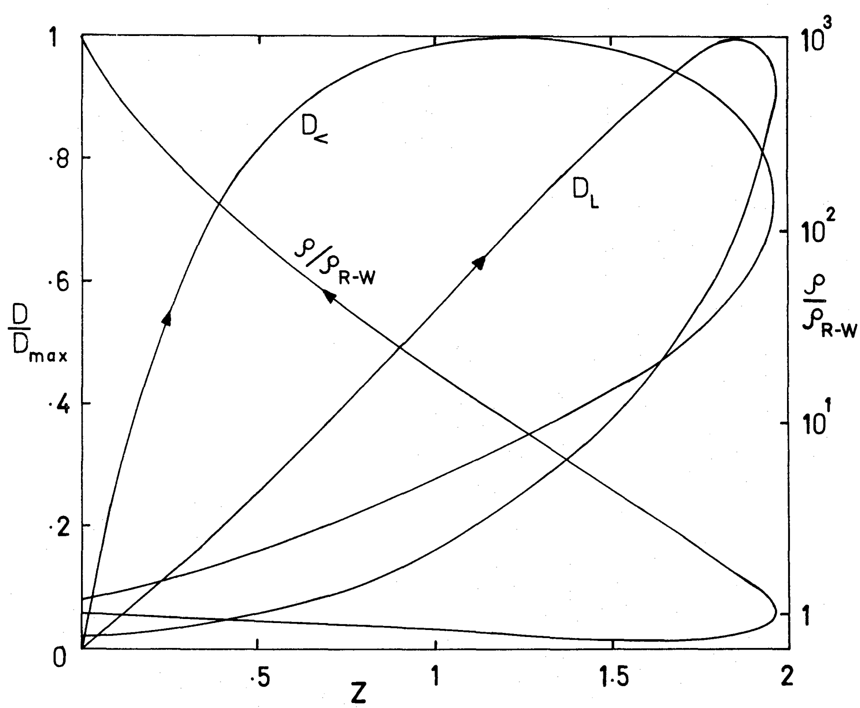

Figure 4. Distance versus redshift and density versus redshift relations for $C = 1$.

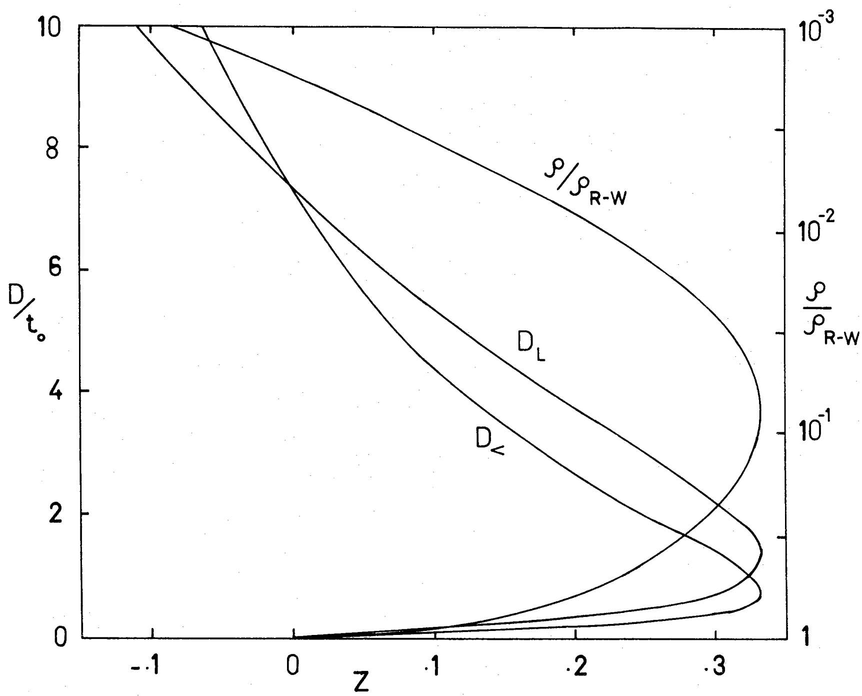

Figure 5. Distance versus redshift and density versus redshift relations for $C = 160$.

Figs 4 & 5, with $C = 1$ and $C = 160$ respectively, give $\mathscr{D}_<$, $\mathscr{D}_L$, and ρ/ρ_{R-W} as functions of z. For $C = 1$, the spacetime diagram is of type II, which of course is similar to type I in the region of interest. In these, the observer actually sees the spacelike singularity, with an infinite blueshift factor. Although the observer sees a density deficit, relative to the homogeneous model, at small separation from him, this deficit eventually becomes a significant enhancement. The observed distances $\mathscr{D}_<$ and $\mathscr{D}_L$ also behave rather oddly in that they both reach maxima, and then decrease to zero at $1 + z = 0$. Hence, unlike the Robertson–Walker models, this model makes the singularity highly observable, at least in the geometric optics limit. This type of solution does not appear a very likely model of the Universe, at least used in its entirety, but used as the central object in a vacuole of a Swiss cheese universe, so that $z_{\max}$ is not exceeded, for example, it is still quite usable as a model for a large density inhomogeneity in the universe.

The solution with $C = 160$ has a spacetime diagram of type III, with an horizon separating the observer from the spacelike singularity. Though this horizon is an infinite blueshift surface, it is also located at an infinite luminosity distance from the observer, so that a highly blueshifted source is seen with very low apparent luminosity. Since this horizon has t/r constant, the observer sees, at large distances, $\mathscr{D}_<$ and $\mathscr{D}_L$ almost proportional to r, while from the constancy of $\rho R^2 \exp(\lambda)$, the density is roughly proportional to r^{-2}, so that one does have $\rho \propto \mathscr{D}_L^{-2}$ for $\mathscr{D}_L \to \infty$. If we were to take the spacelike singularity to be the 'birth' of the Universe, then at sufficiently large r, one is able to see distant galaxies at times when they are older than the central observer, due to the divergence of the horizon and singularity curves in the spacetime diagram.

From these two examples, it is clear that observational astronomy would be somewhat altered if we were the central observers in such a universe, because, for example, the redshift is no longer a monotonic function of separation.

5 Conclusion

We have considered a very simple inhomogeneous model universe chosen partly because the self-similarity of the solution may be related to the hierarchical structure of the universe proposed by de Vaucouleurs. Since observers like ourselves are likely to be located in astronomical systems lying somewhat up the scale of the hierarchy, we have assumed that the observer lies at the centre of the density concentration. The value of the parameter C can be interpreted as a measure of our position in the hierarchy. This model is similar to that of Bonnor, where he also imposes $E(r) \equiv 0$, but instead of self-similarity, he specifies the density function on constant t slices to be roughly that proposed by de Vaucouleurs. Unfortunately it is not simple to derive the observable properties of his model, while the imposition of self-similarity allows direct calculation of these properties.

In a recent paper Wesson (1979) has considered a non-central observer in an $E = 0$ model similar to that considered here. In obtaining his observational relations, he does not make full use of the simplicity arising from the existence of the homothetic Killing vector, and his relations have only limited regions of applicability. He limits his consideration to $t \geqslant 0$, which eliminates consideration of those aspects of the solution which make it quite clear that it is unacceptable as a global solution. The elimination of the $t < 0$ region solves the problems in much the same way that the elimination of the $t \simeq 0$ region of the usual Friedmann solutions would eliminate the problems of the singularity there. It is worth mentioning again that the $r = 0$ world line is non-singular for $t > 0$ (see equation (33)), contrary to the comments of Wesson.

We have seen that the present model is not an acceptable model of the real Universe in the large, though most of the difficulties disappear if we only use this type of solution to represent a limited region of the Universe. This suggests the use of these self-similar dust spheres as the central objects in the vacuoles of the Swiss-cheese models. In this model, we would suppose that both the source and the observer are at the centres of self-similar dust spheres in vacuoles in a Friedmann background universe. The value of the parameter C for each dust sphere would be determined by the position of each of the source and observer on the scale of hierarchy. Thus one could imagine there being a significant difference in C when observing a class of distant highly condensed objects as compared to that when observing very tenuous objects of large physical extent. Hence the use of these self-similar dust spheres may be a useful first step beyond the use of uniform dust spheres as the occupants of the vacuoles in Swiss-cheese model universes.

Acknowledgments

The author is grateful to Professors W. B. Bonnor, E. Honig, and R. C. Roeder, and to Dr K. Lake for helpful discussion, and to the National Research Council of Canada for financial support.

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