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A SPINNING FINITE BEAM ELEMENT OF GENERAL ORIENTATION ANALYZED WITH RAYLEIGH/TIMOSHENKO/SAINT-VENANT THEORY

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ABSTRACT

Linear vibrations are studied for a straight uniform finite beam element of general orientation spinning at a constant angular speed about a fixed axis in the inertial space. The gyroscopic and circulatory matrices and also the geometric stiffness matrix of the beam element are presented. The effect of the centrifugal static axial load on the bending and torsional dynamic stiffnesses is thereby accounted for. The Rayleigh/Timoshenko/Saint-Venant theory is applied, and polynomial shape functions are used in the construction of the deformation fields. Non-zero off-diagonal elements in the gyroscopic and circulatory matrices indicate coupled bending/shearing/torsional/tensional free and forced modes of a generally oriented spinning beam. Two numerical examples demonstrate the use and performance of the beam element.

1. INTRODUCTION

Spinning finite beam elements in structural vibration analysis have been treated by several researchers since the early 1970's. Practical applications are found for turbines, combustion engines, space structures, etc. An early investigation of spinning flexible rotors modelled by use of finite shaft elements was made by Ruhl[1] in 1970. He employed the Euler/Bernoulli beam theory but neglected gyroscopic moments for the finite shaft elements. In 1980 Nelson[2] presented a study of flexible rotors which utilized a finite Rayleigh/Timoshenko spinning shaft element. Rotatory inertia, bending/shearing deformation and gyroscopic forces and also second-order effects of a large static axial load were considered by him. Asymmetric rotors were studied by Kang, Shih & Lee[3]. They developed a finite beam element for modelling asymmetric shafts. A method for calculating matrices for a generally oriented finite beam element was given by Likins[4] in 1972.

A set of governing linear differential equations of motion, in body-fixed coordinates, for a generally oriented spinning Euler/Bernoulli/Saint-Venant finite beam element, with distributed parameters, was established by Wittrick & Williams[5] in 1982. They derived the dynamic stiffness matrix in stationary harmonic vibration for the spinning beam element. In order to obtain simplified differential equations with constant coefficients, they assumed that the static axial sectional force along the beam (from the centrifugal field) did not vary irrespective of the orientation of the element. They also assumed that the torsional motion of the beam element did not couple with the bending/tensional motion. In 1988, Leung & Fung[6] established, by use of an analytical mechanics approach and assumed shape functions, the governing constant-coefficient matrices of a spinning Euler/Bernoulli/Saint-Venant finite beam element, also neglecting the coupling between the torsional and bending/tensional motions.

In 1991, Lundblad[7] advanced an exact (without spatial discretization errors) harmonic dynamic stiffness matrix of a generally oriented spinning Rayleigh/Timoshenko/Saint-Venant beam element considering stiffness, inertia, gyroscopic and circulatory effects. He also included internal and external viscous damping effects. Lundblad showed that couplings exist between the torsional motion and the bending/shearing/tensional motion. Like Wittrick & Williams[5] he approximated the static axial sectional force in the generally oriented element as being constant. Lundblad's method involves extensive numerical work when establishing the dynamic stiffness matrix.

The present work adopts Lundblad's classical mechanics approach using d'Alembert forces in the derivation of the constant-coefficient spinning-speed-dependent gyroscopic, circulatory, and geometric stiffness matrices $\mathbf{G}=\mathbf{G}(\Omega)$, $\mathbf{H}=\mathbf{H}(\Omega)$ and $\mathbf{K}^{\sigma}=\mathbf{K}^{\sigma}(\Omega)$. These matrices appear in the equations of motion of a discretized damped beam element. In matrix form one has

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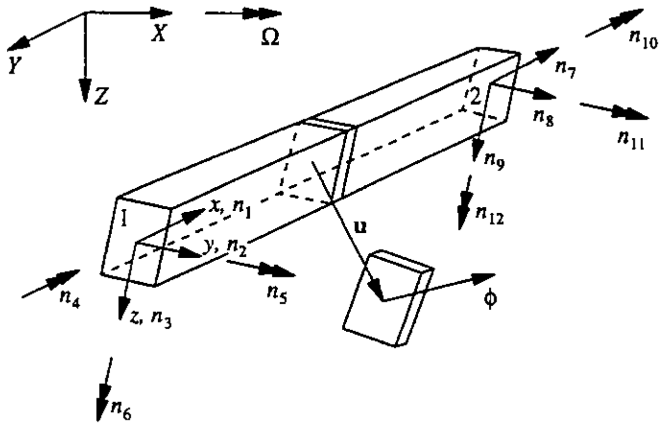

Figure 1. Perspective view of straight uniform beam element 12 in co-rotating global coordinate system *XYZ*. Beam spins at constant angular rate Ω about global X axis the direction of which is fixed in inertial space. Beam end translations and rotations n_1 to n_{12} and vectorially associated end loads N_1 to N_{12} are indicated. Local coordinate system *xyz* (with non-dimensional length coordinate $\xi = x/L$). A beam lamina dx translated **u** and rotated ϕ from nondisplaced position is shown (different scales for geometry and deformation have been used). Beam properties are: mass distribution m, lamina radii of inertia r_x, r_y and r_z, tensional stiffness EA, torsional stiffness GI_t, bending stiffnesses EI_y and EI_z, and shearing stiffnesses k_zGA and k_yGA.

$$\mathbf{M}\,\ddot{\mathbf{n}}(t) + [\mathbf{C} + \mathbf{G}(\Omega)]\,\dot{\mathbf{n}}(t) + [\mathbf{K} + \mathbf{K}^g(\Omega) + \mathbf{H}(\Omega)]\,\mathbf{n}(t) = \mathbf{N}(t) \tag{1}$$

Here **n** is the nodal displacement vector containing the six beam end translations and the six beam end rotations taken in a co-rotating coordinate system. The load vector **N** contains the vectorially associated end forces and end moments. The constant spinning speed is denoted by Ω.

In the derivation of the geometric stiffness matrix, the spatial variation of the axial load (due to centrifugal forces) is accounted for. Rayleigh/Timoshenko/Saint-Venant theory and polynomial shape functions are used throughout. The elastic stiffness matrix **K**, the damping matrix **C** and the mass matrix **M** in Equation (1) are independent of the spinning speed and can be found elsewhere , *e.g.*, in Archer[8] or Przemieniecki[9], and are not reproduced here. The effect of a co-rotating ambient medium of Winkler type along each beam member (as included in the work by Lundblad[7]) can be taken into account in the same manner as used by Sällström[10] for a non-rotating beam element. The case with a rotating beam element in a non-rotating damped ambient medium will not be treated here.

The present work is an extension of that by Leung & Fung[6] in the sense that it considers also rotatory inertia and shear deformation in a manner which is consistent with the assumptions used by Rayleigh and Timoshenko. The matrices derived here can, therefore, be compared to the results by Leung & Fung[6] only in some special cases.

2. COORDINATES AND INERTIA LOADING

A typical finite spinning beam element is shown in Figure 1. The beam element is uniform and initially straight (in the non-spinning state) with its undeformed geometric centre line coinciding with the local coordinate axis x. The shear centre and the mass centre are assumed to coincide with the geometric centre of the beam cross section. The non-dimensional coordinate $\xi = x/L$, varying between zero at beam end number 1 and unity at beam end number 2, will be used in the assumed polynomial shape functions. No external loads act on the element except at its ends 1 and 2. The distributed inertia forces **U** and moments $\mathbf{\Phi}$ acting on a beam lamina of unit length and with cross-sectional area A are

$$\mathbf{U}(\xi) = -\int_A \mathbf{a}(\xi)\rho\, dA \qquad \mathbf{\Phi}(\xi) = -\int_A \mathbf{p}\times\mathbf{a}(\xi)\,\rho\, dA \tag{2a, b}$$

with

$$\mathbf{U}^{\mathrm{T}} = \{U, U_y, U_z\} \qquad \mathbf{\Phi}^{\mathrm{T}} = \{\Phi_x, \Phi_y, \Phi_z\} \tag{3a, b}$$

The forces per unit length in the local directions x, y, z are denoted by U_x, U_y and U_z, and the corresponding moments about the axes x, y, z by Φ_x, Φ_y and Φ_z. The acceleration vector **a** of the lamina element dA is determined by its position and by its relative motion in the co-rotating global coordinate system *XYZ*, and also by the spinning speed Ω of that coordinate system. The position of the lamina element dA is given by the coordinates $Y(\xi)$ and $Z(\xi)$ of the undeformed beam axis in the global coordinate system *XYZ* and by the translation $\mathbf{u}(\xi)$ and rotation $\phi(\xi)$ of the beam cross-section in the local coordinate system *xyz*, and, finally, by the position vector $\mathbf{p}_0$ of the element dA within the non-rotated beam cross-section A. By evaluating the integrals in Equations (2a,b), with the proper acceleration vector **a** and position vector $\mathbf{p} = \mathbf{p}_0 + \phi\times\mathbf{p}_0$ inserted, one obtains the inertia loading as

$$\mathbf{U} = \mathbf{U}_m + \mathbf{U}_g + \mathbf{U}_c + \mathbf{B}_u \tag{4a}$$

$$\mathbf{U}_m = -\mathbf{A}_u^{(2)}\ddot{\mathbf{u}} \qquad \mathbf{U}_g = -\mathbf{A}_u^{(1)}\dot{\mathbf{u}} \qquad \mathbf{U}_c = -\mathbf{A}_u^{(0)}\mathbf{u} \tag{4b-d}$$

$$\mathbf{\Phi} = \mathbf{\Phi}_m + \mathbf{\Phi}_g + \mathbf{\Phi}_c + \mathbf{B}_\phi \tag{4e}$$

$$\mathbf{\Phi}_m = -\mathbf{A}_\phi^{(2)}\ddot{\phi} \qquad \mathbf{\Phi}_g = -\mathbf{A}_\phi^{(1)}\dot{\phi} \qquad \mathbf{\Phi}_c = -\mathbf{A}_\phi^{(0)}\phi \tag{4f-h}$$

The distributed loads $\mathbf{U}_m$ and $\mathbf{\Phi}_m$ are the usual inertia loads of a vibrating non-spinning beam in a fixed coordinate system. The gyroscopic parts, $\mathbf{U}_g$ and $\mathbf{\Phi}_g$, and the circulatory parts, $\mathbf{U}_c$ and $\mathbf{\Phi}_c$, of the inertia loading are used in the following for the derivations of the gyroscopic and circulatory matrices, respectively. State-independent static centrifugal loads $\mathbf{B}_u$ and $\mathbf{B}_\phi$ act on the beam element. These loads will cause an axial sectional force in the beam element. The influence of the axial loading on the vibratory motion will be taken into account by considering the geometric stiffness effect in the analysis.

The matrix $\mathbf{A}_u^{(2)}$ of mass per unit length and the matrix $\mathbf{A}_\phi^{(2)}$ of mass moments of inertia per unit length are used in a mass matrix derivation,

$$\mathbf{A}_u^{(2)}=m\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad \mathbf{A}_\phi^{(2)}=m\begin{bmatrix}r_x^2&0&0\\0&r_y^2&0\\0&0&r_z^2\end{bmatrix}\tag{5a,b}$$

They are given here for completeness but will not be further used in this work. The matrices $\mathbf{A}_u^{(1)}$ and $\mathbf{A}_\phi^{(1)}$. with elements $a_{u,ij}^{(1)}$ and $a_{\phi,ij}^{(1)}$, are associated to Coriolis forces and moments. They can be obtained by use of the projections Ω_x, Ω_y and Ω_z on the local xyz directions of the spin vector $\mathbf{\Omega}$. These skew-symmetric matrices are

$$\mathbf{A}_u^{(1)}=m\begin{bmatrix}0&-2\Omega_z&2\Omega_y\\2\Omega_z&0&-2\Omega_x\\-2\Omega_y&2\Omega_x&0\end{bmatrix}\tag{6a}$$

$$\mathbf{A}_\phi^{(1)}=m\begin{bmatrix}0&-\Omega_z(r_x^2+r_y^2-r_z^2)&\Omega_y(r_x^2+r_z^2-r_y^2)\\\Omega_z(r_x^2+r_y^2-r_z^2)&0&-\Omega_x(r_y^2+r_z^2-r_x^2)\\-\Omega_y(r_x^2+r_z^2-r_y^2)&\Omega_x(r_y^2+r_z^2-r_x^2)&0\end{bmatrix}\tag{6b}$$

For a real beam the matrix element $a_{\phi,23}^{(1)}$ and $a_{\phi,32}^{(1)}$ are zero and they will be omitted in the future presentation of this work. The displacement-dependent centrifugal forces and moments are represented by the two matrices $\mathbf{A}_u^{(0)}$ and $\mathbf{A}_\phi^{(0)}$.

$$\mathbf{A}_u^{(0)}=m\begin{bmatrix}-(\Omega_y^2+\Omega_z^2)&\Omega_x\Omega_y&\Omega_z\Omega_x\\\Omega_x\Omega_y&-(\Omega_z^2+\Omega_x^2)&\Omega_y\Omega_z\\\Omega_z\Omega_x&\Omega_z\Omega_y&-(\Omega_x^2+\Omega_y^2)\end{bmatrix}\tag{7a}$$

$$\mathbf{A}_\phi^{(0)}=m\begin{bmatrix}(\Omega_y^2-\Omega_z^2)&\Omega_x\Omega_y&-\Omega_z\Omega_x\\-\Omega_y\Omega_x&(\Omega_z^2-\Omega_x^2)&\Omega_y\Omega_z\\\Omega_x\Omega_z&-\Omega_z\Omega_y&(\Omega_x^2-\Omega_y^2)\end{bmatrix}\times\begin{bmatrix}(r_y^2-r_z^2)&0&0\\0&(r_z^2-r_x^2)&0\\0&0&(r_x^2-r_y^2)\end{bmatrix}\tag{7b}$$

The elements of these two matrices are denoted by $a_{u,ij}^{(0)}$ and $a_{\phi,ij}^{(0)}$ in the following. It can be noted that the matrix $\mathbf{A}_\phi^{(0)}$ is generally neither symmetric nor skew-symmetric, which results in a circulatory matrix of a general form. The static loads due to centrifugal forces and moments acting on the beam element are here represented by the column vectors $\mathbf{B}_u$ and $\mathbf{B}_\phi$,

$$\mathbf{B}_u=\begin{Bmatrix}\beta_{u1}\\\beta_{u2}\\\beta_{u3}\end{Bmatrix}=-m\Omega^2\,\mathbf{L}\begin{Bmatrix}0\\Y(\xi)\\Z(\xi)\end{Bmatrix}\tag{8a}$$

$$\mathbf{B}_\phi=\begin{Bmatrix}\beta_{\phi1}\\\beta_{\phi2}\\\beta_{\phi3}\end{Bmatrix}=m\begin{Bmatrix}\Omega_y\Omega_z(r_z^2-r_y^2)\\\Omega_z\Omega_x(r_x^2-r_z^2)\\\Omega_x\Omega_y(r_y^2-r_x^2)\end{Bmatrix}\tag{8b}$$

In Equation (8a) the matrix $\mathbf{L}$ (not to be confused with the beam length L) is the transformation matrix from the global XYZ to local xyz coordinates.

3. SHAPE FUNCTIONS AND ASSUMED DEFORMATIONS

The gyroscopic, circulatory and geometric stiffness matrices will all be derived in consistency with an assumed deformation field of the beam element. The static deformation functions of a Rayleigh/Timoshenko/Saint-Venant beam element loaded at its ends in bending/shearing, torsion and tension will be used. The linear and angular displacements of the beam element at a position ξ can be calculated as functions of the nodal translations and rotations (see Figure 1), together with the assumed shape functions,

$$\begin{Bmatrix}\mathbf{u}(\xi)\\\phi(\xi)\end{Bmatrix}=\Psi(\xi)\mathbf{n}\tag{9}$$

Here the shape function matrix Ψ is

$$\Psi^T=\begin{bmatrix}\psi_{1,1}&0&0&0&0&0\\0&\psi_{3,3}(\theta_z)&0&0&0&-\psi_{5,3}(\theta_z)\\0&0&\psi_{3,3}(\theta_y)&0&\psi_{5,3}(\theta_y)&0\\0&0&0&\psi_{1,1}&0&0\\0&0&\psi_{3,5}(\theta_y)&0&\psi_{5,5}(\theta_y)&0\\0&-\psi_{3,5}(\theta_z)&0&0&0&\psi_{5,5}(\theta_z)\\\psi_{1,7}&0&0&0&0&0\\0&\psi_{3,9}(\theta_z)&0&0&0&-\psi_{5,9}(\theta_z)\\0&0&\psi_{3,9}(\theta_y)&0&\psi_{5,9}(\theta_y)&0\\0&0&0&\psi_{1,7}&0&0\\0&0&\psi_{3,11}(\theta_y)&0&\psi_{5,11}(\theta_y)&0\\0&-\psi_{3,11}(\theta_z)&0&0&0&\psi_{5,11}(\theta_z)\end{bmatrix}\tag{10}$$

The shape functions $\psi_{i,j}$ of the beam element used here can be found in the textbook by Przemieniecki[9] and they are repeated for completeness,

$$\psi_{1,1} = 1 - \xi \qquad \psi_{1,7} = \xi \tag{11a,b}$$

$$\psi_{3,3}(\theta_y) = [(1 - 3\xi^2 + 2\xi^3) + \theta_y(1 - \xi)] / (1 + \theta_y) \tag{11c}$$

$$\psi_{3,5}(\theta_y) = [(-\xi + 2\xi^2 - \xi^3)L + \theta_y(-\xi + \xi^2)L / 2] / (1 + \theta_y) \tag{11d}$$

$$\psi_{3,9}(\theta_y) = [(3\xi^2 - 2\xi^3) + \theta_y \xi] / (1 + \theta_y) \tag{11e}$$

$$\psi_{3,11}(\theta_y) = [(\xi^2 - \xi^3)L + \theta_y(\xi - \xi^2)L / 2] / (1 + \theta_y) \tag{11f}$$

$$\psi_{5,3}(\theta_y) = [(6\xi - 6\xi^2) / L] / (1 + \theta_y) \tag{11g}$$

$$\psi_{5,5}(\theta_y) = [(1 - 4\xi + 3\xi^2) + \theta_y(1 - \xi)] / (1 + \theta_y) \tag{11h}$$

$$\psi_{5,9}(\theta_y) = [(-6\xi + 6\xi^2) / L] / (1 + \theta_y) \tag{11i}$$

$$\psi_{5,11}(\theta_y) = [(-2\xi + 3\xi^2) + \theta_y \xi] / (1 + \theta_y) \tag{11j}$$

The bending/shearing ratio parameters are defined as

$$\theta_y = \frac{12EI_y}{k_z GAL^2} \qquad \theta_z = \frac{12EI_z}{k_y GAL^2} \tag{12a,b}$$

The shape functions of an Euler/Bemoulli beam element can be obtained as the special case for which both θ_y and θ_z are zero. The gyroscopic, circulatory and geometric stiffness matrices of a spinning Euler/Bemoulli beam element can thus easily be recovered, if required, from the matrices presented below.

4. GYROSCOPIC AND CIRCULATORY MATRICES

The consistent nodal load vector $\mathbf{N}_g$ due to the gyroscopic parts of the distributed inertia loading is given by the relation, see Cook, Malkus & Plesha[11],

$$\mathbf{N}_g = \int_0^1 \Psi^T \begin{Bmatrix} \mathbf{U}_g \\ \mathbf{\Phi}_g \end{Bmatrix} L d\xi \tag{13}$$

Using Equations (4c) and (4g) for the gyroscopic part of the inertia load, together with the assumed linear and angular velocities $\{\dot{\mathbf{u}}^T \dot{\boldsymbol{\phi}}^T\}^T = \Psi \,\dot{\mathbf{n}}$, one obtains

$$\mathbf{N}_g = -\mathbf{G}\,\dot{\mathbf{n}} \qquad \mathbf{G} = \int_0^1 \Psi^T \begin{bmatrix} \mathbf{A}_u^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_\phi^{(1)} \end{bmatrix} \Psi L d\xi \tag{14a,b}$$

Equation (14b) gives the 12×12 gyroscopic matrix of the beam element. The 96 non-zero elements of this skew-symmetric matrix are given in Appendix.

The consistent nodal load vector $\mathbf{N}_c$ due to the circulatory parts of the distributed inertia loading is

$$\mathbf{N}_c = \int_0^1 \Psi^T \begin{Bmatrix} \mathbf{U}_c \\ \mathbf{\Phi}_c \end{Bmatrix} L d\xi \tag{15}$$

Using Equations (4d) and (4h) for the circulatory part of the inertia load, together with the assumed linear and angular displacements in Equation (9), one obtains

$$\mathbf{N}_c = -\mathbf{H}\,\mathbf{n} \qquad \mathbf{H} = \int_0^1 \Psi^T \begin{bmatrix} \mathbf{A}_u^{(0)} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_\phi^{(0)} \end{bmatrix} \Psi L d\xi \tag{16a,b}$$

Equation (16b) gives the non-symmetric circulatory 12×12 matrix $\mathbf{H}$. This matrix generally holds 136 non-zero elements as given in Appendix. This implies that couplings exist in all possible combinations of bending/shearing, tensional and torsional vibrations.

5. AXIAL LOAD DISTRIBUTION

When a large static axial compressive/tensional load act on a beam element of a frame structure, the stiffnesses in bending and torsion of that element are decreased/increased. Since the static distributed centrifugal loads $\mathbf{B}_u$ and $\mathbf{B}_\phi$ make the axial loads dependent on the spinning speed Ω of the structure, large spinning speeds may cause buckling instability of the structure. The calculation of the axial loads must be treated as a non-linear problem calling for an iterative solution method for spinning speeds causing large deformations.

However, for small deformations a linear analysis can be made and the compressive axial loads can be obtained by superimposing the solutions of two complementary static problems. In the first problem the consistent loads $\mathbf{N}^{cent}$ pertaining to the distributed inertia loads $\mathbf{B}_u$ and $\mathbf{B}_\phi$ are calculated for each beam element. These loads are applied to the joints of the structure. The corresponding static displacements $\mathbf{p}^{st}$ can then easily be calculated by use of the assembled global static stiffness matrix $\mathbf{K}$ of the structure. In the second problem the distributed inertia loads $\mathbf{B}_u$ and $\mathbf{B}_\phi$ are applied together with fictitious so-called fixed-loads, *i. e.*, the loads that should be applied to the joints of the structure to prevent them from moving. These nodal loads are equal to the reversed consistent nodal loads, $-\mathbf{N}^{cent}$. This means that each beam element can be studied separately. Superposition of the solutions of these two complementary problems gives the solution of the actual problem and the total actual static axial load distribution $H(\xi)$.

In the first problem a constant static compressive axial load H_0 for each beam element is easily determined since the axial deformation of each beam element is implicitly known through the vector $\mathbf{p}^{st}$. Here both the displacement vector $\mathbf{p}^{st}$ and the axial load H_0 in each beam element are proportional to the square of the spinning speed Ω.

The consistent nodal loads due to the static centrifugal loading can be calculated as

$$\mathbf{N}^{cent} = \int_0^1 \Psi^T \begin{Bmatrix} \mathbf{B}_u \\ \mathbf{B}_\phi \end{Bmatrix} L d\xi \tag{17}$$

The force and moment elements of the consistent nodal load vector $\mathbf{N}^{cent}$ can be calculated using Equations (8a,b) for static centrifugal load. One finds

$$N_1^{cent} = \frac{L}{6}(2\beta_{u1}' + \beta_{u1}'') \tag{18a}$$

$$N_2^{cent} = \frac{L}{60}(21\beta_{u2}'\theta_{z1} + 20\beta_{u2}'\theta_{z2} + 9\beta_{u2}''\theta_{z1} + 10\beta_{u2}''\theta_{z2}) - \beta_{\phi3}\theta_{z1} \tag{18b}$$

$$N_3^{cent} = \frac{L}{60}(21\beta_{u3}'\theta_{y1} + 20\beta_{u3}'\theta_{y2} + 9\beta_{u3}''\theta_{y1} + 10\beta_{u3}''\theta_{y2}) + \beta_{\phi2}\theta_{y1} \tag{18c}$$

$$N_4^{cent} = N_{10}^{cent} = \frac{L}{2}\beta_{\phi1} \tag{18d,e}$$

$$N_5^{cent} = -\frac{L^2}{120}(6\beta_{u3}'\theta_{y1} + 5\beta_{u3}'\theta_{y2} + 4\beta_{u3}''\theta_{y1} + 5\beta_{u3}''\theta_{y2}) + \frac{L}{2}\beta_{\phi2}\theta_{y2} \tag{18f}$$

$$N_6^{cent} = \frac{L^2}{120}(6\beta_{u2}'\theta_{z1} + 5\beta_{u2}'\theta_{z2} + 4\beta_{u2}''\theta_{z1} + 5\beta_{u2}''\theta_{z2}) + \frac{L}{2}\beta_{\phi3}\theta_{z2} \tag{18g}$$

$$N_7^{cent} = \frac{L}{6}(\beta_{u1}' + 2\beta_{u1}'') \tag{18h}$$

$$N_8^{cent} = \frac{L}{60}(9\beta_{u2}'\theta_{z1} + 10\beta_{u2}'\theta_{z2} + 21\beta_{u2}''\theta_{z1} + 20\beta_{u2}''\theta_{z2}) + \beta_{\phi3}\theta_{z1} \tag{18i}$$

$$N_9^{cent} = \frac{L}{60}(9\beta_{u3}'\theta_{y1} + 10\beta_{u3}'\theta_{y2} + 21\beta_{u3}''\theta_{y1} + 20\beta_{u3}''\theta_{y2}) - \beta_{\phi2}\theta_{y1} \tag{18j}$$

$$N_{11}^{cent} = \frac{L^2}{120}(4\beta_{u3}'\theta_{y1} + 5\beta_{u3}'\theta_{y2} + 6\beta_{u3}''\theta_{y1} + 5\beta_{u3}''\theta_{y2}) + \frac{L}{2}\beta_{\phi2}\theta_{y2} \tag{18k}$$

$$N_{12}^{cent} = -\frac{L^2}{120}(4\beta_{u2}'\theta_{z1} + 5\beta_{u2}'\theta_{z2} + 6\beta_{u2}''\theta_{z1} + 5\beta_{u2}''\theta_{z2}) + \frac{L}{2}\beta_{\phi3}\theta_{z2} \tag{18l}$$

Here the β_{ui}' and β_{ui}'' are the static centrifugal loads in the direction i at the beam ends 1 and 2, respectively, see Equation (8). The additional bending/shearing parameters θ_{y1}, θ_{y2}, θ_{z1} and θ_{z2} are defined in Equation (A.2) in Appendix.

In the second problem the axial load variation $H^{fix}(\xi)$ within each beam element depending on the distributed inertia loads $\mathbf{B}_u$ and $\mathbf{B}_\phi$ will be determined. The centrifugal force distributions are given by Equations (8a,b). Their dependence on the beam element orientation is given by the 3×3 transformation matrix $\mathbf{L}$ with elements l_{ij}. The force distribution along the beam axis can be extracted as

$$U_x = m\Omega^2[\, l_{12}Y(\xi) + l_{13}Z(\xi)] \tag{19}$$

As the beam lamina position $(Y(\xi), Z(\xi))$ varies linearly with the beam coordinate ξ, one has

$$Y(\xi) = Y_1 + \Delta Y \xi \qquad \Delta Y = Y_2 - Y_1 \tag{20a, b}$$

$$Z(\xi) = Z_1 + \Delta Z \xi \qquad \Delta Z = Z_2 - Z_1 \tag{20c, d}$$

Here (Y_1, Z_1) and (Y_2, Z_2) are the YZ-coordinates of the beam ends 1 and 2. The axial load distribution $H^{fix}(\xi)$ of a beam with both ends fixed and with the applied external load given by Equation (19) can now be calculated with basically an axial equilibrium consideration as

$$H^{fix} = H_1^{fix}(2\xi - 1) + H_2^{fix}(3\xi^2 - 1) \tag{21a}$$

$$H_1^{fix} = \frac{1}{2}mL\Omega^2(l_{12}Y_1 + l_{13}Z_1) \tag{21b}$$

$$H_2^{fix} = \frac{1}{3}mL\Omega^2(l_{12}\Delta Y + l_{13}\Delta Z) \tag{21c}$$

The total compressive axial load distribution is thus given by

$$H(\xi) = H_0 + H_1^{fix}(2\xi - 1) + H_2^{fix}(3\xi^2 - 1) \tag{22}$$

6. GEOMETRIC STIFFNESS MATRIX

The geometric stiffness matrix for a beam studied in the xz-plane can be derived, see Cook, Malkus & Plesha[11], as

$$\mathbf{K}^g = -\int_0^1 \Psi_{3,.}^T\, H(\xi)\Psi_{3,.}\, d\xi / L \tag{23}$$

The row vector $\Psi_{3,.}$ is the third row of the matrix Ψ in Equation (10). The elements of the geometric stiffness matrix pertaining to the displacement in the xz-plane are given in Appendix. The stiffness in torsion is also affected by the axial force in a beam element developed for a spinning structure. The torsional stiffness GI_t of an element is reduced with the factor $H_0 i_x^2$. Here, the parameter i_x is the radius of area inertia and H_0 is the constant axial force calculated when the consistent loads of Equation (18) are applied.

7. NUMERICAL EXAMPLES

Two example problems have been studied, and the numerical results are presented below. The first example has been chosen to verify the stiffness reduction represented by the geometric stiffness matrix $\mathbf{K}^g = \mathbf{K}^g(\Omega)$. In the second example, the bending/torsional coupling effect for a straight beam is studied. Only beams having a doubly-symmetric cross-section will be considered here.

EXAMPLE A: CANTILEVER BEAM ON A ROTATING RIGID RING

The fundamental eigenfrequency and the stability of a spinning cantilever beam are studied. The Euler/Bernoulli beam theory is employed. The cantilever beam is clamped to a rigid ring and directed along a radius towards the centre, see Figure 2.

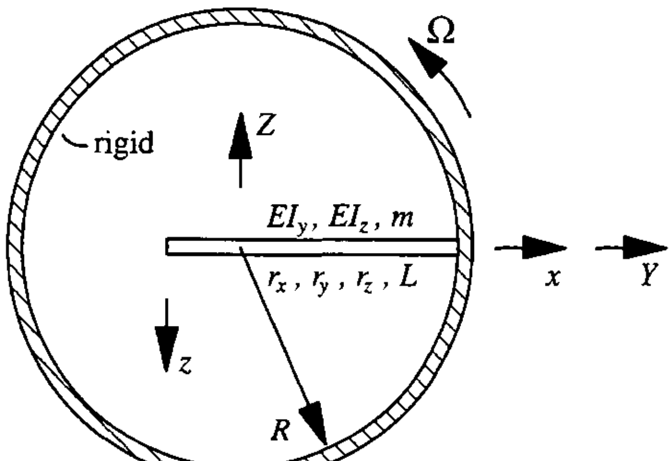

Figure 2. Spinning rigid ring of radius R with a radially oriented cantilever beam of length L=1.0 m. Beam has circular cross-section with diameter $2r$=10.0 mm. Young's modulus is E=200 GPa. Density is ρ=8000 kg/m^3. Interesting beam properties are: bending stiffnesses EI_y=EI_z=98.2 Nm^2, mass per unit length m=0.628 kg/m, radii of inertia r_x=3.54 mm and r_y=r_z=2.5 mm. Fundamental frequency for zero spinning speed is ω_{10}=43.9 rad/s.

The ring spins in its plane around the centre. In Figure 3 the beam is studied for the large ratio α=R/L=100 of radius to beam length. The beam is then subjected to a nearly uniformly distributed axial load and the classical Greenhill[12] buckling load should be recovered. Instability by divergence occurs when the fundamental eigenfrequency has decreased to zero. The calculated fundamental eigenfrequency is plotted versus the spinning speed for four cases in Figure 3. Subdivisions of the beam into one and four finite elements of equal length are made. The finite elements used have a geometric stiffness matrix either taking into account a linearly varying compressive axial load or assuming an average constant axial load in each beam element. The Greenhill buckling load q_c=$7.837EI/L^3$ corresponds to the critical spinning speed Ω_c=$0.0798\omega_{10}$ with ω_{10}=$0.3562\pi^2(EI_z/mL^4)^{1/2}$ =43.9 rad/s. Figure 3 shows the significance of allowing a linearly varying compressive axial load. One finite element with a linear axial load gives better results than four finite elements with an average constant axial load in each beam element. In Figure 4 the beam is studied for the ratio α=R/L equal to 1.0 and 0.5. Also here the fundamental eigenfrequency is plotted versus the spinning speed. The beam is modelled with four elements taking into account a parabolically varying compressive axial load. The calculated speeds which give instability by divergence for the two cases are indicated in Figure 4. The numerical results presented in Figure 4 coincide with those given by Gürgöze[13]. Various similar structures for other parameter values have also been studied of Bauer & Eidel[14].

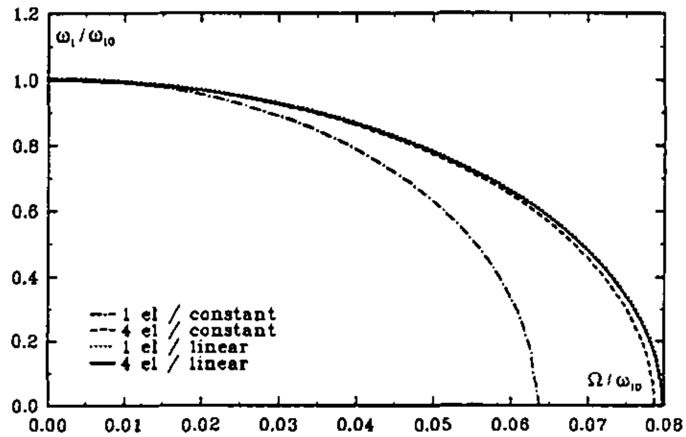

Figure 3. Calculated dimensionless fundamental frequency ω_1/ω_{10} plotted versus dimensionless spinning speed Ω_1/ω_{10} of cantilever beam in Figure 2 for ratio α=R/L=100. Beam is modelled with one and four finite elements, respectively, of equal length. Axial compressive load is taken as constant within each element or as linearly varying within each element.

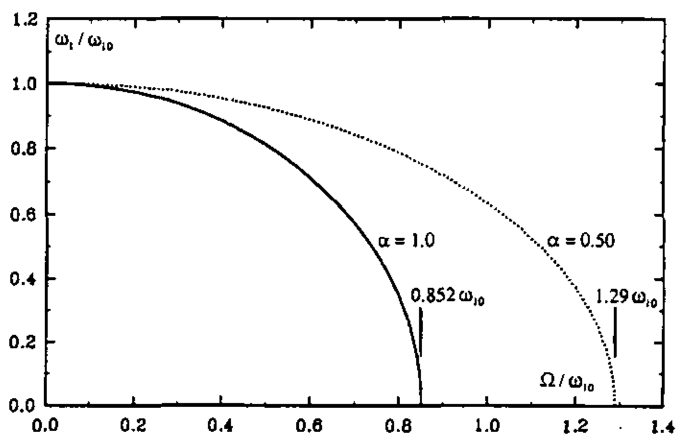

Figure 4. Calculated dimensionless fundamental frequency ω_1/ω_{10} plotted versus dimensionless spinning speed Ω/ω_{10} of cantilever beam in Figure 2 for ratios α=R/L equal to 1.0 and 0.50. Beam is modelled with four finite elements of equal length. Axial compressive load varies parabolically within each beam element.

EXAMPLE B: CANTILEVER BEAM SPINNING AS A BLADE

Free and forced vibration of a spinning cantilever beam system is studied. The beam is oriented perpendicularly to the spin axis, which passes through the clamped end of the beam, see Figure 5. The beam has a cruciform cross-section with a low torsional stiffness GI_t as compared to the bending stiffnesses EI_y and EI_z. A similar beam was studied by Leung & Fung[6] but their beam was much stiffer in torsion. Figure 6 shows the calculated lowest six natural frequencies of the beam plotted

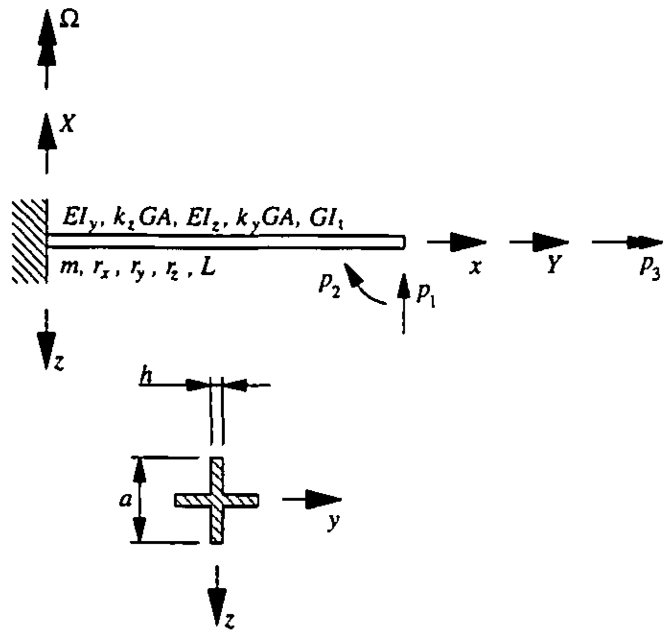

Figure 5. Spinning cantilever beam of length L=8.0 m oriented perpendicularly to spin axis passing through clamped end. Beam has doubly-symmetric cruciform cross-section with dimensions a=0.20 m and h=3.0 mm. Young's modulus is E=200 GPa, density ρ=8000 kg/m^3, and Poisson's ratio ν=0.30. Beam properties are: bending stiffnesses EI_y=EI_z=0.40 MNm^2, shearing stiffnesses k_zGA=k_yGA=38.8 MN, torsional stiffness GI_t=0.277 kNm^2, mass per unit length m=9.6 kg/m, radii of inertia r_x=57.7 mm and r_y=r_z=40.8 mm. Displacements p_1, p_2 and p_3 of beam end are shown.

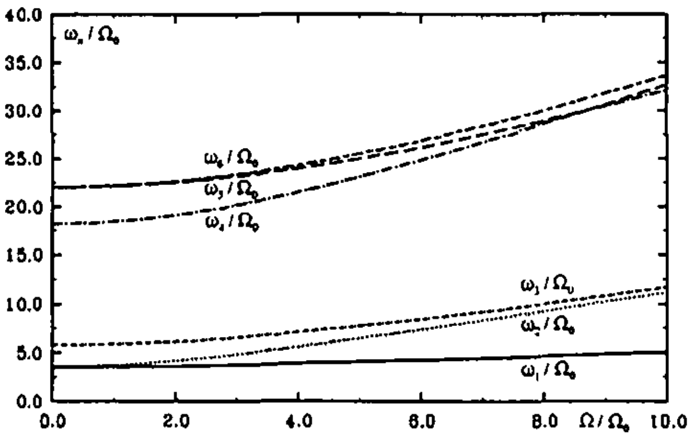

Figure 6. Calculated lowest six dimensionless eigenfrequencies ω_n/Ω_0 of beam in Figure 5 plotted versus dimensionless spinning speed Ω/Ω_0 with $\Omega_0=(EI_y/mL^4)^{1/2}$=3.19 rad/s.

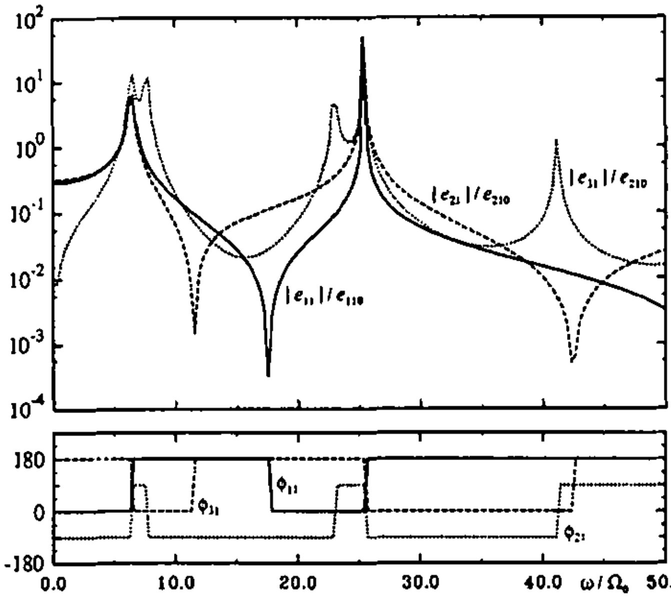

Figure 7. Calculated magnitudes and phases of three complex dimensionless flexibilities are plotted versus dimensionless load frequency ω/Ω_0. Spinning speed is $\Omega=5\Omega_0$. Normalizing flexibilities are $e_{110}=L^3/3EI_y$ and $e_{210}=L^2/2EI_y$.

versus the spinning speed. The four natural frequencies ω_1, ω_2, ω_5 and ω_6 pertaining to bending coincide with the results presented by Leung & Fung[6]. For zero spinning speed Ω the eigenfrequencies pertaining to eigenmodes describing a bending motion in the XY-plane coincide with the corresponding eigenfrequencies for the YZ-plane. The two distinct eigenfrequencies ω_3 and ω_4 for zero spinning speed Ω in Figure 6 pertain to torsional motion. When the spinning speed Ω increases the system becomes stiffer mainly because of the increasing tensional axial load depending on the centrifugal acceleration. For a non-zero spinning speed Ω, separate eigenmodes in bending and torsion do no longer exist, *i.e.*, the eigenmodes describe a coupled motion in both bending in the XY-plane and torsion about the Y-direction, or bending in the YZ-plane and tension in the Y-direction. Note that the eigenmodes discussed here are described in a co-rotating coordinate system. In Figure 7 the magnitudes and phases of three displacements are plotted versus the frequency of a harmonic load applied to the beam in Figure 5 in the p_1-direction. Note that both the load and the displacements are taken in a co-rotating coordinate system. The coupling between bending in the XY-plane and torsion about the Y-direction shown in Figure 7 is not included in the paper by Leung & Fung[6]. It should be observed that the Rayleigh/Timoshenko beam theory has been used in the present study.

8. CONCLUDING REMARKS

The gyroscopic, circulatory and geometric stiffness matrices of a spinning finite Rayleigh/Timoshenko/Saint-Venant beam element have been derived. In combination with the classical stiffness, damping and mass matrices of such a beam element, these matrices constitute the discrete counterpart to the exact dynamic stiffness matrices presented by Wittrick & Williams[5] (Euler/Bernoulli beam theory) and Lundblad[7] (Rayleigh/Timoshenko beam theory). The ability to represent a spatially varying axial load distinguishes the present discretized approach from the 'exact' approach presented by the above authors. Another advantage of using the discrete parameter element over the exact distributed parameter element is that the dynamic stiffness can be computed with a drastically reduced effort and time. The matrices are also well suited for a transient vibration analysis with time-marching algorithms.

The coupling elements of the gyroscopic and circulatory matrices predict a coupled bending/shearing/tensional/torsional vibration of the generally oriented spinning beam. Some of the coupling elements are, to the authors' knowledge, presented here for the first time for a discretized element. The coupled free vibration of a spinning beam system with the influence on the bending/shearing motion by the torsional coupling has been demonstrated in a numerical example.

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APPENDIX

The 96 non-zero elements of the skew-symmetric gyroscopic matrix **G** of the beam element defined by Equation (14b) are, except for the common divisor 840,

$$g_{1,2}=g_{7,8}=-g_{2,1}=-g_{8,7}=a_{u,12}^{(1)}(294\theta_{z1}+280\theta_{z2})\,L \tag{A.1a}$$

$$g_{1,3}=g_{7,9}=-g_{3,1}=-g_{9,7}=a_{u,13}^{(1)}(294\theta_{y1}+280\theta_{y2})\,L \tag{A.1b}$$

$$g_{1,5}=-g_{7,11}=-g_{5,1}=g_{11,7}=-a_{u,13}^{(1)}(42\theta_{y1}+35\theta_{y2})\,L^2 \tag{A.1c}$$

$$g_{1,6}=-g_{7,12}=-g_{6,1}=g_{12,7}=a_{u,12}^{(1)}(42\theta_{z1}+35\theta_{z2})\,L^2 \tag{A.1d}$$

$$g_{1,8}=-g_{2,7}=-g_{8,1}=g_{7,2}=a_{u,12}^{(1)}(126\theta_{z1}+140\theta_{z2})\,L \tag{A.1e}$$

$$g_{1,9}=-g_{3,7}=-g_{9,1}=g_{7,3}=a_{u,13}^{(1)}(126\theta_{y1}+140\theta_{y2})\,L \tag{A.1f}$$

$$g_{1,11}=g_{5,7}=-g_{11,1}=-g_{7,5}=a_{u,13}^{(1)}(28\theta_{y1}+35\theta_{y2})\,L^2 \tag{A.1g}$$

$$g_{1,12}=g_{6,7}=-g_{12,1}=-g_{7,6}=-a_{u,12}^{(1)}(28\theta_{z1}+35\theta_{z2})\,L^2 \tag{A.1h}$$

$$\begin{aligned} g_{2,4} &= g_{2,10}=g_{4,8}=-g_{8,10}=-g_{4,2}=-g_{10,2}=-g_{8,4}=g_{10,8} \\ &= 420a_{\phi,13}^{(1)}\theta_{z1} \end{aligned} \tag{A.1i}$$

$$\begin{aligned} g_{3,4} &= g_{3,10}=g_{4,9}=-g_{9,10}=-g_{4,3}=-g_{10,3}=-g_{9,4}=g_{10,9} \\ &= -420a_{\phi,12}^{(1)}\theta_{y1} \end{aligned} \tag{A.1j}$$

$$g_{4,5}=g_{10,11}=-g_{5,4}=-g_{11,10}=70a_{\phi,12}^{(1)}(\theta_{y1}+4\theta_{y2})\,L \tag{A.1k}$$

$$g_{4,6}=g_{10,12}=-g_{6,4}=-g_{12,10}=70a_{\phi,13}^{(1)}(\theta_{z1}+4\theta_{z2})\,L \tag{A.1l}$$

$$g_{4,11}=-g_{5,10}=-g_{11,4}=g_{10,5}=-70a_{\phi,12}^{(1)}(\theta_{y1}-2\theta_{y2})\,L \tag{A.1m}$$

$$g_{4,12}=-g_{6,10}=-g_{12,4}=g_{10,6}=-70a_{\phi,13}^{(1)}(\theta_{z1}-2\theta_{z2})\,L \tag{A.1n}$$

$$\begin{aligned} g_{2,3}=g_{8,9}=-g_{3,2}=-g_{9,8} &= a_{u,23}^{(1)}(312\theta_{y1}\theta_{z1}+294\theta_{y1}\theta_{z2}+294\theta_{y2}\theta_{z1} \\ &\quad +280\theta_{z2}\theta_{y2})\,L \end{aligned} \tag{A.1o}$$

$$g_{2,5} = -g_{8,11} = -g_{5,2} = g_{11,8} = -a^{(1)}_{u,23}(44\theta_{y1}\theta_{z1} + 42\theta_{y1}\theta_{z2} + 35\theta_{y2}\theta_{z1} + 35\theta_{y2}\theta_{z2})\,L^2 \qquad \text{(A.1p)}$$

$$g_{2,9} = -g_{3,8} = -g_{9,2} = g_{8,3} = a^{(1)}_{u,23}(108\theta_{y1}\theta_{z1} + 126\theta_{y1}\theta_{z2} + 126\theta_{y2}\theta_{z1} + 140\theta_{y2}\theta_{z2})\,L \qquad \text{(A.1q)}$$

$$g_{2,11} = g_{5,8} = -g_{11,2} = -g_{8,5} = a^{(1)}_{u,23}(26\theta_{y1}\theta_{z1} + 28\theta_{y1}\theta_{z2} + 35\theta_{y2}\theta_{z1} + 35\theta_{y2}\theta_{z2})\,L^2 \qquad \text{(A.1r)}$$

$$g_{3,6} = -g_{9,12} = -g_{6,3} = g_{12,9} = -a^{(1)}_{u,23}(44\theta_{y1}\theta_{z1} + 35\theta_{y1}\theta_{z2} + 42\theta_{y2}\theta_{z1} + 35\theta_{y2}\theta_{z2})\,L^2 \qquad \text{(A.1s)}$$

$$g_{3,12} = g_{6,9} = -g_{12,3} = -g_{9,6} = a^{(1)}_{u,23}(26\theta_{y1}\theta_{z1} + 35\theta_{y1}\theta_{z2} + 28\theta_{y2}\theta_{z1} + 35\theta_{y2}\theta_{z2})\,L^2 \qquad \text{(A.1t)}$$

$$g_{5,8} = g_{11,12} = -g_{6,5} = -g_{12,11} = a^{(1)}_{u,23}(8\theta_{y1}\theta_{z1} + 7\theta_{y1}\theta_{z2} + 7\theta_{y2}\theta_{z1} + 7\theta_{y2}\theta_{z2})\,L^3 \qquad \text{(A.1u)}$$

$$g_{5,12} = -g_{6,11} = -g_{12,5} = g_{11,6} = -a^{(1)}_{u,23}(6\theta_{y1}\theta_{z1} + 7\theta_{y1}\theta_{z2} + 7\theta_{y2}\theta_{z1} + 7\theta_{y2}\theta_{z2})\,L^3 \qquad \text{(A.1v)}$$

The additional bending/shearing parameters are defined as

$$\theta_{y1} = 1/(1+\theta_y) \qquad \theta_{y2} = \theta_y/(1+\theta_y) \qquad \text{(A.2a, b)}$$

$$\theta_{z1} = 1/(1+\theta_z) \qquad \theta_{z2} = \theta_z/(1+\theta_z) \qquad \text{(A.2c, d)}$$

It should be noted that none of the 96 elements above vanishes when the Euler/Bernoulli theory ($\theta_{y2} = \theta_{z2} = 0$) is employed.

The 136 non-zero elements of the non-symmetric circulatory 12×12 matrix **H** in Equation (16b) are, except for the common divisor $2520L$,

$$h_{1,1} = h_{7,7} = 840a^{(0)}_{u,11}\,L^2 \qquad \text{(A.3a)}$$

$$h_{1,2} = h_{7,8} = h_{2,1} = h_{8,7} = a^{(0)}_{u,12}(882\theta_{z1} + 840\theta_{z2})\,L^2 \qquad \text{(A.3b)}$$

$$h_{1,3} = h_{7,9} = h_{3,1} = h_{9,7} = a^{(0)}_{u,13}(882\theta_{y1} + 840\theta_{y2})\,L^2 \qquad \text{(A.3c)}$$

$$h_{1,5} = -h_{7,11} = h_{5,1} = -h_{11,7} = -a^{(0)}_{u,13}(126\theta_{y1} + 105\theta_{y2})\,L^3 \qquad \text{(A.3d)}$$

$$h_{1,6} = -h_{7,12} = h_{6,1} = -h_{12,7} = a^{(0)}_{u,12}(126\theta_{z1} + 105\theta_{z2})\,L^3 \qquad \text{(A.3e)}$$

$$h_{1,7} = h_{7,1} = 420a^{(0)}_{u,11}\,L^2 \qquad \text{(A.3f)}$$

$$h_{1,8} = h_{2,7} = h_{8,1} = h_{7,2} = a^{(0)}_{u,12}(378\theta_{z1} + 420\theta_{z2})\,L^2 \qquad \text{(A.3g)}$$

$$h_{1,9} = h_{3,7} = h_{9,1} = h_{7,3} = a^{(0)}_{u,13}(378\theta_{y1} + 420\theta_{y2})\,L^2 \qquad \text{(A.3h)}$$

$$h_{1,11} = -h_{5,7} = h_{11,1} = -h_{7,5} = a^{(0)}_{u,13}(84\theta_{y1} + 105\theta_{y2})\,L^3 \qquad \text{(A.3i)}$$

$$h_{1,12} = -h_{6,7} = h_{12,1} = -h_{7,6} = -a^{(0)}_{u,12}(84\theta_{z1} + 105\theta_{z2})\,L^3 \qquad \text{(A.3j)}$$

$$h_{2,2} = h_{8,8} = a^{(0)}_{u,22}(936\theta^2_{z1} + 1764\theta_{z1}\theta_{z2} + 840\theta^2_{z2})\,L^2 + 3024a^{(0)}_{\phi,33}\theta^2_{z1} \qquad \text{(A.3k)}$$

$$h_{2,3} = h_{8,9} = a^{(0)}_{u,23}(936\theta_{y1}\theta_{z1} + 882\theta_{y1}\theta_{z2} + 882\theta_{y2}\theta_{z1} + 840\theta_{y2}\theta_{z2})\,L^2 - 3024a^{(0)}_{\phi,32}\theta_{y1}\theta_{z1} \qquad \text{(A.3l)}$$

$$h_{3,2} = h_{9,8} = a^{(0)}_{u,23}(936\theta_{y1}\theta_{z1} + 882\theta_{y1}\theta_{z2} + 882\theta_{y2}\theta_{z1} + 840\theta_{y2}\theta_{z2})\,L^2 - 3024a^{(0)}_{\phi,23}\theta_{y1}\theta_{z1} \qquad \text{(A.3m)}$$

$$h_{2,4} = h_{2,10} = -h_{8,10} = -h_{8,4} = -1260a^{(0)}_{\phi,31}\theta_{z1}\,L \qquad \text{(A.3n)}$$

$$h_{4,2} = h_{10,2} = -h_{10,8} = -h_{4,8} = -1260a^{(0)}_{\phi,13}\theta_{z1}\,L \qquad \text{(A.3o)}$$

$$h_{2,5} = -h_{8,11} = -a^{(0)}_{u,23}(132\theta_{y1}\theta_{z1} + 126\theta_{y1}\theta_{z2} + 105\theta_{y2}\theta_{z1} + 105\theta_{y2}\theta_{z2})\,L^3 + 252a^{(0)}_{\phi,32}\theta_{z1}(\theta_{y1} - 5\theta_{y2})\,L \qquad \text{(A.3p)}$$

$$h_{5,2} = -h_{11,8} = -a^{(0)}_{u,23}(132\theta_{y1}\theta_{z1} + 126\theta_{y1}\theta_{z2} + 105\theta_{y2}\theta_{z1} + 105\theta_{y2}\theta_{z2})\,L^3 + 252a^{(0)}_{\phi,23}\theta_{z1}(\theta_{y1} - 5\theta_{y2})\,L \qquad \text{(A.3q)}$$

$$h_{2,6} = -h_{8,12} = h_{6,2} = -h_{12,8} = a^{(0)}_{u,22}(132\theta^2_{z1} + 231\theta_{z1}\theta_{z2} + 105\theta^2_{z2})\,L^3 + 252a^{(0)}_{\phi,33}\theta_{z1}(\theta_{z1} - 5\theta_{z2})\,L \qquad \text{(A.3r)}$$

$$h_{2,8} = h_{8,2} = a^{(0)}_{u,22}(324\theta^2_{z1} + 756\theta_{z1}\theta_{z2} + 420\theta^2_{z2})\,L^2 - 3024a^{(0)}_{\phi,33}\theta^2_{z1} \qquad \text{(A.3s)}$$

$$h_{2,9} = h_{8,3} = a^{(0)}_{u,23}(324\theta_{y1}\theta_{z1} + 378\theta_{y1}\theta_{z2} + 378\theta_{y2}\theta_{z1} + 420\theta_{y2}\theta_{z2})\,L^2 + 3024a^{(0)}_{\phi,32}\theta_{y1}\theta_{z1} \qquad \text{(A.3t)}$$

$$h_{2,11} = -h_{8,5} = a^{(0)}_{u,23}(78\theta_{y1}\theta_{z1} + 84\theta_{y1}\theta_{z2} + 105\theta_{y2}\theta_{z1} + 105\theta_{y2}\theta_{z2})\,L^3 + 252a^{(0)}_{\phi,32}\theta_{z1}(\theta_{y1} - 5\theta_{y2})\,L \qquad \text{(A.3u)}$$

$$h_{5,8} = -h_{11,2} = -a^{(0)}_{u,23}(78\theta_{y1}\theta_{z1} + 84\theta_{y1}\theta_{z2} + 105\theta_{y2}\theta_{z1} + 105\theta_{y2}\theta_{z2})\,L^3 - 252a^{(0)}_{\phi,23}\theta_{z1}(\theta_{y1} - 5\theta_{y2})\,L \qquad \text{(A.3v)}$$

$$h_{2,12} = -h_{6,8} = h_{12,2} = -h_{8,6} = -a^{(0)}_{u,22}(78\theta^2_{z1} + 189\theta_{z1}\theta_{z2} + 105\theta^2_{z2})\,L^3 + 252a^{(0)}_{\phi,33}\theta_{z1}(\theta_{z1} - 5\theta_{z2})\,L \qquad \text{(A.3w)}$$

$$h_{3,3}=h_{9,9}=a^{(0)}_{u,33}(936\theta^2_{y1}+1764\theta_{y1}\theta_{y2}+840\theta^2_{y2})\,L^2+3024a^{(0)}_{\phi,22}\theta^2_{y1} \tag{A.3x}$$

$$h_{3,4}=h_{3,10}=-h_{9,10}=-h_{9,4}=1260a^{(0)}_{\phi,21}\theta_{y1}\,L \tag{A.3y}$$

$$h_{4,3}=h_{10,3}=-h_{10,9}=-h_{4,9}=1260a^{(0)}_{\phi,12}\theta_{y1}\,L \tag{A.3z}$$

$$h_{3,5}=-h_{9,11}=h_{5,3}=-h_{11,9}=-a^{(0)}_{u,33}(132\theta^2_{y1}+231\theta_{y1}\theta_{y2}+105\theta^2_{y2})\,L^3-252a^{(0)}_{\phi,22}\theta_{y1}(\theta_{y1}-5\theta_{y2})\,L \tag{A.3A}$$

$$h_{3,6}=-h_{9,12}=a^{(0)}_{u,23}(132\theta_{y1}\theta_{z1}+105\theta_{y1}\theta_{z2}+126\theta_{y2}\theta_{z1}+105\theta_{y2}\theta_{z2})\,L^3-252a^{(0)}_{\phi,23}\theta_{y1}(\theta_{z1}-5\theta_{z2})\,L \tag{A.3B}$$

$$h_{6,3}=-h_{12,9}=a^{(0)}_{u,23}(132\theta_{y1}\theta_{z1}+105\theta_{y1}\theta_{z2}+126\theta_{y2}\theta_{z1}+105\theta_{y2}\theta_{z2})\,L^3-252a^{(0)}_{\phi,32}\theta_{y1}(\theta_{z1}-5\theta_{z2})\,L \tag{A.3C}$$

$$h_{3,8}=h_{9,2}=a^{(0)}_{u,23}(324\theta_{y1}\theta_{z1}+378\theta_{y1}\theta_{z2}+378\theta_{y2}\theta_{z1}+420\theta_{y2}\theta_{z2})\,L^2+3024a^{(0)}_{\phi,23}\theta_{y1}\theta_{z1} \tag{A.3D}$$

$$h_{3,9}=h_{9,3}=a^{(0)}_{u,33}(324\theta^2_{y1}+756\theta_{y1}\theta_{y2}+420\theta^2_{y2})\,L^2-3024a^{(0)}_{\phi,22}\theta^2_{y1} \tag{A.3E}$$

$$h_{3,11}=-h_{5,9}=h_{11,3}=-h_{9,5}=a^{(0)}_{u,33}(78\theta^2_{y1}+189\theta_{y1}\theta_{y2}+105\theta^2_{y2})\,L^3-252a^{(0)}_{\phi,22}\theta_{y1}(\theta_{y1}-5\theta_{y2})\,L \tag{A.3F}$$

$$h_{3,12}=-h_{9,6}=-a^{(0)}_{u,23}(78\theta_{y1}\theta_{z1}+105\theta_{y1}\theta_{z2}+84\theta_{y2}\theta_{z1}+105\theta_{y2}\theta_{z2})\,L^3-252a^{(0)}_{\phi,23}\theta_{y1}(\theta_{z1}-5\theta_{z2})\,L \tag{A.3G}$$

$$h_{4,4}=h_{10,10}=840a^{(0)}_{\phi,11}\,L^2 \tag{A.3H}$$

$$h_{4,5}=h_{10,11}=210a^{(0)}_{\phi,12}(\theta_{y1}+4\theta_{y2})\,L^2 \tag{A.3I}$$

$$h_{5,4}=h_{11,10}=210a^{(0)}_{\phi,21}(\theta_{y1}+4\theta_{y2})\,L^2 \tag{A.3J}$$

$$h_{4,6}=h_{10,12}=210a^{(0)}_{\phi,13}(\theta_{z1}+4\theta_{z2})\,L^2 \tag{A.3K}$$

$$h_{6,4}=h_{12,10}=210a^{(0)}_{\phi,31}(\theta_{z1}+4\theta_{z2})\,L^2 \tag{A.3L}$$

$$h_{4,10}=h_{10,4}=420a^{(0)}_{\phi,11}\,L^2 \tag{A.3M}$$

$$h_{4,11}=h_{10,5}=-210a^{(0)}_{\phi,12}(\theta_{y1}-2\theta_{y2})\,L^2 \tag{A.3N}$$

$$h_{5,10}=h_{11,4}=-210a^{(0)}_{\phi,21}(\theta_{y1}-2\theta_{y2})\,L^2 \tag{A.3O}$$

$$h_{4,12}=h_{10,6}=-210a^{(0)}_{\phi,13}(\theta_{z1}-2\theta_{z2})\,L^2 \tag{A.3P}$$

$$h_{6,10}=h_{12,4}=-210a^{(0)}_{\phi,31}(\theta_{z1}-2\theta_{z2})\,L^2 \tag{A.3Q}$$

$$h_{5,5}=h_{11,11}=a^{(0)}_{u,33}(24\theta^2_{y1}+42\theta_{y1}\theta_{y2}+21\theta^2_{y2})\,L^4+84a^{(0)}_{\phi,22}(4\theta^2_{y1}+5\theta_{y1}\theta_{y2}+10\theta^2_{y2})\,L^2 \tag{A.3R}$$

$$h_{5,6}=h_{11,12}=-a^{(0)}_{u,23}(24\theta_{y1}\theta_{z1}+21\theta_{y1}\theta_{z2}+21\theta_{y2}\theta_{z1}+21\theta_{y2}\theta_{z2})\,L^4+42a^{(0)}_{\phi,23}(8\theta_{y1}\theta_{z1}+5\theta_{y2}\theta_{z1}+5\theta_{y1}\theta_{z2}+20\theta_{y2}\theta_{z2})\,L^2 \tag{A.3S}$$

$$h_{6,5}=h_{12,11}=-a^{(0)}_{u,23}(24\theta_{y1}\theta_{z1}+21\theta_{y1}\theta_{z2}+21\theta_{y2}\theta_{z1}+21\theta_{y2}\theta_{z2})\,L^4+42a^{(0)}_{\phi,32}(8\theta_{y1}\theta_{z1}+5\theta_{y2}\theta_{z1}+5\theta_{y1}\theta_{z2}+20\theta_{y2}\theta_{z2})\,L^2 \tag{A.3T}$$

$$h_{5,11}=h_{11,5}=-a^{(0)}_{u,33}(18\theta^2_{y1}+42\theta_{y1}\theta_{y2}+21\theta^2_{y2})\,L^4-84a^{(0)}_{\phi,22}(\theta^2_{y1}+5\theta_{y1}\theta_{y2}-5\theta^2_{y2})\,L^2 \tag{A.3U}$$

$$h_{5,12}=h_{11,6}=a^{(0)}_{u,23}(18\theta_{y1}\theta_{z1}+21\theta_{y1}\theta_{z2}+21\theta_{y2}\theta_{z1}+21\theta_{y2}\theta_{z2})\,L^4-42a^{(0)}_{\phi,23}(2\theta_{y1}\theta_{z1}+5\theta_{y2}\theta_{z1}+5\theta_{y1}\theta_{z2}-10\theta_{y2}\theta_{z2})\,L^2 \tag{A.3V}$$

$$h_{6,11}=h_{12,5}=a^{(0)}_{u,23}(18\theta_{y1}\theta_{z1}+21\theta_{y1}\theta_{z2}+21\theta_{y2}\theta_{z1}+21\theta_{y2}\theta_{z2})\,L^4-42a^{(0)}_{\phi,32}(2\theta_{y1}\theta_{z1}+5\theta_{y2}\theta_{z1}+5\theta_{y1}\theta_{z2}-10\theta_{y2}\theta_{z2})\,L^2 \tag{A.3W}$$

$$h_{6,6}=h_{12,12}=a^{(0)}_{u,22}(24\theta^2_{z1}+42\theta_{z1}\theta_{z2}+21\theta^2_{z2})\,L^4+84a^{(0)}_{\phi,33}(4\theta^2_{z1}+5\theta_{z1}\theta_{z2}+10\theta^2_{z2})\,L^2 \tag{A.3X}$$

$$h_{6,9}=-h_{12,3}=a^{(0)}_{u,23}(78\theta_{y1}\theta_{z1}+105\theta_{y1}\theta_{z2}+84\theta_{y2}\theta_{z1}+105\theta_{y2}\theta_{z2})\,L^3+252a^{(0)}_{\phi,32}\theta_{y1}(\theta_{z1}-5\theta_{z2})\,L \tag{A.3Y}$$

$$h_{6,12}=h_{12,6}=-a^{(0)}_{u,22}(18\theta^2_{z1}+42\theta_{z1}\theta_{z2}+21\theta^2_{z2})\,L^4-84a^{(0)}_{\phi,33}(\theta^2_{z1}+5\theta_{z1}\theta_{z2}-5\theta^2_{z2})\,L^2 \tag{A.3Z}$$

It should be noted that several elements representing a coupling between motions in tension, torsion, and bending/shearing in the two planes *xy* and *xz*, will vanish in the case that the spin axis *X* is parallel to any of the principal axes *x*, *y* and *z*.

The symmetric geometric stiffness matrix for displacement in the *xz*-plane has been derived from Equation (23). The contribution to the geometric stiffness matrix from the constant part of the axial loading is, except for the common divisor 60*L*,

$$k^{ga}_{3,3}=-k^{ga}_{3,9}=k^{ga}_{9,9}=-H_a(72\theta^2_{y1}+120\theta_{y1}\theta_{y2}+60\theta^2_{y2}) \tag{A.4a}$$

$$k^{ga}_{3,5}=k^{ga}_{3,11}=-k^{ga}_{5,9}=-k^{ga}_{9,11}=6H_a\theta^2_{y1}\,L \tag{A.4b}$$

$$k^{ga}_{5,5}=k^{ga}_{11,11}=-H_a(8\theta^2_{y1}+10\theta_{y1}\theta_{y2}+5\theta^2_{y2})\,L^2 \tag{A.4c}$$

$$k^{ga}_{5,11}=H_a(2\theta^2_{y1}+10\theta_{y1}\theta_{y2}+5\theta^2_{y2})\,L^2 \tag{A.4d}$$

Here, the constant axial load is $H_a=H_0-H_1^{fix}-H_2^{fix}$. The linear variation of the axial loading $H_b=2H_1^{fix}$ contributes to the geometric stiffness with, except for the common divisor 840*L*,

$$k^{gb}_{3,3} = -k^{gb}_{3,9} = k^{gb}_{9,9} = -H_b(504\theta^2_{y1} + 840\theta_{y1}\theta_{y2} + 420\theta^2_{y2}) \tag{A.5a}$$

$$k^{gb}_{3,5} = -k^{gb}_{5,9} = H_b(84\theta^2_{y1} + 112\theta_{y1}\theta_{y2} + 70\theta^2_{y2})\,L \tag{A.5b}$$

$$k^{gb}_{3,11} = -k^{gb}_{9,11} = -H_b(112\theta_{y1}\theta_{y2} + 70\theta^2_{y2})\,L \tag{A.5c}$$

$$k^{gb}_{5,5} = -H_b(28\theta^2_{y1} + 42\theta_{y1}\theta_{y2} + 35\theta^2_{y2})\,L^2 \tag{A.5d}$$

$$k^{gb}_{5,11} = H_b(14\theta^2_{y1} + 70\theta_{y1}\theta_{y2} + 35\theta^2_{y2})\,L^2 \tag{A.5e}$$

$$k^{gb}_{11,11} = -H_b(84\theta^2_{y1} + 98\theta_{y1}\theta_{y2} + 35\theta^2_{y2})\,L^2 \tag{A.5f}$$

The contribution from the quadratic variation $H_c=3H_2^{fix}$ of the axial loading is, except for the common divisor $420L$,

$$k^{gc}_{3,3} = -k^{gc}_{3,9} = k^{gc}_{9,9} = -H_c(144\theta^2_{y1} + 252\theta_{y1}\theta_{y2} + 140\theta^2_{y2}) \tag{A.6a}$$

$$k^{gc}_{3,5} = -k^{gc}_{5,9} = H_c(30\theta^2_{y1} + 49\theta_{y1}\theta_{y2} + 35\theta^2_{y2})\,L \tag{A.6b}$$

$$k^{gc}_{3,11} = -k^{gc}_{9,11} = -H_c(12\theta^2_{y1} + 63\theta_{y1}\theta_{y2} + 35\theta^2_{y2})\,L \tag{A.6c}$$

$$k^{gc}_{5,5} = -H_c(8\theta^2_{y1} + 14\theta_{y1}\theta_{y2} + 14\theta^2_{y2})\,L^2 \tag{A.6d}$$

$$k^{gc}_{5,11} = H_c(6\theta^2_{y1} + 28\theta_{y1}\theta_{y2} + 14\theta^2_{y2})\,L^2 \tag{A.6e}$$

$$k^{gc}_{11,11} = -H_c(36\theta^2_{y1} + 42\theta_{y1}\theta_{y2} + 14\theta^2_{y2})\,L^2 \tag{A.6f}$$