A Split-Combination Approach to Merging Knowledge Bases in Possibilistic Logic*

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Abstract

We propose an adaptive approach to merging possibilistic knowledge bases that deploys multiple operators instead of a single operator in the merging process. The merging approach consists of two steps: the splitting step and the combination step. The splitting step splits each knowledge base into two subbases and then in the second step, different classes of subbases are combined using different operators. Our merging approach is applied to knowledge bases which are self-consistent and results in a knowledge base which is also consistent. Two operators are proposed based on two different splitting methods. Both operators result in a possibilistic knowledge base which contains more information than that obtained by the *t-conorm* (such as the maximum) based merging methods. In the flat case, one of the operators provides a good alternative to syntax-based merging operators in classical logic.

Keywords: Knowledge representation, merging of knowledge bases, inconsistency handling, possibilistic logic.

1 Introduction

In many cases we confront the problem of merging inconsistent information from different sources [10, 11, 14, 15, 31, 24, 25, 30, 38]. When merging different data sources, we often need to consider uncertainty. Possibilistic logic [17] provides a good framework for dealing with fusion problems when information is pervaded with inconsistency and uncertainty or where only partial or incomplete information is available [4, 6, 7, 8, 10, 11]. There are two different views for merging possibilistic knowledge bases. According to the first view, inconsistency is considered unacceptable and the conflicting information between different sources should be resolved after merging [4, 8, 9]. By contrast, the second view claims that inconsistency is unavoidable and so the resulting possibilistic knowledge base can be inconsistent after merging [1, 10, 11, 37]. Although possibilistic inference is

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inconsistency tolerant, it suffers from the "drowning problem" [3, 7]. That is, those formulas whose necessity degrees are below the inconsistency degree of a possibilistic knowledge base cannot be used to infer conclusions. In [1], the argumentation based merging approach has been proposed to overcome the drowning problem. However, it has been shown that argumentation based inference is computationally very hard [18]. Therefore, in this paper, we focus on the first view - that is, we require that inconsistency should be resolved after merging.

In [4, 8, 9, 10] some merging operators are proposed. Among them, two merging operators, the *maximum* (or more generally, *t-conorm*) based merging operator and the *minimum* (or more generally, *t-norm*) based merging operator, are used to combine inconsistent and consistent sources of information respectively. Given two *possibilistic knowledge bases* $\mathcal{B}_1 = \{(\phi_i, \alpha_i),\ i = 1, ..., n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j), j = 1, ..., m\}$, where ϕ_i and ψ_j are classical propositional formulas, and α_i and β_j are their necessity degrees respectively (with $\alpha_i,\ \beta_j \in [0, 1]$), the syntactical results of merging $\mathcal{B}_1$ and $\mathcal{B}_2$ by the maximum based merging operator and the minimum based merging operator are $\mathcal{B}_{dm} = \{(\phi_i \vee \psi_j, min(\alpha_i, \beta_j))|(\phi_i, \alpha_i) \in \mathcal{B}_1,\ (\psi_j, \beta_j) \in \mathcal{B}_2\}$ and $\mathcal{B}_{cm} = \mathcal{B}_1 \cup \mathcal{B}_2$ respectively. $\mathcal{B}_{dm}$ is always consistent provided that $\mathcal{B}_1$ or $\mathcal{B}_2$ is consistent, whilst $\mathcal{B}_{cm}$ is consistent only if the union of $\mathcal{B}_1$ and $\mathcal{B}_2$ is consistent. So the maximum based merging operator is preferable to the minimum based merging operator for dealing with inconsistency. However, when the union of $\mathcal{B}_1$ and $\mathcal{B}_2$ is consistent, the minimum based merging operator results in a more *specific* possibilistic knowledge base. That is, the possibility distribution of the combination of $\mathcal{B}_1$ and $\mathcal{B}_2$ by the minimum based merging operator is more *specific* than that of the maximum based merging operator. Therefore, the maximum (or more generally, t-conorm) based merging operator is often considered to be too *cautious* for merging possibilistic knowledge bases that are consistent with each other.

In this paper we propose two Split-Combination (S-C for short) operators that follow the first view on possibilistic merging. We divide the fusion process into two steps: the splitting step and the combination step. The splitting step splits each knowledge base into two subbases and then in the second step, different classes of subbases are combined using different operators.

We first introduce an Incremental Split-Combination (*I-S-C* for short) merging operator. Given two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$ (where $\mathcal{B}_1 \cup \mathcal{B}_2$ is inconsistent but each of them is individually consistent), we first split each of them into two subbases such that $\mathcal{B}_1 = \mathcal{C}_1 \cup \mathcal{D}_1$ and $\mathcal{B}_2 = \mathcal{C}_2 \cup \mathcal{D}_2$ with regard to a value obtained by an incremental algorithm. $\mathcal{C}_i$ $(i = 1, 2)$ contain formulas in $\mathcal{B}_i$ whose weights are less than or equal to the splitting value and $\mathcal{D}_i$ are the complements of $\mathcal{C}_i$ *w.r.t* $\mathcal{B}_i$. In the second step, we combine $\mathcal{C}_1$ and $\mathcal{C}_2$ using a t-conorm based merging operator, while combining $\mathcal{D}_1$ and $\mathcal{D}_2$ using a t-norm based merging operator. Finally, the union of the possibilistic knowledge bases obtained by the second step is taken as the result of the combination of $\mathcal{B}_1$ and $\mathcal{B}_2$. We prove that the new merging operator reduces to the t-norm based merging operator when no conflict exists and that its resulting possibilistic knowledge base contains more important information than that of the t-conorm based merging operator. Furthermore, we adapt the set of postulates for merging propositional knowledge bases in [24] to possibilistic logic and discuss the logical properties of our merging operator.

The *I-S-C* merging operator encounters problems when applied to the merging of flat (or classical) knowledge bases, i.e., knowledge bases without any priorities between their elements, because the weight of a formula used to split the possibilistic knowledge bases is related to priority. Therefore we propose an alternative approach to splitting the knowledge bases which do not involve priority. The corresponding split-combination operator, called the Free-formula based Split-Combination (*F-S-C* for short) merging operator, can then be applied to merge classical knowledge bases. We compare our *F-S-C* merging operator with propositional merging operators in the flat case and conclude that it is a good alternative to syntax-based merging operators in classical logic.

We compare our merging operators with existing possibilistic merging operators by considering the following criteria,

(i) Rationality. We generalize the set of rationality postulates in the propositional setting in [24] and discuss the logical properties of different possibilistic merging operators. The generalized postulates are not used to give a normative definition or characterization of possibilistic merging. The reason why we propose them is that we think they can help users to choose among different possibilistic merging operators.

(ii) Inference power of the resulting possibilistic knowledge base. Given two merging operators, we prefer the one leading to a merged base which can non-trivially infer more information.

(iii) Compatibility with merging operators in the classical setting. That is, we prefer possibilistic merging operators which are well-behaved in classical logic to those which are not.

(iv) Computational complexity. This criterion has been adopted to evaluate alternative approaches to computing a solution in many AI problems, such as belief revision and non-monotonic reasoning. It is clear that operators that compute a solution more efficiently are preferred to more complex ones.

This paper is organized as follows. Section 2 provides some preliminary definitions in possibilistic logic. We then give a brief survey of existing merging methods in possibilistic logic in Section 3. In Section 4, we propose an Incremental Split Combination (*I-S-C* for short) merging operator. In Section 5, we discuss the semantic aspect of the *I-S-C* operator. A second split-combination operator, called the Free-formula based Split-Combination operator, is proposed in Section 6. Section 7 discusses related work. Finally, we conclude this paper in Section 8.

2 Background on Possibilistic Logic

In this section, we give a brief overview of possibilistic logic. More details on possibilistic logic can be found in [17, 10].

Throughout the paper, $\mathcal{L}$ is a propositional language formed in the usual way from a finite set of propositional symbols $\mathcal{P}$. An interpretation is a truth assignment to the atoms in $\mathcal{P}$, i.e. a mapping from $\mathcal{P}$ to $\{true, false\}$. We denote the set of classical interpretations by Ω, and the classical consequence relation by $\models$. Propositional symbols are denoted by p, q, r..., and propositional formulas are denoted by Greek letters ϕ, ψ, χ,... The satisfiability relation $\omega \models \phi$ is defined as usual between an interpretation ω and a formula ϕ. A *knowledge base* K is a finite set of propositional formulas and can be represented as the formula ϕ, which is the conjunction of the formulas in K. A *knowledge profile* is then defined as a multi-set E consisting of a finite number of knowledge bases, that is, $E = \{K_1, ..., K_n\}$, where K_i may be the same as K_j for $i \neq j$. Two knowledge profiles E_1 and E_2 are equivalent, denoted $E_1 \equiv E_2$, iff there exists a bijection f between E_1 and E_2 such that for each $K \in E_1$, $f(K) \equiv K$. The union of knowledge bases in E is defined as $\cup E = \cup_{i=1}^{n} K_i$, and the symbol $\sqcup$ denotes the union of multi-sets.

2.1 Semantics of possibilistic logic

The semantics of possibilistic logic is based on the notion of a *possibility distribution* π which is a mapping from Ω to the interval [0,1]. The unit interval is not necessary and can be replaced by any totally ordered scale. $\pi(\omega)$ represents the degree of compatibility of the interpretation ω with the available beliefs about the real world. $\pi(\omega) = 0$ means that it is impossible that ω to be the real world, and $\pi(\omega) = 1$ means that nothing prevents ω from being the real world, while $0 < \pi(\omega) < 1$ means that it is only somewhat possible for ω to be the real world. When $\pi(\omega) > \pi(\omega')$, ω is more plausible than ω' for being the real world. A possibility distribution is said to be normal if $\exists\omega\in\Omega$, such that $\pi(\omega) = 1$. Given two possibility distributions π and π', π is said to be less specific (or less informative) than π' if $\forall\omega$, $\pi(\omega)\geq\pi'(\omega)$, and π is said to be strictly less specific (or less informative) than π' if $\forall\omega$, $\pi(\omega) \geq \pi'(\omega)$ and $\exists\omega$, $\pi(\omega) > \pi'(\omega)$.

From a *possibility distribution* π, two measures defined on a set of propositional formulas can be determined. One is the possibility degree of formula ϕ, denoted as $\Pi_\pi(\phi) = max\{\pi(\omega) : \omega \models \phi\}$. The other is the necessity degree of formula ϕ, and is defined as $N_\pi(\phi) = 1 - \Pi_\pi(\neg\phi)$. The possibility degree of ϕ evaluates the extent to which ϕ is consistent with knowledge expressed by π and the necessity degree of ϕ evaluates the extent to which ϕ is entailed by the available knowledge. $N_\pi(\phi) = 1$ means that ϕ is a totally certain piece of knowledge, while $N_\pi(\phi) = 0$ expresses the complete lack of knowledge of priority about ϕ, but does not mean that ϕ is false. We have $N_\pi(true) = 1$ and $N_\pi(\phi\wedge\psi) = min(N_\pi(\phi), N_\pi(\psi))$ for all ϕ and ψ.

2.2 Possibilistic knowledge bases

At the syntactic level, a formula, called a *possibilistic formula*, is represented by a pair (ϕ, α) where ϕ is a propositional formula and $\alpha \in [0, 1]$, which means that the necessity degree of ϕ is at least equal to α, i.e. $N(\phi) \geq \alpha$. Uncertain pieces of information can then be represented by a *possibilistic knowledge base* which is a finite set of *possibilistic formulas* of the form $\mathcal{B} = \{(\phi_i, \alpha_i) : i = 1, ..., n\}$. A *possibilistic knowledge profile* $\mathcal{E}$ is a multi-set of possibilistic knowledge bases. In this paper we only consider possibilistic knowledge bases where every formula ϕ is a classical propositional formula. The weights attached to formulas are denoted by α, β, γ, ... The classical base associated with $\mathcal{B}$ is denoted as $\mathcal{B}^*$, namely $\mathcal{B}^* = \{\phi_i|(\phi_i, \alpha_i) \in \mathcal{B}\}$. A possibilistic knowledge base $\mathcal{B}$ is consistent if and only if its classical base $\mathcal{B}^*$ is consistent.

Definition 1 *Let $\mathcal{B}$ be a possibilistic knowledge base, and $\alpha \in [0, 1]$. The α-cut (respectively strict α-cut) of $\mathcal{B}$ is $\mathcal{B}_{\geq\alpha} = \{\phi\in\mathcal{B}^*|(\phi, \beta)\in\mathcal{B}$ and $\beta\geq\alpha\}$ (respectively $\mathcal{B}_{>\alpha} = \{\phi\in\mathcal{B}^*|(\phi, \beta)\in\mathcal{B}$ and $\beta>\alpha\}$).*

Given a possibilistic knowledge base $\mathcal{B}$, we can associate with it a semantics *w.r.t* possibility distributions.

Definition 2 *Let $\mathcal{B}$ be a possibilistic knowledge base and $(\phi, \alpha)\in\mathcal{B}$. A possibility distribution π is said to be compatible with (ϕ, α) if $N_\pi(\phi)\geq\alpha$ and it is compatible with $\mathcal{B}$ if for each $(\phi_i, \alpha_i)\in\mathcal{B}$, we have $N_\pi(\phi_i)\geq\alpha_i$.*

Generally, there are several possibility distributions compatible with a possibilistic knowledge base $\mathcal{B}$. However, a unique possibility distribution, denoted by $\pi_\mathcal{B}$ can be obtained by the principle of minimum specificity. That is, among the possibility distributions compatible with $\mathcal{B}$, we choose the one which is the least specific, i.e. there is no possibility distribution π' such that π' is less specific than π. This possibility distribution can be computed as follows [17]. For all $\omega \in \Omega$,

$$\pi_\mathcal{B}(\omega) = \begin{cases} 1 & \text{if } \forall(\phi_i,\ \alpha_i) \in \mathcal{B}, \omega \models \phi_i, \\ 1 - max\{\alpha_i|\omega \not\models \phi_i\} & \text{otherwise.} \end{cases} \quad (1)$$

It is clear that a possibilistic knowledge base $\mathcal{B}$ is consistent iff its associated possibility $\pi_\mathcal{B}$ is normal.

Let us look at an example.

Example 1 *Let* $\mathcal{B} = \{(p, 0.9), (q, 0.6), (\neg q \vee r, 0.5), (\neg r, 0.5), (r, 0.3)\}$ *be a possibilistic knowledge base. By Equation 1, the least specific possibility distribution associated with* $\mathcal{B}$ *is defined by* $\pi_{\mathcal{B}}(pqr) = 0.5$*,* $\pi_{\mathcal{B}}(pq\neg r) = 0.5$*,* $\pi_{\mathcal{B}}(p\neg qr) = 0.4$*,* $\pi_{\mathcal{B}}(p\neg q\neg r) = 0.4$*,* $\pi_{\mathcal{B}}(\neg pqr) = 0.1$*,* $\pi_{\mathcal{B}}(\neg pq\neg r) = 0.1$*,* $\pi_{\mathcal{B}}(\neg p\neg qr) = 0.1$*,* $\pi_{\mathcal{B}}(\neg p\neg q\neg r) = 0.1$*.*

Two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$ are said to be equivalent, denoted by $\mathcal{B}_1 \equiv_s \mathcal{B}_2$ iff $\pi_{\mathcal{B}_1} = \pi_{\mathcal{B}_2}$, that is, their associated possibility distributions are the same. The equivalence of two possibilistic knowledge bases can also be defined as $\mathcal{B}_1 \equiv_s \mathcal{B}_2$ iff $\forall\alpha\in[0, 1]$, $(\mathcal{B}_1)_{\geq\alpha}\equiv(\mathcal{B}_2)_{\geq\alpha}$ [11]. Moreover, two possibilistic knowledge profiles $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent, denoted $\mathcal{E}_1\equiv_s\mathcal{E}_2$ iff there exists a bijection f between $\mathcal{E}_1$ and $\mathcal{E}_2$ such that for each $\mathcal{B}\in\mathcal{E}_1$, $f(\mathcal{B})\equiv_s\mathcal{B}$.

2.3 Possibilistic inference

Given a possibilistic knowledge base $\mathcal{B}$, we can define its level of inconsistency as follows.

Definition 3 *Let* $\mathcal{B}$ *be a possibilistic knowledge base. The* inconsistency degree *of* $\mathcal{B}$ *is:*

$$Inc(\mathcal{B}) = max\{\alpha_i : \mathcal{B}_{\geq\alpha_i} \text{ is inconsistent}\}.$$

That is, the inconsistency degree of $\mathcal{B}$ is the largest weight α_i such that the α_i-cut of $\mathcal{B}$ is inconsistent. It can be equivalently defined by the possibility distribution associated with $\mathcal{B}$ as $Inc(\mathcal{B}) = 1 - max_\omega\pi_{\mathcal{B}}(\omega)$. When $\mathcal{B}$ is consistent, we have $Inc(\mathcal{B}) = 0$. So if the possibility distribution $\pi_{\mathcal{B}}$ is subnormal, i.e. $\forall\omega, \pi_{\mathcal{B}}(\omega) < 1$, $\mathcal{B}$ is not consistent.

The possibilistic consequence relation is defined as follows.

Definition 4 *[11] Let* $\mathcal{B}$ *be a possibilistic knowledge base. A formula* ϕ *is said to be a consequence of* $\mathcal{B}$*, denoted by* $\mathcal{B} \vdash_\pi (\phi, \alpha)$*, iff*

(i) $\mathcal{B}_{\geq\alpha}$ *is consistent,*

(ii) $\mathcal{B}_{\geq\alpha} \models \phi$*,*

(iii) $\forall\beta>\alpha$, $\mathcal{B}_{\geq\beta} \not\models \phi$*.*

It is required that weights of possibilistic formulas which are consequences of $\mathcal{B}$ be greater than the inconsistency degree of $\mathcal{B}$. This is because for any possibilistic formula (ϕ, α), if $\alpha\leq Inc(\mathcal{B})$, then $\mathcal{B}_{\geq\alpha} \models \phi$. That is, (ϕ, α) can be inferred from $\mathcal{B}$ trivially.

Subsumption can be defined as follows:

Definition 5 *Let* (ϕ, α) *be a possibilistic formula in* $\mathcal{B}$*.* (ϕ, α) *is said to be subsumed by* $\mathcal{B}$ *if* $(\mathcal{B} \setminus \{(\phi, \alpha)\})_{\geq\alpha} \models \phi$*.*

Subsumed formulas can be viewed as redundant by the following proposition.

Proposition 1 *[10] Let* (ϕ, α) *be a subsumed formula in* $\mathcal{B}$*. Then* $\mathcal{B}$ *and* $\mathcal{B}\setminus\{(\phi, \alpha)\}$ *are equivalent.*

By Proposition 1, possibilistic formulas with a null degree, i.e. having the form $(\phi_i, 0)$, are subsumed in any possibilistic knowledge base. So they are not included in the knowledge base explicitly.

It has been shown in [29] that the computational complexity of possibilistic inference is similar to that of classical logic, that is, it needs $\lceil log_2 n\rceil$ satisfiability checks, where n is the number of certainty levels used in $\mathcal{B}$.

Although possibilistic inference is inconsistency tolerant, it suffers from the "drowning problem" [3, 7]. That is, given an inconsistent possibilistic knowledge base $\mathcal{B}$, formulas whose certainty degrees are not larger than $Inc(\mathcal{B})$ are completely useless for nontrivial deductions. For instance, let $\mathcal{B} = \{(p, 0.9), (\neg p, 0.8), (r, 0.6), (q, 0.7)\}$, it is clear that $\mathcal{B}$ is equivalent to $\mathcal{B} = \{(p, 0.9), (\neg p, 0.8)\}$ because $Inc(\mathcal{B}) = 0.8$. So $(q, 0.7)$ and $(r, 0.6)$ are not used in the possibilistic inference. Since possibilistic inference has the drowning problem, inconsistency is not desirable and should be avoided if possible.

3 Merging Approaches in Possibilistic Logic

Many approaches have been proposed for merging prioritized knowledge bases in possibilistic logic [4, 8, 9, 10, 11]. There are two different views for merging possibilistic knowledge bases. The first view considers inconsistency as unacceptable and claims that conflicting information between different sources should be resolved after merging [4, 8, 9]. By contrast, the second view claims that inconsistency is unavoidable and so the resulting possibilistic knowledge base can be inconsistent after merging [10, 11].

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two possibilistic knowledge bases and let π_1 and π_2 be their associated possibility distributions. Semantically, a two place function $\oplus$ from [0,1]×[0,1] to [0,1], is applied to aggregate the two possibility distributions π_1 and π_2 into a new one $\pi_\oplus$, i.e. $\pi_\oplus(\omega) = \pi_1(\omega) \oplus \pi_2(\omega)$. Generally, the operator $\oplus$ is very weakly constrained, i.e. the only requirements for it are the following properties [9, 10]:

1. $1 \oplus 1 = 1$, and
2. if $\alpha_1 \geq \beta_1$, $\alpha_2 \geq \beta_2$ then $\alpha_1 \oplus \alpha_2 \geq \beta_1 \oplus \beta_2$, where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$ (monotonicity).

The first property states that if two sources agree that an interpretation ω is fully possible, then the result of merging should confirm it. The second property is the monotonicity condition, that is, a degree resulting from a combination cannot decrease if the degrees to be combined increase.

In the case of n sources $\mathcal{B}_1$,...,$\mathcal{B}_n$, the semantic combination of their possibility distributions π_1,...,π_n can be performed easily when $\oplus$ is associative. That is, we have $\pi_\oplus(\omega) = (...((\pi_1(\omega) \oplus \pi_2(\omega)) \oplus \pi_3(\omega)) \oplus ...) \oplus \pi_n(\omega)$. When the operator is not associative, it needs to be generalized as a unary operator defined on a vector $(\pi_1, ..., \pi_n)$ of possibility distributions such that:

1. $\oplus(1, ..., 1) = 1$, and
2. if $\forall i = 1, ..., n,\ \pi_i(\omega) \geq \pi_i(\omega')$ then $\oplus(\pi_1(\omega), ..., \pi_n(\omega)) \geq \oplus(\pi_1(\omega'), ..., \pi_n(\omega'))$.

Two basic operators are the maximum and the minimum. Given two possibility distributions π_1 and π_2, let $\pi_{dm}(\omega) = max(\pi_1(\omega), \pi_2(\omega))$ and $\pi_{cm}(\omega) = min(\pi_1(\omega), \pi_2(\omega))$. The merging operators based on the maximum and the minimum have no reinforcement effect. That is, given an interpretation ω, if expert 1 assigns possibility $\pi_1(\omega) < 1$ and expert 2 assigns possibility $\pi_2(\omega) < 1$ to ω, then overall $\pi_{dm}(\omega) = \pi_2(\omega)$ (or $\pi_{cm}(\omega) = \pi_1(\omega)$) if $\pi_1(\omega) < \pi_2(\omega)$, regardless of the value of $\pi_1(\omega)$ (or $\pi_2(\omega)$). To obtain a reinforcement effect, we can use a triangular norm operator other than the minimum for conjunctive combination, and a triangular conorm operator other than the maximum for disjunctive combination.

Definition 6 *[23] A triangular norm (t-norm) tn is a two place real-valued function* $tn : [0, 1] \times [0, 1] \rightarrow [0, 1]$ *which satisfies the following conditions:*

1. *tn(0,0)=0, and tn(α,1)=tn(1,α)=α, for every α (boundary condition);*
2. *tn(α_1,α_2)$\leq$tn(β_1,β_2) whenever $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$ (monotonicity);*

3. *tn(α,β)=tn(β,α) (symmetry);*

4. *tn(α,tn(β,γ))=tn(tn(α,β),γ) (associativity).*

A triangular conorm (t-conorm) is a two place real-valued function $ct : [0,1] \times [0,1] \rightarrow [0,1]$ *which satisfies the conditions 2-4 given in Definition 6 plus the following revised boundary conditions:*

1'. *ct(1,1)=1,ct(α,0)=ct(0,α)=α.*

Any t-conorm ct can be generated from a t-norm through the duality transformation:

$$ct(\alpha, \beta) = 1 - tn(1 - \alpha, 1 - \beta)$$

and conversely.

It is easy to check that the maximum operator is a t-conorm and the minimum operator is a t-norm. Other frequently used t-norms are the product operator $\alpha\beta$ and the *Lukasiewicz t-norm* ($max(0, \alpha + \beta - 1)$). The duality relation yields the following t-conorms respectively: the *probabilistic sum* ($\alpha + \beta - \alpha\beta$), and the *bounded sum* ($min(1, \alpha + \beta)$).

Given two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$ with possibility distributions π_1 and π_2 respectively, the semantic results of their combination by a t-norm tn and a t-conorm ct are

$$\forall\omega,\ \pi_{tn}(w) = tn(\pi_1(w), \pi_2(w)), \quad (2)$$

$$\forall\omega,\ \pi_{ct}(w) = ct(\pi_1(w), \pi_2(w)). \quad (3)$$

When at least one original possibility distribution is normal, the merging methods based on t-conorms preserve normalization. However, the merging methods based on t-norms may result in subnormal results, i.e. $\forall\omega, \pi_{tn}(\omega) < 1$ (or equivalently, the possibilistic knowledge base associated with π_{tn} is inconsistent). In that case, we may think of renormalizing π_{tn} (that is, if we follow the first views on possibilistic merging). Let π be a possibility distribution which is subnormal, π_N be the possibility distribution renormalized from π. Then π_N should satisfy the following conditions:

1. $\exists\omega,\ \pi_N(\omega) = 1$,

2. if π is normal then $\pi_N = \pi$,

3. $\forall\omega, \omega',\ \pi(\omega) < \pi(\omega')$ if and only if $\pi_N(\omega) < \pi_N(\omega')$.

For example, let $h(\pi_{tn}) = max_{\omega\in\Omega}\{\pi_{tn}(\omega)\}$, the following equation provides a normalization rule.

$$\pi_{N,tn}(\omega) = \begin{cases} 1 & \text{if } \pi_{tn}(\omega) = h(\pi_{tn}), \\ \pi_{tn}(\omega) & \text{otherwise.} \end{cases} \quad (4)$$

The normalization rule defined by Equation 4 resolves inconsistency because the inconsistency degree of any possibilistic knowledge base associated with $\pi_{N,tn}$ is zero. Other normalization rules can be found in [4].

The syntactical counterpart of the fusion of π_1 and π_2 is to obtain a possibilistic knowledge base whose possibility distribution is $\pi_\oplus$. In [9], it has been shown that this knowledge base has the following form:

$$\mathcal{B}_\oplus = \{(\phi_i, 1 - (1 - \alpha_i)\oplus 1) : (\phi_i, \alpha_i)\in\mathcal{B}_1\}\cup\{(\psi_j, 1 - 1\oplus(1 - \beta_j)) : (\psi_j, \beta_j)\in\mathcal{B}_2\}$$
$$\cup\{(\phi_i \vee \psi_j, 1 - (1 - \alpha_i)\oplus(1 - \beta_j)) : (\phi_i, \alpha_i)\in\mathcal{B}_1 \textit{ and } (\psi_j, \beta_j)\in\mathcal{B}_2\}. \quad (5)$$

That is, we have $\pi_{\mathcal{B}_\oplus}(\omega) = \pi_\oplus(\omega) = \pi_1(\omega)\oplus\pi_2(\omega)$, where $\pi_{\mathcal{B}_\oplus}$ is the possibility distribution associated with $\mathcal{B}_\oplus$. It is clear that when $\oplus$ is associative, the syntactic computation of the resulting base is easily generalized to n sources. The syntactic generalization for a non-associative operator can be carried out as follows.

Proposition 2 *[10] Let $\mathcal{E} = \{\mathcal{B}_1, ..., \mathcal{B}_n\}$ be a set of n possibilistic knowledge bases and $(\pi_1, ..., \pi_n)$ be their associated possibility distributions. Let $\pi_{\mathcal{B}_\oplus}$ be the result of combining $(\pi_1, ..., \pi_n)$ with $\oplus$. The possibilistic knowledge base associated with $\pi_{\mathcal{B}_\oplus}$ is:*

$$\mathcal{B}_\oplus = \{(D_j, 1 - \oplus(x_1, ..., x_n)) : j = 1, ..., n\}, \tag{6}$$

where D_j are disjunctions of size j between formulas taken from different $\mathcal{B}_i$'s ($i = 1, ..., n$) and x_i is equal to $1 - \alpha_i$ if ϕ_i belongs to D_j and 1 if it does not.

By Equation 6, the possibilistic knowledge bases, which are the syntactical counterparts of semantic merging using the maximum and the minimum operators, are

$$\mathcal{B}_{dm} = \{(\phi_i \vee \psi_j, min(\alpha_i, \beta_j))|(\phi_i, \alpha_i) \in \mathcal{B}_1,\ and\ (\psi_j, \beta_j) \in \mathcal{B}_2\}, \tag{7}$$

$$\mathcal{B}_{cm} = \mathcal{B}_1 \cup \mathcal{B}_2, \tag{8}$$

respectively. B_{dm} and B_{cm} are referred to as the results of *disjunctive* and *conjunctive* combination respectively. More generally, the syntactic results associated with π_{tn} and π_{ct} are the following knowledge bases respectively [4]:

$$\mathcal{B}_{tn} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \{(\phi_i \vee \psi_j, ct(\alpha_i, \beta_j))|(\phi_i, \alpha_i) \in \mathcal{B}_1\ and\ (\psi_j, \beta_j) \in \mathcal{B}_2\}, \tag{9}$$

$$\mathcal{B}_{ct} = \{(\phi_i \vee \psi_j, tn(\alpha_i, \beta_j))|(\phi_i, \alpha_i) \in \mathcal{B}_1\ and\ (\psi_j, \beta_j) \in \mathcal{B}_2\}. \tag{10}$$

By Equation 9, the possibilistic knowledge base $\mathcal{B}_{tn}$ may be inconsistent. Let $\pi_{N,tn}$ be the possibility distribution obtained by Equation 4, then the possibilistic knowledge base associated with it has the following form:

$$\mathcal{B}_{N,tn} = \{(\phi_i, \alpha_i) : (\phi_i, \alpha_i) \in \mathcal{B}_{tn}\ and\ \alpha_i > Inc(\mathcal{B}_{tn})\}. \tag{11}$$

$\mathcal{B}_{N,tn}$ restores consistency of $\mathcal{B}_{tn}$ by dropping formulas whose weights are less than or equal to the inconsistency degree of $\mathcal{B}_{tn}$. We call the merging operator obtained by Equation 11 a renormalization based merging operator. It is clear that $\mathcal{B}_{N,tn}$ may drop too much information from $\mathcal{B}_{tn}$ if $Inc(\mathcal{B}_{tn})$ is large, for example, 0.8.

Example 2 *Let $\mathcal{B}_1 = \{(p, 0.9), (q, 0.7)\}$ and $\mathcal{B}_2 = \{(\neg p, 0.8), (r, 0.6), (p\vee q, 0.5)\}$. Suppose the operator is the maximum, then by Equation 7, we have $\mathcal{B}_{dm} = \{(p\vee r, 0.6), (p\vee q, 0.5), (\neg p\vee q, 0.7), (q\vee r, 0.6)\}$. It is clear that the maximum based merging operator is very* cautious*, that is, all the formulas are weakened as disjunctions. In contrast, if we choose the minimum, then by Equation 8, we have $\mathcal{B}_{cm} = \{(p, 0.9), (\neg p, 0.8), (q, 0.7), (r, 0.6), (p\vee q, 0.5)\}$. $\mathcal{B}_{cm}$ is inconsistent. Suppose we apply the normalization rule (Equation 4) to the possibility distribution associated with $\mathcal{B}_{cm}$, then by Equation 11 the possibilistic knowledge base associated with the normalized possibility distribution is $\mathcal{B}_{N,cm} = \{(p, 0.9)\}$. $(q, 0.7)$ and $(r, 0.6)$ are not involved in conflict between $\mathcal{B}_1$ and $\mathcal{B}_2$, but they are deleted after merging.*

Example 2 illustrates that the merging methods based on t-conorms are too cautious when most of the formulas are not involved in conflict while the renormalization based merging method may delete too much original information from the resulting knowledge base. According to [12], when knowledge bases are consistent with each other, it is preferable to use a t-norm based merging method. The maximum based merging method is preferable to the minimum based merging method (or any other t-norm based merging method) only if the inconsistency degree of $\mathcal{B}_1 \cup ... \cup \mathcal{B}_n$ is 1; that is, if there is a strong conflict among the sources of information[1].

4 Incremental Split-Combination Merging Operator

In this section, we introduce an Incremental Split-Combination (*I-S-C*) operator for merging possibilistic knowledge bases. We follow the first view on possibilistic merging, that is, the resulting knowledge base should be consistent. We further assume that the original possibilistic knowledge bases are individually consistent. According to the analysis in Section 3, the t-conorm (for example, the maximum) based merging methods can be used to weaken conflicting information, while the t-norm (for example, the minimum) based merging methods exploit the symbolic complementarities between sources, i.e. all the symbolic information is recovered. In this section and Section 6, we propose two split-combination operators for merging individually consistent possibilistic knowledge bases by utilizing both t-norm and t-conorm based merging operators.

The general idea of the *S-C* approach can be described as follows. Given two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$, in the first step, we split them into $\mathcal{B}_1 =< \mathcal{C}_1, \mathcal{D}_1 >$ and $\mathcal{B}_2 =< \mathcal{C}_2, \mathcal{D}_2 >$ with regard to a splitting method such that $\mathcal{C}_i$ $(i = 1, 2)$ contain information which would be weakened and $\mathcal{D}_i$ $(i = 1, 2)$ contain formulas which are "safe" to keep. In the second step, we combine $\mathcal{C}_1$ and $\mathcal{C}_2$ by a t-conorm operator (the result is a possibilistic knowledge base $\mathcal{C}$) and combine $\mathcal{D}_1$ and $\mathcal{D}_2$ by a t-norm operator (the result is a possibilistic knowledge base $\mathcal{D}$). The final result of the *S-C* combination approach, denoted $\mathcal{B}_{S-C}$, is $\mathcal{C} \cup \mathcal{D}$. Different S-C methods can be developed by incorporating different ways of splitting the knowledge bases, while retaining the general S-C approach.

4.1 Incremental *S-C* Operator

Our first splitting method is to split possibilistic knowledge bases using necessity degrees in the possibilistic knowledge bases. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two possibilistic knowledge bases, and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{(\varphi_i, \alpha_i) : i = 1, ..., n\}$. Let $\alpha_{max} = max(\alpha_i : i = 1, ..., n)$. Suppose that the weights of the formulas in $\mathcal{B}$ are rearranged in such a way that $\alpha_1 = \alpha_{max} > \alpha_2 > ... > \alpha_{n_1}$. Let $\alpha_{n_1+1} = 0$. Suppose $\mathcal{B}_i$ $(i = 1, 2)$ are split *w.r.t* some α_m $(1 \leq m \leq n_1 + 1)$ into $\mathcal{B}_i = \langle \mathcal{C}_i, \mathcal{D}_i \rangle$, where $\mathcal{C}_i = \{(\phi_i, \alpha_i) : (\phi_i, \alpha_i) \in \mathcal{B}_i, \alpha_i \leq \alpha_m\}$ and $\mathcal{D}_i = \mathcal{B}_i \setminus \mathcal{C}_i$. Suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ are combined [2] by a t-norm operator tn, and $\mathcal{C}_1$ and $\mathcal{C}_2$ are combined by a t-conorm operator ct. By Equation 9 and 10, we have $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \{(\phi_i \vee \psi_j, ct(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{D}_1 \; and \; (\psi_j, \beta_j) \in \mathcal{D}_2\}$ and $\mathcal{C} = \{(\phi_i \vee \psi_j, tn(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{C}_1 \; and \; (\psi_j, \beta_j) \in \mathcal{C}_2\}$. The final result of the S-C method is $\mathcal{B}_{S-C} = \mathcal{C} \cup \mathcal{D}$. The main problem is how to select the splitting point. Let $Inc(\mathcal{B}) = \alpha_k$. When $\alpha_m < \alpha_k$, by Definition 3, $\mathcal{D}$ is inconsistent. So $\mathcal{B}_{S-C} = \mathcal{C} \cup \mathcal{D}$ is inconsistent. Therefore, we cannot use α_m such that $\alpha_m < \alpha_k$ as the splitting point. Suppose α_k is the splitting point, then both $\mathcal{C}$ and $\mathcal{D}$ are consistent. However, $\mathcal{B}_{S-C}$ may still be inconsistent. Let us look at an example.

[1] We are indebted to one of the reviewers for pointing this out.

[2] When we say that two possibilistic knowledge bases are combined by a t-norm (or t-conorm) operator, we mean they are syntactically combined using Equation 9 (or Equation 10).

Example 3 *Let* $\mathcal{B}_1 = \{(p, 0.9), (\neg r, 0.7)\}$ *and* $\mathcal{B}_2 = \{(r, 0.8), (\neg p, 0.6)\}$*. It is easy to check that* $Inc(\mathcal{B}_1 \cup \mathcal{B}_2) = 0.7$*. Suppose we select 0.7 as the splitting point, then* $\mathcal{B}_1$ *is split into* $\mathcal{D}_1 = \{(p, 0.9)\}$ *and* $\mathcal{C}_1 = \{(\neg r, 0.7)\}$*, and* $\mathcal{B}_2$ *is split into* $\mathcal{D}_2 = \{(r, 0.8)\}$ *and* $\mathcal{C}_2 = \{(\neg p, 0.6)\}$*. Suppose* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are combined by the minimum and* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are combined by the maximum. Then* $\mathcal{D} = \{(p, 0.9), (r, 0.8)\}$ *and* $\mathcal{C} = \{(\neg p \vee \neg r, 0.6)\}$*. So* $\mathcal{B}_{S-C} = \mathcal{C} \cup \mathcal{D} = \{(p, 0.9), (r, 0.8), (\neg p \vee \neg r, 0.6)\}$*. It is clear that* $\mathcal{B}_{S-C}$ *is inconsistent.*

There is no guarantee that the final merged knowledge base is consistent, when selecting an α_i such that $\alpha_i = \alpha_k$ as the splitting point, where $\alpha_k = Inc(\mathcal{B}_1 \cup \mathcal{B}_2)$. We need to find the point incrementally. To do so, we give the following algorithm to find the value step by step and use it to split both $\mathcal{B}_1$ and $\mathcal{B}_2$.

Algorithm 1

Input: two possibilistic knowledge bases $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, ..., n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, ..., m\}$, a t-conorm ct and a t-norm tn.

Output: a splitting point γ.

Step 1 Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{(\varphi_i, \gamma_i) : i = 1, ..., n+m\}$. Rearrange the weights of formulas in $\mathcal{B}$ such that $\gamma_1 > \gamma_2 > ... > \gamma_{n_1}$. Let $\gamma_{n_1+1} = 0$.

Step 2 Compute $Inc(\mathcal{B})$. Assume $Inc(\mathcal{B}) = \gamma_k$. Let $l = k$.

Step 3 Split $\mathcal{B}_1$ and $\mathcal{B}_2$ with regard to γ_l such that $\mathcal{B}_1 = < \mathcal{C}_1, \mathcal{D}_1 >$ and $\mathcal{B}_2 = < \mathcal{C}_2, \mathcal{D}_2 >$, where $\mathcal{C}_1 = \{(\phi_i, \alpha_i) : (\phi_i, \alpha_i) \in \mathcal{B}_1, \alpha_i \leq \gamma_l\}$ and $\mathcal{D}_1 = \mathcal{B}_1 \setminus \mathcal{C}_1$, and $\mathcal{C}_2 = \{(\psi_j, \beta_j) : (\psi_j, \beta_j) \in \mathcal{B}_2, \beta_j \leq \gamma_l\}$ and $\mathcal{D}_2 = \mathcal{B}_2 \setminus \mathcal{C}_2$.

Step 4 Combine $\mathcal{C}_1$ and $\mathcal{C}_2$ by ct and combine $\mathcal{D}_1$ and $\mathcal{D}_2$ by tn, as shown by Equation 10 and Equation 9. The results are respectively

$$\mathcal{C} = \{(\phi_i \vee \psi_j, tn(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{C}_1 \text{ and } (\psi_j, \beta_j) \in \mathcal{C}_2\}, \quad (12)$$

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \{(\phi_i \vee \psi_j, ct(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{D}_1 \text{ and } (\psi_j, \beta_j) \in \mathcal{D}_2\}. \quad (13)$$

Step 5 Let $\mathcal{B}_{S-C} = \mathcal{C} \cup \mathcal{D}$.

Step 6 If $\mathcal{B}_{S-C}$ is inconsistent, let $l = l - 1$ and go to Step 3.

Step 7 Return γ_l.

In Algorithm 1, we first rearrange all the weights of formulas in the union of $\mathcal{B}_1$ and $\mathcal{B}_2$ in decreasing order. We then search the weights incrementally until we find the minimal weight such that the resulting possibilistic knowledge base of the split-combination approach is consistent.

We now define the *I-S-C* merging operator based on Algorithm 1.

Definition 7 *Let* $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, ..., n\}$ *and* $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, ..., m\}$ *be two possibilistic knowledge bases, ct be a t-conorm and tn be a t-norm. Let* γ *be the splitting point obtained by Algorithm 1. Suppose* $\mathcal{B}_i$ *(*$i = 1, 2$*) are split into* $\mathcal{B}_i = \langle \mathcal{C}_i, \mathcal{D}_i \rangle$ *w.r.t* γ*, and* $\mathcal{C}$ *and* $\mathcal{D}$ *are obtained by Equations 12 and 13 respectively. The resulting possibilistic knowledge base of the I-S-C merging operator* [3]*, denoted* $\mathcal{B}_{I-S-C}$*, is defined as* $\mathcal{B}_{I-S-C} = \mathcal{C} \cup \mathcal{D}$*.*

Let us look at an example.

[3] The merging operator is based on a t-conorm *ct* and a t-norm *tn*. For notational simplicity, we omit *ct* and *tn* and simply call it the *I-S-C* operator from now on.

Example 4 *Given two possibilistic knowledge bases* $\mathcal{B}_1 = \{(\neg\phi\vee\varphi, 0.8), (\neg\phi\vee\psi, 0.6), (\phi, 0.5)\}$ *and* $\mathcal{B}_2 = \{(\phi \vee \delta,\ 0.9), (\neg\psi \vee \delta, 0.7), (\delta \vee \varphi, 0.5), (\neg\phi \vee \neg\psi, 0.4), (\psi, 0.3)\}$*. Let* $ct = max$ *and* $tn = min$*. Let* $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{(\phi \vee \delta, 0.9), (\neg\phi \vee \varphi, 0.8), (\neg\psi \vee \delta, 0.7), (\neg\phi \vee \psi, 0.6), (\phi, 0.5), (\delta \vee \varphi, 0.5), (\neg\phi \vee \neg\psi, 0.4), (\psi, 0.3)\}$*. The weights of formulas in* $\mathcal{B}$ *are rearranged as* $\gamma_1 = 0.9 > \gamma_2 = 0.8 > \gamma_3 = 0.7 > \gamma_4 = 0.6 > \gamma_5 = 0.5 > \gamma_6 = 0.4 > \gamma_7 = 0.3$*. Let* $\gamma_8 = 0$*. The inconsistency degree of* $\mathcal{B}_1 \cup \mathcal{B}_2$ *is* $Inc(\mathcal{B}_1 \cup \mathcal{B}_2) = 0.4$*. Let* $l = 6$*.* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are split with regard to* $\gamma_6 = 0.4$ *into*

$$\mathcal{C}_1 = \emptyset,\ \mathcal{D}_1 = \{(\neg\phi \vee \varphi, 0.8), (\neg\phi \vee \psi, 0.6), (\phi, 0.5)\}$$

$$\mathcal{C}_2 = \{(\neg\phi \vee \neg\psi, 0.4), (\psi, 0.3)\},\ \mathcal{D}_2 = \{(\phi \vee \delta, 0.9), (\neg\psi \vee \delta, 0.7), (\delta \vee \varphi, 0.5)\}.$$

Combining $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *using the maximum and combining* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *using the minimum respectively give*

$$\mathcal{C} = \emptyset,\ \mathcal{D} = \{(\phi \vee \delta, 0.9), (\neg\phi \vee \varphi, 0.8), (\neg\psi \vee \delta, 0.7), (\neg\phi \vee \psi, 0.6), (\delta \vee \varphi, 0.5), (\phi, 0.5)\}.$$

So $\mathcal{B}_{S-C} = \{(\phi\vee\delta, 0.9), (\neg\phi\vee\varphi, 0.8), (\neg\psi\vee\delta, 0.7), (\neg\phi\vee\psi, 0.6), (\delta\vee\varphi, 0.5), (\phi, 0.5)\}$*. Since* $\mathcal{B}_{S-C}$ *is consistent,* $\gamma = 0.4$ *is the splitting point and* $\mathcal{B}_{I-S-C} = \{(\phi\vee\delta, 0.9), (\neg\phi\vee\varphi, 0.8), (\neg\psi\vee\delta, 0.7), (\neg\phi\vee\psi, 0.6), (\delta \vee \varphi, 0.5), (\phi, 0.5)\}$*.*

4.2 Properties

4.2.1 Upper bound of the splitting point

Algorithm 1 terminates in a finite number of steps because there are a finite number of certainty levels in $\mathcal{B}_1 \cup \mathcal{B}_2$. In this section, we introduce an *upper free degree* and show that it is an upper bound of the splitting point obtained by Algorithm 1. We first introduce some definitions found in [5, 7].

Definition 8 *Let* K *be a classical knowledge base. A subbase* K' *of* K *is said to be minimally inconsistent if and only if it satisfies the following two requirements:* (1) $K' \models false$ *and* (2) $\forall\phi \in K'$*,* $K' - \{\phi\} \not\models false$*.*

Definition 9 *Let* K *be a classical knowledge base. A formula* ϕ *is said to be* free *in* K *if it does not belong to any minimally inconsistent subbase of* K *and is said to be* in conflict *otherwise.* $Free(K)$ *denotes the set of free formulas in* K*.*

Definition 10 *Let* $\mathcal{B}$ *be a possibilistic knowledge base. A possibilistic formula* (ϕ, α) *is said to be free in* $\mathcal{B}$ *if* ϕ *is free in* $\mathcal{B}^*$ *and it is said to be in conflict otherwise.*

We now define the upper free degree of a possibilistic knowledge base.

Definition 11 *The upper free degree of a possibilistic knowledge base* $\mathcal{B} = \{(\phi_i, \alpha_i) : i = 1, ..., n\}$ *is defined as:*

$$Free_{upp}(\mathcal{B}) = min\{\alpha \in \{\alpha_1, ...\alpha_n\} : \mathcal{B}_{>\alpha} \text{ does not contain any conflicting formulas in } \mathcal{B}^*\}. \quad (14)$$

$Free_{upp}(\mathcal{B}) = 0$ when $\mathcal{B}$ is consistent. $\mathcal{B}_{>Free_{upp}(\mathcal{B})}$ contains some free formulas of $\mathcal{B}$, but not all of them.

Definition 12 *(upper-free-degree-based splitting) Given a possibilistic knowledge base* $\mathcal{B}$*, the splitting of* $\mathcal{B}$ *with regard to* $Free_{upp}(\mathcal{B})$ *is defined as a pair* $< \mathcal{C}, \mathcal{D} >$ *such that* $\mathcal{B} = \mathcal{C} \cup \mathcal{D}$*, where*

$$\mathcal{C} = \{(\phi, \alpha) \in \mathcal{B} \mid \alpha \leq Free_{upp}(\mathcal{B})\} \text{ and } \mathcal{D} = \{(\phi, \alpha) \in \mathcal{B} \mid \alpha > Free_{upp}(\mathcal{B})\}.$$

It is clear that $Free_{upp}(\mathcal{B}) \geq Inc(\mathcal{B})$, for each possibilistic knowledge base $\mathcal{B}$. By Definition 12, $\mathcal{C}$ is inconsistent if $Free_{upp}(\mathcal{B}) > 0$ and $\mathcal{D}$ is always consistent.

Let us look at an example to illustrate how to split a possibilistic knowledge base *w.r.t* the upper free degree.

Example 5 *Given a possibilistic knowledge base* $\mathcal{B} = \{(\neg\psi \vee \delta,\ 0.9), (\phi \vee \delta,\ 0.7), (\neg\phi \vee \neg\varphi, 0.6), (\neg\psi \vee \varphi,\ 0.5), (\phi,\ 0.4), (\neg\phi \vee \psi,\ 0,3)\}$*, by Definition 12, the upper free degree of* $\mathcal{B}$ *is 0.6.* $\mathcal{B}$ *is then split into* $< \mathcal{C}, \mathcal{D} >$ *such that* $\mathcal{C} = \{(\neg\phi \vee \neg\varphi, 0.6), (\neg\psi \vee \varphi,\ 0.5), (\phi,\ 0.4), (\neg\phi \vee \psi,\ 0,3)\}$*, and* $\mathcal{D} = \{(\neg\psi \vee \delta,\ 0.9), (\phi \vee \delta,\ 0.7)\}$.

Suppose the splitting method is the upper-free-degree-based splitting method, the corresponding *S-C* method can be defined as follows.

Definition 13 *Let* $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, ..., n\}$ *and* $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, ..., m\}$ *be two possibilistic knowledge bases. Suppose* $\mathcal{B}_i$ *are split into* $\mathcal{B}_i = \langle \mathcal{C}_i, \mathcal{D}_i \rangle$ *w.r.t* $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$*, and* $\mathcal{C}$ *and* $\mathcal{D}$ *are obtained by Equation 12 and 13 respectively. The resulting possibilistic knowledge base of the upper-free-degree based S-C (U-S-C for short) merging operator, denoted* $\mathcal{B}_{Upper-S-C}$*, is defined as* $\mathcal{B}_{Upper-S-C} = \mathcal{C} \cup \mathcal{D}$.

Example 6 *(Continue Example 4) The upper free degree of* $\mathcal{B}_1 \cup \mathcal{B}_2$ *is 0.6. Therefore we split* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *into* $< \mathcal{C}_1, \mathcal{D}_1 >$ *and* $< \mathcal{C}_2, \mathcal{D}_2 >$ *such that*

$$\mathcal{C}_1 = \{(\neg\phi \vee \psi,\ 0.6), (\phi,\ 0.5)\}, \quad \mathcal{D}_1 = \{(\neg\phi \vee \varphi,\ 0.8)\},$$

$$\mathcal{C}_2 = \{(\delta \vee \varphi,\ 0.5), (\neg\phi \vee \neg\psi,\ 0.4), (\psi,\ 0.3)\},\ \mathcal{D}_2 = \{(\phi \vee \delta,\ 0.9), (\neg\psi \vee \delta,\ 0.7)\}.$$

Combining $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *using the maximum operator and combining* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *using the minimum operator gives*

$$\mathcal{C} = \{(\phi \vee \delta \vee \varphi,\ 0.5), (\neg\phi \vee \psi \vee \delta \vee \varphi,\ 0.5), (\phi \vee \psi,\ 0.3), (\neg\phi \vee \psi,\ 0.3)\},$$

$$\mathcal{D} = \{(\phi \vee \delta,\ 0.9), (\neg\phi \vee \varphi,\ 0.8), (\neg\psi \vee \delta,\ 0.7)\}.$$

So we have

$$\begin{aligned}\mathcal{B}_{Upper-S-C} = \ & \{(\phi \vee \delta,\ 0.9), (\neg\phi \vee \varphi,\ 0.8), (\neg\psi \vee \delta,\ 0.7), (\neg\phi \vee \psi \vee \delta \vee \varphi,\ 0.5), \\ & (\phi \vee \delta \vee \varphi,\ 0.5), (\phi \vee \psi,\ 0.3), (\neg\phi \vee \psi,\ 0.3)\}.\end{aligned}$$

Given two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$, if $\mathcal{B}_1 \cup \mathcal{B}_2$ is consistent, by Definition 12, we have $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2) = 0$. When we split $\mathcal{B}_1$ and $\mathcal{B}_2$ using $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, we obtain $\mathcal{C}_1 = \emptyset$, $\mathcal{D}_1 = \mathcal{B}_1$ and $\mathcal{C}_2 = \emptyset$, $\mathcal{D}_2 = \mathcal{B}_2$, which results in $\mathcal{B}_{Upper-S-C} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \{(\phi_i \vee \psi_j, ct(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{B}_1 \ and \ (\psi_j, \beta_j) \in \mathcal{B}_2\}$. Therefore, the *U-S-C* operator is equivalent to the t-norm based merging operator when sources are consistent. Next we give some properties of the *U-S-C* operator when $\mathcal{B}_1 \cup \mathcal{B}_2$ is inconsistent.

Proposition 3 *The resulting possibilistic knowledge base* $\mathcal{B}_{Upper-S-C}$ *of the U-S-C operator is consistent.*

Proofs of some of the propositions are in the Appendix.

Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{(\varphi_i, \alpha_i) : i = 1, ..., n\}$. Suppose the weights of the formulas in $\mathcal{B}$ are rearranged such that $\alpha_1 = 1 > \alpha_2 > ... > \alpha_{n_1}$. Let $\alpha_{n_1+1} = 0$. Since $Inc(\mathcal{B}) \leq Free_{upp}(\mathcal{B})$, if $Inc(\mathcal{B}) = \alpha_k$ and $Free_{upp}(\mathcal{B}) = \alpha_m$, we have $m \leq k$, then by Proposition 3, Algorithm 1 terminates after at most k-m+1 iterations.

Proposition 4 *Given two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$, suppose $\mathcal{B}_{Upper-S-C}$ and $\mathcal{B}_{I-S-C}$ are the results of merging $\mathcal{B}_1$ and $\mathcal{B}_2$ using the U-S-C operator and the I-S-C operator respectively, then we have*

$$\mathcal{B}_{I-S-C} \vdash_\pi (\phi, \alpha), \quad \textit{for all } (\phi, \alpha) \in \mathcal{B}_{Upper-S-C},$$

but not vice versa.

Proposition 4 shows that the upper-free-degree of $\mathcal{B}_1 \cup \mathcal{B}_2$ is the upper bound of the splitting point of Algorithm 1, that is, Algorithm 1 must terminate when the splitting point reaches the upper-free-degree of $\mathcal{B}_1 \cup \mathcal{B}_2$, if it has not terminated beforehand.

Proposition 5 *Given two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$, let $\mathcal{B}_{Upper-S-C}$ be the possibilistic knowledge base obtained by the U-S-C operator (which is based on a t-norm tn and a t-conorm ct) and $\mathcal{B}_{ct}$ be the resulting possibilistic knowledge base of merging using ct, then*

$$\mathcal{B}_{Upper-S-C} \vdash_\pi (\phi, \alpha), \quad \textit{for all } (\phi, \alpha) \in \mathcal{B}_{ct} \tag{15}$$

The converse of Proposition 5 is false. Let us look at a counter-example.

Example 7 *(Continue Example 6) By Example 6, we have*

$$\begin{aligned}\mathcal{B}_{Upper-S-C} \;=\; & \{(\phi \vee \delta,\ 0.9), (\neg\phi \vee \varphi,\ 0.8), (\neg\psi \vee \delta,\ 0.7), (\neg\phi \vee \psi \vee \delta \vee \varphi,\ 0.5),\\ & (\phi \vee \delta \vee \varphi,\ 0.5), (\phi \vee \psi,\ 0.3), (\neg\phi \vee \psi,\ 0.3)\}.\end{aligned}$$

If we combine $\mathcal{B}_1$ and $\mathcal{B}_2$ by the maximum operator, the result is

$$\begin{aligned}\mathcal{B}_{dm} \;=\; & \{(\neg\phi \vee \neg\psi \vee \delta \vee \varphi,\ 0.7), (\phi \vee \neg\psi \vee \delta,\ 0.5),\ (\neg\phi \vee \psi \vee \delta \vee \varphi,\ 0.5), (\phi \vee \delta,\ 0.5),\\ & (\neg\phi \vee \delta \vee \varphi,\ 0.5), (\phi \vee \delta \vee \varphi,\ 0.5), (\neg\phi \vee \neg\psi \vee \varphi,\ 0.4),\ (\neg\phi \vee \psi,\ 0.3),\\ & (\neg\phi \vee \psi \vee \varphi,\ 0.3), (\phi \vee \psi,\ 0.3)\}\end{aligned}$$

It is easy to check that all the possibilistic formulas in $\mathcal{B}_{dm}$ can be inferred from $\mathcal{B}_{Upper-S-C}$. However, $\mathcal{B}_{Upper-S-C}$ contains $(\phi \vee \delta,\ 0.9), (\neg\phi \vee \varphi,\ 0.8), (\neg\psi \vee \delta,\ 0.7)$, which are not involved in conflict in $\mathcal{B}_1 \cup \mathcal{B}_2$ and cannot be inferred from $\mathcal{B}_{dm}$.

Proposition 5 and Example 7 show that the resulting possibilistic knowledge base of the *U-S-C* method contains more formulas in the original knowledge bases than that of the t-conorm based merging operator.

Clearly we have the following corollary.

Corollary 1 *Given two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$, let $\mathcal{B}_{I-S-C}$ be the possibilistic knowledge base obtained by the I-S-C operator (which is based on a t-norm tn and a t-conorm ct) and $\mathcal{B}_{ct}$ be the possibilistic knowledge base that results from merging using ct, then*

$$\mathcal{B}_{I-S-C} \vdash_\pi (\phi, \alpha), \quad \textit{for all } (\phi, \alpha) \in \mathcal{B}_{ct}, \tag{16}$$

but not vice versa.

We have shown that the *I-S-C* operator maintains more original information than the t-conorm based merging operator. Next, we compare our *I-S-C* operator with the renormalization based merging operator (see Equation 11).

We have the following proposition.

Proposition 6 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two possibilistic knowledge bases. Suppose tn is the minimum and ct is an arbitrary t-conorm in Algorithm 1. Let γ be the splitting point obtained by Algorithm 1. Suppose $\gamma = Inc(\mathcal{B}_1 \cup \mathcal{B}_2)$, then $\mathcal{B}_{N,min} \subseteq \mathcal{B}_{I-S-C}$, but not vice versa.*

Proposition 6 states that, when the t-norm is the minimum operator, the resulting knowledge base of our merging operator contains more information than that of the renormalization based merging operator. However, this conclusion does not hold for other t-norms generally. Let us look at an example.

Example 8 *Let $\mathcal{B}_1 = \{(\psi, 0.7), (\phi, 0.7)\}$ and $\mathcal{B}_2 = \{(\neg\phi, 0.6), (\psi, 0.4)\}$. The inconsistency degree of $\mathcal{B}_1 \cup \mathcal{B}_2$ is 0.6. So $\mathcal{B}_1$ and $\mathcal{B}_2$ are split w.r.t 0.6 into $\langle \mathcal{C}_1, \mathcal{D}_1 \rangle$ and $\langle \mathcal{C}_2, \mathcal{D}_2 \rangle$ respectively, where $\mathcal{C}_1 = \emptyset$, $\mathcal{D}_1 = \mathcal{B}_1$, $\mathcal{C}_2 = \mathcal{B}_2$ and $\mathcal{D}_2 = \emptyset$. Combining $\mathcal{C}_1$ and $\mathcal{C}_2$ using the probabilistic sum and combining $\mathcal{D}_1$ and $\mathcal{D}_2$ using the product operator we get $\mathcal{C} = \emptyset$ and $\mathcal{D} = \mathcal{B}_1$ respectively. So $\mathcal{B}_{S-C} = \mathcal{B}_1$. It is clear that $\mathcal{B}_{S-C}$ is consistent. So $\mathcal{B}_{I-S-C} = \mathcal{B}_1$. In contrast, suppose we combine $\mathcal{B}_1$ and $\mathcal{B}_2$ using the product operator, then the resulting possibilistic knowledge base is $\{(\neg\phi \vee \psi, 0.88), (\psi, 0.82), (\phi \vee \psi, 0.82), (\psi, 0.7), (\phi, 0.7), (\neg\phi, 0.6), (\psi, 0.4)\}$, which is equivalent to $\{(\neg\phi \vee \psi, 0.88), (\psi, 0.82), (\phi, 0.7), (\neg\phi, 0.6)\}$. So we have $\mathcal{B}_{N,tn} \equiv_s \{(\neg\phi \vee \psi, 0.88), (\psi, 0.82), (\phi, 0.7)\}$, where tn is the product operator. It is clear that every formula in $\mathcal{B}_{I-S-C}$ can be inferred from $\mathcal{B}_{N,tn}$, while the converse is false.*

When the original knowledge bases are consistent, the *I-S-C* operator is equivalent to the t-norm based operator. That is, we have the following proposition.

Proposition 7 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two possibilistic knowledge bases. If $\mathcal{B}_1 \cup \mathcal{B}_2$ is consistent, then $\mathcal{B}_{tn} = \mathcal{B}_{I-S-C}$.*

The proof of Proposition 7 is trivial.

4.2.2 Logical properties

In this section, we discuss the logical properties of possibilistic merging operators by generalizing postulates or logical properties for propositional merging operators. It is not our intention to use generalized postulates to give a normative definition or characterization of possibilistic merging. The normative definition of possibilistic merging is left as an open problem. In fact, as we will see, none of the possibilistic merging operators satisfy all of the generalized postulates. The reason why we propose these postulates is that we think they are helpful for users wishing to choose among different possibilistic merging operators.

We first introduce the postulates for characterizing a propositional merging operator proposed in [24].

Definition 14 *[24] Let Δ be an operator which assigns to each knowledge profile E a knowledge base $\Delta(E)$. Let E_1 and E_2 be two knowledge profiles, K and K' be two knowledge bases. Δ is a merging operator iff it satisfies the following postulates:*

(A1) $\Delta(E)$ *is consistent.*

(A2) If E is consistent, then $\Delta(E) \equiv \bigwedge E$, where $\bigwedge E = \wedge_{K_i \in E} K_i$.

(A3) If $E_1 \equiv E_2$, then $\Delta(E_1) \equiv \Delta(E_2)$.

(A4) If $K \wedge K'$ is not consistent, then $\Delta(\{K\} \sqcup \{K'\}) \not\models K$.

(A5) $\Delta(E_1) \wedge \Delta(E_2) \models \Delta(E_1 \sqcup E_2)$.

(A6) If $\Delta(E_1) \wedge \Delta(E_2)$ *is consistent, then* $\Delta(E_1 \sqcup E_2) \models \Delta(E_1) \wedge \Delta(E_2)$.

(A1) requires that the resulting knowledge base of a merging operator be consistent. (A2) says that the resulting knowledge base should be the conjunction of the original knowledge bases when they are consistent with each other. (A3) is the principle of irrelevance of syntax. (A4) is the fairness postulate, which means that if two knowledge bases are in conflict, merging operators must not give preference to either of them. (A5) and (A6) together state that if there are two subgroups whose merged results are consistent, then the result of merging of the global group is the conjunction of merged results of subgroups. (A5) can be equivalently expressed as

($A5$) If $\Delta(E_1) \wedge \Delta(E_2)$ is consistent, $\Delta(E_1) \wedge \Delta(E_2) \models \Delta(E_1 \sqcup E_2)$.

So (A5) and (A6) can be merged as the following single postulate,

($A7$) If $\Delta(E_1) \wedge \Delta(E_2)$ is consistent, $\Delta(E_1) \wedge \Delta(E_2) \equiv \Delta(E_1 \sqcup E_2)$.

We now propose some postulates for a possibilistic merging operator based on postulates in Definition 14. In [11], some work has been done to adapt propositional postulates for the possibilistic logic framework. However, the postulates there are based on the assumption that the result of merging can be inconsistent, i.e. they follow the second view of merging. We propose some postulates for a possibilistic merging operator which follow the first view of merging

In the following, $\mathcal{E}$, $\mathcal{E}_1$ and $\mathcal{E}_2$ denote possibilistic knowledge profiles, and $\mathcal{B}_1$ and $\mathcal{B}_2$ denote possibilistic knowledge bases.

(P1) $\Delta(\mathcal{E})$ is consistent.

(P2) Let $\mathcal{E} = \{\mathcal{B}_1, ..., \mathcal{B}_n\}$. If $\mathcal{B}_1 \cup .. \cup \mathcal{B}_n$ is consistent, then $(\Delta(\mathcal{E}))^* \equiv (\mathcal{B}_1 \cup ... \cup \mathcal{B}_n)^*$ and $\forall \phi$, if $\exists i$ such that $\mathcal{B}_i \vdash_\pi (\phi, \alpha)$ then $\exists \beta$ such that $\beta \geq \alpha$ and $\Delta(\mathcal{E}) \vdash_\pi (\phi, \beta)$.

(P3) If $\mathcal{E}_1 \equiv_s \mathcal{E}_2$, then $\Delta(\mathcal{E}_1) \equiv_s \Delta(\mathcal{E}_2)$.

(P4) $\Delta(\{\Delta(\mathcal{E}_1)\}, \{\Delta(\mathcal{E}_2)\}) \vdash_\pi (\phi, \alpha)$ for all $(\phi, \alpha) \in \Delta(\mathcal{E}_1 \sqcup \mathcal{E}_2)$.

(P5) $\Delta(\mathcal{E}_1 \sqcup \mathcal{E}_2) \vdash_\pi (\phi, \alpha)$ for all $(\phi, \alpha) \in \Delta(\{\Delta(\mathcal{E}_1)\}, \{\Delta(\mathcal{E}_2)\})$.

(P6) If $\Delta(\mathcal{E}_1) \cup \Delta(\mathcal{E}_2)$ is consistent, then $\Delta(\mathcal{E}_1)^* \wedge \Delta(\mathcal{E}_2)^* \equiv (\Delta(\mathcal{E}_1 \sqcup \mathcal{E}_2))^*$ and if $\Delta(\mathcal{E}_1) \cup \Delta(\mathcal{E}_2) \vdash_\pi (\phi, \alpha)$, then $\exists \beta$ such that $\beta \geq \alpha$ and $\Delta(\mathcal{E}_1 \sqcup \mathcal{E}_2) \vdash_\pi (\phi, \beta)$.

(P1) is a mandatory condition because we adopt the first view on possibilistic merging. (P2) is adapted from the postulate W2 in [11]. The difference between (P2) and W2 in [11] is that we require that if $\mathcal{B}_1 \cup .. \cup \mathcal{B}_n$ be consistent, then $(\Delta(\mathcal{E}))^* \equiv (\mathcal{B}_1 \cup ... \cup \mathcal{B}_n)^*$. It is stronger than (A2) in that when there is no conflict among the original knowledge bases, it requires not only all the original information be restored, but also the weights of the formulas in original knowledge bases should not be decreased after merging. (P3) is the principle of irrelevance of syntax, that is, it is a generalizations of (A3). Note we do not consider the generalization of (A4) because it is a controversial postulate (see [27] for a discussion) and not intuitive in the context of possibilistic merging because weights are attached to formulas. (P4) and (P5) together are the associativity condition, that is, the fusion of n bases can be decomposed into several steps. (P6) is a generalization of (A7). We do not use $\Delta(\mathcal{E}_1) \cup \Delta(\mathcal{E}_2) \equiv_s \Delta(\mathcal{E}_1 \sqcup \mathcal{E}_2)$ because it is equivalent to assuming that we combine $\Delta(\mathcal{E}_1)$ and $\Delta(\mathcal{E}_2)$ using the minimum based operator, which violates (P2).

Before checking the logical properties of our *I-S-C* operator, we need to generalize it to n possibilistic knowledge bases. It is easy to check that the *I-S-C* operator is not *associative*, that is, the order of combination will influence the result of merging. For example, given three possibilistic knowledge bases $\mathcal{B}_1 = \{(\phi, 0.6)\}$, $\mathcal{B}_2 = \{(\neg\phi, 0.7)\}$, and $\mathcal{B}_3 = \{(\phi, 0.7)\}$. Let tn be the product operator and $ct = max$. Suppose we combine $\mathcal{B}_1$ and $\mathcal{B}_2$ first, the result of combination is $\mathcal{B}_4 = \{(\neg\phi, 0.7)\}$. We then combine $\mathcal{B}_4$ and $\mathcal{B}_3$, the result of combination is $\mathcal{B} = \{(\top, 0.7)\}$. If we combine $\mathcal{B}_1$ and $\mathcal{B}_3$ first, the result of combination is $\mathcal{B}_5 \equiv_s \{(\phi, 0.88)\}$. By combining $\mathcal{B}_5$ and $\mathcal{B}_2$ we get $\mathcal{B}' = \{(\phi, 0.88)\}$. It is clear $\mathcal{B} \not\equiv_s \mathcal{B}'$.

In the following, we generalize Algorithm 1 to compute a splitting point for n possibilistic knowledge bases.

Algorithm 2

Input: a set of possibilistic knowledge bases $\{\mathcal{B}_1, ..., \mathcal{B}_n\}$, a t-conorm ct and a t-norm tn.
Output: a splitting point γ.

Step 1 Let $\mathcal{B} = \cup_{i=1}^{n}\mathcal{B}_i = \{(\varphi_i, \gamma_i) : i = 1, ..., m\}$. Rearrange the weights of formulas in $\mathcal{B}$ such that $\gamma_1 > \gamma_2 > ... > \gamma_{n_1}$. Let $\gamma_{n_1+1} = 0$.
Step 2 Compute $Inc(\mathcal{B})$. Assume $Inc(\mathcal{B}) = \gamma_k$. Let $l = k$.
Step 3 Split $\mathcal{B}_i$ with regard to γ_l such that $\mathcal{B}_i = < \mathcal{C}_i, \mathcal{D}_i >$, where

$$\begin{aligned} \mathcal{C}_i &= \{(\phi_{ij}, \alpha_{ij}) : (\phi_{ij}, \alpha_{ij}) \in \mathcal{B}_i, \alpha_{ij} \leq \gamma_l\} \\ \mathcal{D}_i &= \mathcal{B}_i \setminus \mathcal{C}_i. \end{aligned}$$

Step 4 Combine $\mathcal{C}_i$ $(i = 1, ..., n)$ by ct (the result is $\mathcal{C}$) and combine $\mathcal{D}_i$ $(i = 1, ..., n)$ by tn (the result is $\mathcal{D}$).
Step 5 Let $\mathcal{B}_{S-C} = \mathcal{C} \cup \mathcal{D}$.
Step 6 If $\mathcal{B}_{S-C}$ is inconsistent, let $l = l - 1$ and go to Step 3.
Step 7 Return γ_l.

Definition 15 *Let $\mathcal{E} = \{\mathcal{B}_1, ..., \mathcal{B}_n\}$ be a set of n possibilistic knowledge bases. Let ct be a t-conorm and tn be a t-norm. Let γ be the splitting point obtained by Algorithm 2. Suppose $\mathcal{B}_i$ are split into $\mathcal{B}_i = \langle \mathcal{C}_i, \mathcal{D}_i \rangle$ w.r.t γ, and $\mathcal{C}$ and $\mathcal{D}$ are obtained by merging $\mathcal{C}_i$ $(i = 1, ..., n)$ by ct and merging $\mathcal{D}_i$ $(i = 1, ..., n)$ by tn respectively. The resulting possibilistic knowledge base of the generalized incremental S-C (G-I-S-C for short) merging operator, denoted $\mathcal{B}_{G-I-S-C}$, is defined as $\mathcal{B}_{G-I-S-C} = \mathcal{C} \cup \mathcal{D}$.*

Example 9 *Let $\mathcal{B}_1 = \{(\phi, 0.7), (\psi, 0.5), (\varphi, 0.4)\}$, $\mathcal{B}_2 = \{(\phi \to \neg\psi, 0.6), (\delta, 0.5)\}$, and $\mathcal{B}_3 = \{(\phi, 0.8), (\varphi, 0.7)\}$ be three possibilistic knowledge bases. The union of $\mathcal{B}_i$ is $\mathcal{B} = \{(\phi, 0.8), (\varphi, 0.7), (\phi, 0.7), (\phi \to \neg\psi, 0.6), (\psi, 0.5), (\delta, 0.5), (\varphi, 0.4)\}$ and its inconsistency degree is 0.5. The weights of formulas in $\mathcal{B}$ are rearranged as $\gamma_1 = 0.8 > \gamma_2 = 0.7 > \gamma_3 = 0.6 > \gamma_4 = 0.5 > \gamma_5 = 0.4$. Let $\gamma_6 = 0$. So $Inc(\mathcal{B}) = \gamma_4$. Let $l = 4$. $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ are split w.r.t $\gamma_4 = 0.5$ as $\mathcal{D}_1 = \{(\phi, 0.7)\}$ and $\mathcal{C}_1 = \{(\psi, 0.5), (\varphi, 0.4)\}$, $\mathcal{D}_2 = \{(\phi \to \neg\psi, 0.6)\}$ and $\mathcal{C}_2 = \{(\delta, 0.5)\}$, and $\mathcal{D}_3 = \{(\phi, 0.8), (\varphi, 0.7)\}$ and $\mathcal{C}_3 = \emptyset$ respectively. Combining $\mathcal{C}_i$ by the bounded sum and combining $\mathcal{D}_i$ by the Lukasiewicz t-norm, we get $\mathcal{D} \equiv_s \{(\phi, 1), (\neg\phi \vee \neg\psi \vee \varphi, 1), (\varphi, 0.7), (\phi \to \neg\psi, 0.6)\}$ and $\mathcal{C} = \emptyset$. Since $\mathcal{D} \cup \mathcal{C}$ is consistent, we have $\gamma = 0.5$. So the final result of merging is $\mathcal{B}_{G-I-S-C} = \mathcal{D} \equiv_s \{(\phi, 1), (\neg\phi \vee \neg\psi \vee \varphi, 1), (\varphi, 0.7), (\phi \to \neg\psi, 0.6)\}$. In $\mathcal{B}_{G-I-S-C}$, the weight of ϕ is increased to 1 because there is a reinforcement between $(\phi, 0.7)$ and $(\phi, 0.8)$.*

We have compared the results of merging using our operator with the results obtained from the t-conorm and renormalization based operators. We now compare their logical properties. To make the notation consistent with those used in (P1)-(P5), we use $\Delta_{G-I-S-C}$, Δ_{ct} and $\Delta_{N,tn}$ to denote *G-I-S-C*, the t-conorm based, and renormalization based merging operators respectively.

Proposition 8 *The G-I-S-C operator* $\Delta_{G-I-S-C}$ *satisfies (P1)-(P3). It does not satisfy (P4)-(P6) in general.*

Proposition 9 *The t-conorm based merging operator* Δ_{ct} *satisfies (P1),(P3)-(P5). It does not satisfy (P2) and (P6) in general. The operator* $\Delta_{N,tn}$ *satisfies (P1)-(P3). It does not satisfy (P4)-(P6) in general.*

Proposition 8 and Proposition 9 show that the *G-I-S-C* operator and the renormalization based operator satisfy (P1)-(P3). However, neither satisfy (P4)-(P6) in general. In contrast, the t-conorm based operator satisfies (P4) and (P5), but it does not satisfy (P2) and (P6) in general.

4.2.3 Computational complexity

We analyze the computational complexity of the *I-S-C* operator by the following proposition.

Proposition 10 *Generating a consistent possibilistic knowledge base using the I-S-C operator is in* $F\Delta_2^p\ (\mathcal{O}(n))$*, where* $\Delta_2^p(\mathcal{O}(n))$ *denotes the set of decision problems decidable in polynomial time with no more than* $\mathcal{O}(n)$ *calls to an* NP *oracle,* n *is the number of different valuations involved in* $\mathcal{B}$ *[22], and "F" in* $F\Delta_2^p(\mathcal{O}(n))$ *stands for* function *and is intended to turn a complexity class for decision problems into one for* search problems*, i.e., problems that have answers.*

We also have the following results for the computational complexity of t-conorm and renormalization based operators.

Proposition 11 *Generating a consistent possibilistic knowledge base by a t-conorm based merging operator is in* FP *and generating a consistent possibilistic knowledge base by a renormalization based merging operator is in* $F\Delta_2^p$*.*

By Proposition 10 and Proposition 11, the computational complexity of the *I-S-C* operator and that of the renormalization based merging operator lie in the same level of the boolean hierarchy, and the computational complexity of the t-conorm based merging operator is tractable.

5 Semantic Aspects of the *I-S-C* Operator

In this section, we provide a semantic analysis of the *I-S-C* operator. We first give a definition of splitting a possibility distribution *w.r.t* a weight.

Definition 16 *Suppose* $\mathcal{B} = \{(\phi_i, \alpha_i) : i = 1, ..., n\}$ *is a possibilistic knowledge base and* $\pi_{\mathcal{B}}$ *is the possibility distribution associated with it. For weight* α_k *of formula* ϕ_k*, we can split* $\pi_{\mathcal{B}}$ *w.r.t* α_k *as* $\langle \pi_1, \pi_2 \rangle$*, where*

$$\pi_1(\omega) = \begin{cases} 1 & \text{if } \omega \models \phi_i, \forall(\phi_i, \alpha_i) \in \mathcal{B} \text{ where } \alpha_i \leq \alpha_k, \\ 1 - max\{\alpha_i | \omega \not\models \phi_i, (\phi_i, \alpha_i) \in \mathcal{B} \text{ and } \alpha_i \leq \alpha_k\} & \text{otherwise.} \end{cases} \tag{17}$$

and

$$\pi_2(\omega) = \begin{cases} 1 & \text{if } \omega \models \phi_i, \forall(\phi_i, \alpha_i) \in \mathcal{B} \text{ where } \alpha_i > \alpha_k, \\ 1 - max\{\alpha_i | \omega \not\models \phi_i, (\phi_i, \alpha_i) \in \mathcal{B} \text{ and } \alpha_i > \alpha_k\} & \text{otherwise.} \end{cases} \tag{18}$$

Clearly, we have the following propositions.

Proposition 12 *Suppose $\mathcal{B} = \{(\phi_i, \alpha_i) : i = 1, ..., n\}$ is a possibilistic knowledge base and $\pi_\mathcal{B}$ is the possibility distribution associated with it. (ϕ_i, α_i) is a formula of $\mathcal{B}$. Suppose $\mathcal{B}$ is split w.r.t α_i as $\langle \mathcal{C}, \mathcal{D} \rangle$, $\pi_\mathcal{B}$ is split w.r.t α_i as $\langle \pi_1, \pi_2 \rangle$. We then have $\pi_1 = \pi_\mathcal{C}$ and $\pi_2 = \pi_\mathcal{D}$, where $\pi_\mathcal{C}$ and $\pi_\mathcal{D}$ are the possibility distributions of $\mathcal{C}$ and $\mathcal{D}$ respectively.*

Proposition 13 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two equivalent possibilistic knowledge bases and $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ be possibility distributions associated with them. Let $(\phi_k, \alpha_k) \in \mathcal{B}_1$. Suppose π_{11} and π_{12} are possibility distributions obtained by splitting $\pi_{\mathcal{B}_1}$ w.r.t α_k using Equation 17 and Equation 18 respectively. Suppose π_{21} and π_{22} are possibility distributions obtained by splitting $\pi_{\mathcal{B}_2}$ w.r.t α_k using Equation 17 and Equation 18 respectively. We then have $\pi_{11}(\omega) = \pi_{21}(\omega)$ and $\pi_{12}(\omega) = \pi_{22}(\omega)$ for all ω.*

Proposition 13 shows that the splitting of the possibility distribution associated with a possibilistic knowledge base using Equation 17 and Equation 18 is syntax-independent.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two possibilistic knowledge bases and $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ be their respective associated possibility distributions. The idea of the semantic *I-S-C* method can be described as follows. We first find a splitting point by the algorithm below and then split $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ into $\langle \pi_{11}, \pi_{12} \rangle$ and $\langle \pi_{21}, \pi_{22} \rangle$ respectively by the splitting point (π_{i1} are obtained by Equation 17 and π_{i2} are obtained by Equation 18). After that, we combine π_{11} and π_{21} using a t-conorm operator (the result is a possibility distribution π_1) and combine π_{12} and π_{22} using a t-norm operator (the result is a possibility distribution π_2). Finally, the resulting possibility distribution of our semantic *I-S-C* operator is defined as $\pi_{I-S-C}(\omega) = min(\pi_1(\omega), \pi_2(\omega))$. The following algorithm is the semantic counterpart of Algorithm 1.

Algorithm 3

Input: two possibility distributions $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ which are associated with $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, ..., n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, ..., m\}$ respectively.

Output: a splitting point γ.

Step 1 Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{(\varphi_i, \gamma_i) : i = 1, ..., n+m\}$. Rearrange the weights of formulas in $\mathcal{B}$ such that $\gamma_1 > \gamma_2 > ... > \gamma_{n_1}$. Let $\gamma_{n_1+1} = 0$.

Step 2 Compute $Inc(\mathcal{B})$, assume $Inc(\mathcal{B}) = \gamma_k$. Let $l = k$.

Step 3 Split $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ with regard to γ_l into $\pi_{\mathcal{B}_1} = \langle \pi_{11}, \pi_{12} \rangle$ and $\pi_{\mathcal{B}_2} = \langle \pi_{21}, \pi_{22} \rangle$, where π_{i1} ($i = 1, 2$) are obtained by Equation 17 and π_{i2} ($i = 1, 2$) are obtained by Equation 18.

Step 4 Combine π_{11} and π_{21} using a t-conorm operator *ct* and combine π_{12} and π_{22} using a t-norm operator *tn*. The results are $\pi_1(\omega) = tn(\pi_{11}(\omega), \pi_{21}(\omega))$ and $\pi_2(\omega) = ct(\pi_{12}(\omega), \pi_{22}(\omega))$, for all ω.

Step 5 Let $\pi_{S-C}(\omega) = min(\pi_1(\omega), \pi_2(\omega))$ for all ω.

Step 6 If π_{S-C} is subnormal, let $l = l - 1$ and go to Step 3.

Step 7 Return γ_l.

The semantic *I-S-C* operator is then defined as follows.

Definition 17 *Let $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, ..., n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, ..., m\}$ be two possibilistic knowledge bases, $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ be their associated possibility distributions. Let γ be the splitting point obtained by Algorithm 3. Suppose $\pi_{\mathcal{B}_i}$ ($i = 1, 2$) are split into $\pi_i = \langle \pi_{i1}, \pi_{i2} \rangle$ w.r.t γ, where π_{i1} and π_{i2} are obtained by Equation 17 and 18 respectively. Suppose π_1 and π_2 are defined by $\pi_1(\omega) = ct(\pi_{11}(\omega), \pi_{21}(\omega))$ and $\pi_2(\omega) = tn(\pi_{12}(\omega), \pi_{22}(\omega))$, for all ω. The resulting possibility distribution of the semantic incremental S-C operator, denoted as π_{I-S-C}, is defined as $\pi_{I-S-C}(\omega) = min(\pi_1(\omega), \pi_2(\omega))$, for all ω.*

The following proposition shows that the semantic *I-S-C* operator results in a possibility distribution which is equivalent to that associated with the possibilistic knowledge base obtained by the *I-S-C* merging operator.

Proposition 14 *Let* $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, ..., n\}$ *and* $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, ..., m\}$ *be two possibilistic knowledge bases and let* $\pi_{\mathcal{B}_1}$ *and* $\pi_{\mathcal{B}_2}$ *be their respective associated possibility distributions. We have* $\pi_{\mathcal{B}_{I-S-C}}(\omega) = \pi_{I-S-C}(\omega)$*, for all* ω*.*

We have the following relationship between the resulting possibility distributions of the semantic *I-S-C* operator and t-conorm based merging operator.

Proposition 15 *Let* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be two possibilistic knowledge bases and let* $\pi_{\mathcal{B}_1}$ *and* $\pi_{\mathcal{B}_2}$ *be their respective associated possibility distributions. Let* π_{I-S-C} *be the resulting possibility distribution of merging by the semantic I-S-C operator and* π_{ct} *be the resulting possibility distribution by the t-conorm, then* π_{I-S-C} *is more specific than* π_{ct}*, that is* $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$ *for all* $\omega \in \Omega$*.*

6 An Alternative Way to Split Possibilistic Knowledge Bases

6.1 An alternative splitting approach

When defining the *I-S-C* operator, given two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$, we split each of them using the weight obtained by Algorithm 1 such that $\mathcal{B}_1 = \mathcal{C}_1 \cup \mathcal{D}_1$ and $\mathcal{B}_2 = \mathcal{C}_2 \cup \mathcal{D}_2$. We then combine $\mathcal{C}_1$ and $\mathcal{C}_2$ by a t-conorm operator to weaken conflicting information. Since $\mathcal{C}_1 \cup \mathcal{C}_2$ consists of possibilistic formulas in $\mathcal{B}_1 \cup \mathcal{B}_2$ with necessity degrees less than the splitting point, there may exist some free formulas in $\mathcal{C}_1$ or $\mathcal{C}_2$. So, when we combine $\mathcal{C}_1$ and $\mathcal{C}_2$ by a t-conorm operator, these free formulas are combined with other formulas as disjunctive forms. However, we know that free formulas will not cause inconsistency, so it may be safe to keep them unchanged. Therefore, we propose the following alternative approach to splitting the knowledge bases.

Definition 18 *(free-formulas-based splitting) Given a possibilistic knowledge base* $\mathcal{B}$*, the splitting of* $\mathcal{B}$ *with regard to* $Free(\mathcal{B})$ *is a pair* $< \mathcal{C}_{Con}, \mathcal{D}_{Free} >$ *such that* $\mathcal{B} = \mathcal{C}_{Con} \cup \mathcal{D}_{Free}$*, where*

$$\mathcal{D}_{Free} = \{(\phi, \alpha) | (\phi, \alpha) \in Free(\mathcal{B})\},$$

$$\mathcal{C}_{Con} = \mathcal{B} \setminus \mathcal{D}_{Free} = \{(\phi, \alpha) | (\phi, \alpha) \notin Free(\mathcal{B})\}.$$

That is, $\mathcal{D}_{Free}$ contains all the free formulas, and $\mathcal{C}_{Con}$ contains all the conflicting formulas in $\mathcal{B}$.

Lemma 1 *Let* $\mathcal{B}$ *be a possibilistic knowledge base. Let* $\mathcal{B}$ *be split by the upper-free-degree approach and free-formulas approach respectively, with the splitting results as* $\mathcal{B} = \mathcal{C} \cup \mathcal{D}$ *and* $\mathcal{B} = \mathcal{C}_{Con} \cup \mathcal{D}_{Free}$*. Then* $\mathcal{D} \subseteq \mathcal{D}_{Free}$*, and* $\mathcal{C}_{Con} \subseteq \mathcal{C}$*.*

We omit the proof of Lemma 1, as it is easy to prove.

Now we define the free-formulas-based *S-C* operator.

Definition 19 *Let* $\mathcal{B}_1 = \{(\phi_1,\ \alpha_1), ..., (\phi_n,\ \alpha_n)\}$ *and* $\mathcal{B}_2 = \{(\psi_1,\ \beta_1), ..., (\psi_m,\ \beta_m)\}$ *be two possibilistic knowledge bases. Let* $< \mathcal{C}', \mathcal{D}' >$ *be the result of splitting* $\mathcal{B}_1 \cup \mathcal{B}_2$ *with respect to* $Free(\mathcal{B}_1 \cup \mathcal{B}_2)$*. Suppose* $\mathcal{B}_1$ *is split into a pair* $< \mathcal{C}_{Con_1}, \mathcal{D}_{Free_1} >$ *such that* $\mathcal{C}_{Con_1} = \mathcal{C}' \cap \mathcal{B}_1$ *and* $\mathcal{D}_{Free_1} = \mathcal{D}' \cap \mathcal{B}_1$*, and* $\mathcal{B}_2$ *is split into a pair* $< \mathcal{C}_{Con_2}, \mathcal{D}_{Free_2} >$ *such that* $\mathcal{C}_{Con_2} = \mathcal{C}' \cap \mathcal{B}_2$ *and* $\mathcal{D}_{Free_2} = \mathcal{D}' \cap \mathcal{B}_2$*. Let* $\mathcal{C}_{Con}$ *and* $\mathcal{D}_{Free}$ *be the possibilistic knowledge bases obtained by merging* $\mathcal{C}_{Con_1}$ *and* $\mathcal{C}_{Con_2}$ *using a t-conorm operator and merging* $\mathcal{D}_{Free_1}$ *and* $\mathcal{D}_{Free_2}$ *using a t-norm operator respectively. The result of the F-S-C merging operator, denoted* $\mathcal{B}_{F-S-C}$*, is defined as* $\mathcal{B}_{F-S-C} = \mathcal{C}_{Con} \cup \mathcal{D}_{Free}$*.*

The resulting possibilistic knowledge base of the *F-S-C* operator is always consistent.

Lemma 2 *The possibilistic knowledge base $\mathcal{B}_{F-S-C}$ obtained by the F-S-C method is consistent.*

The proof of Lemma 2 is similar to that of Proposition 3.

Example 10 *Suppose there are two persons whose beliefs on Tweety the penguin are expressed by two possibilistic knowledge bases*

$$\begin{aligned}
\mathcal{B}_1 &= \{(penguin(Tweety), 1), (bird(Tweety), 0.8), (eatfish(Tweety), 0.8)\}, \\
\mathcal{B}_2 &= \{(bird(Tweety) \rightarrow fly(Tweety), 1), (\neg fly(Tweety), 0.8), \\
&\quad (eatfish(Tweety) \rightarrow swim(Tweety), 0.6)\}.
\end{aligned}$$

Since

$$\begin{aligned}
Free(\mathcal{B}_1 \cup \mathcal{B}_2) &= \{(penguin(Tweety), 1), (eatfish(Tweety), 0.8), \\
&\quad (eatfish(Tweety) \rightarrow swim(Tweety), 0.6)\},
\end{aligned}$$

$\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are split into* $\langle \mathcal{C}_{Con_1}, \mathcal{D}_{Free_1} \rangle$ *and* $\langle \mathcal{C}_{Con_2}, \mathcal{D}_{Free_2} \rangle$*, where*

$$\begin{aligned}
\mathcal{C}_{Con_1} &= \{(bird(Tweety), 0.8)\}, \\
\mathcal{D}_{Free_1} &= \{(penguin(Tweety), 1), (eatfish(Tweety), 0.8)\}, \\
\mathcal{C}_{Con_2} &= \{bird(Tweety) \rightarrow fly(Tweety), 1), (\neg fly(Tweety), 0.8)\} \\
\mathcal{D}_{Free_2} &= \{(eatfish(Tweety) {\rightarrow} swim(Tweety), 0.6)\}.
\end{aligned}$$

Suppose the t-conorm operator used is the probabilistic sum and the t-norm operator used is the product operator. Then combining $\mathcal{C}_{Con_1}$ *and* $\mathcal{C}_{Con_2}$ *using the probabilistic sum and combining* $\mathcal{D}_{Free_1}$ *and* $\mathcal{D}_{Free_2}$ *using the product operator we get*

$$\begin{aligned}
\mathcal{C}_{Con} &= \{(bird(Tweety) \vee \neg fly(Tweety), 0.64)\}, \\
\mathcal{D}_{Free} &= \{(penguin(Tweety) {\vee} (eatfish(Tweety) {\rightarrow} swim(Tweety)), 1), \\
&\quad (penguin(Tweety), 1), (eatfish(Tweety), 0.8), \\
&\quad (eatfish(Tweety) {\rightarrow} swim(Tweety), 0.6)\}.
\end{aligned}$$

So

$$\begin{aligned}
\mathcal{B}_{F-S-C} &= \{(penguin(Tweety) {\vee} (eatfish(Tweety) {\rightarrow} swim(Tweety)), 1), \\
&\quad (penguin(Tweety), 1), (bird(Tweety) {\vee} \neg fly(Tweety), 0.64), \\
&\quad (eatfish(Tweety), 0.8), (eatfish(Tweety) {\rightarrow} swim(Tweety), 0.6)\}.
\end{aligned}$$

The conflicting formulas $(bird(Tweety), 0.8)$*,* $(bird(Tweety) \rightarrow fly(Tweety), 1)$*,* $(\neg fly(Tweety), 0.8)$ *are weakened to be* $(bird(Tweety) \vee \neg fly(Tweety), 0.64)$*. That is, after merging, we are moderately confident that either Tweety is a bird or Tweety cannot fly.*

The resulting knowledge base of the *F-S-C* operator and that of the *I-S-C* operator are not comparable in general. Let us look at Example 8 again.

Example 11 *(Continue Example 8)* $Free(\mathcal{B}_1 \cup \mathcal{B}_2) = \{(\psi, 0.7), (\psi, 0.4)\}$. *So* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are split into* $< \mathcal{C}_{Con_1}, \mathcal{D}_{Free_1} >$ *and* $< \mathcal{C}_{Con_2}, \mathcal{D}_{Free_2} >$, *where* $\mathcal{C}_{Con_1} = \{(\phi, 0.7)\}$, $\mathcal{D}_{Free_1} = \{(\psi, 0.7)\}$, *and* $\mathcal{C}_{Con_2} = \{(\neg\phi, 0.6)\}$, $\mathcal{D}_{Free_2} = \{(\psi, 0.4)\}$. *Combining* $\mathcal{C}_{Con_1}$ *and* $\mathcal{C}_{Con_2}$ *using the probabilistic sum and combining* $\mathcal{D}_{Free_1}$ *and* $\mathcal{D}_{Free_2}$ *using the product operator we get* $\mathcal{C}_{Con} = \{(\top, 0.42)\}$ *and* $\mathcal{D}_{Free} \equiv_s \{(\psi, 0.88)\}$. *Therefore* $\mathcal{B}_{F-S-C} \equiv_s \{(\psi, 0.88)\}$. *Clearly,* $\mathcal{B}_{F-S-C}$ *is not comparable with* $\mathcal{B}_{I-S-C}$ *in Example 8.*

However, we have the following proposition.

Proposition 16 *Given two possibilistic knowledge bases* $\mathcal{B}_1$ *and* $\mathcal{B}_2$, *if* $\mathcal{B}_{F-S-C}$ *is the possibilistic knowledge base obtained by the F-S-C method and* $\mathcal{B}_{Upper-S-C}$ *is the possibilistic knowledge base obtained by the U-S-C method, then*

$$\mathcal{B}_{F-S-C} \vdash_\pi (\phi, \alpha), \quad \textit{for all } (\phi, \alpha) \in \mathcal{B}_{Upper-S-C}, \tag{19}$$

but not vice verse.

6.2 Properties

We now consider the logical properties of the *F-S-C* operator. It is easy to generalize the *F-S-C* operator to n possibilistic knowledge bases. We denote the generalized *F-S-C* operator by $\triangle_{F-S-C}$. We have the following proposition for the logical properties of the $\triangle_{F-S-C}$ operator.

Proposition 17 *The F-S-C operator* $\triangle_{F-S-C}$ *satisfies (P1), (P2), (P4). It does not satisfy (P3), (P5) and (P6) in general.*

Since the *F-S-C* operator does not satisfy (P3), it is syntax-dependent. However, it satisfies the following important postulate, which is falsified by all the other possibilistic merging operators.

(P7) Let $(\phi, \alpha) \in \cup\mathcal{E}$. If (ϕ, α) is free in $\cup\mathcal{E}$, then $\Delta(\mathcal{E}) \vdash_\pi (\phi, \beta)$, where $\beta \geq \alpha$.

Proposition 18 $\triangle_{F-S-C}$ *satisfies (P7). The I-S-C operator, t-conorm based operator, and renormalization based operator do not satisfy (P7).*

The computational complexity of the *F-S-C* operator is worse than that of other operators (under the usual assumption in complexity theory).

Proposition 19 *Generating a knowledge base by the F-S-C method is* $F\Sigma_2^p$*-complete.*

6.3 Application of the *F-S-C* operator to merge flat knowledge bases

6.3.1 *F-S-C* operator: flat case

It has been pointed out in [17] that when the necessity degrees of all the possibilistic formulas are taken as 1, possibilistic logic regresses to classical logic. So classical logic is a special case of possibilistic logic in which all the formulas have the same level of priority. That is, given a set of formulas $K = \{\phi_1, ..., \phi_n\}$ in classical logic, we can relate it to a set of possibilistic formulas $\mathcal{K} = \{(\phi_1,\ 1), ...(\phi_n,\ 1)\}$. Therefore, our *F-S-C* method can be applied to merge flat (or classical) knowledge bases. For notational simplicity, we omit "flat" and use "knowledge base" only in this section.

Definition 20 *Let K a knowledge base. The splitting of K w.r.t $Free(K)$ is a pair $< K_{Con}, K_{Free} >$ such that $K_{Free} = \{\phi | \phi \in Free(K)\}$ and $K_{Con} = K \setminus K_{Free}$.*

It is clear that we have the following proposition.

Proposition 20 *Let $E = \{K_1, ..., K_n\}$ be a set of n knowledge bases. Let $< K', K'' >$ be the splitting of $\cup E$ w.r.t $Free(\cup E)$. Suppose K_i is split into a pair $< K_{Con_i}, K_{Free_i} >$ such that $K_{Con_i} = K' \cap K_i$ and $K_{Free_i} = K'' \cap K_i$. Let $K_{Con} = \{(\vee_{i=1}^{n} \phi_i : \phi_i \in K_{Con_i}\}$ and $K_{Free} = \cup_{i=1}^{n} K_{Free_i}$. Suppose the result of the F-S-C merging operator in the flat case is K_{F-S-C}, we then have $K_{F-S-C} = K_{Con} \cup K_{Free}$.*

In [5], a consequence relation called a *free consequence relation* is defined to cope with inconsistency in knowledge bases.

Definition 21 *A formula ϕ is said to be a free consequence of a knowledge base K, denoted $K \models_{Free} \phi$, if and only if ϕ is logically entailed by $Free(K)$, namely,*

$$K \models_{Free} \phi, \quad iff \quad Free(K) \models \phi$$

Given two knowledge bases K_1 and K_2, a method is introduced in [5] to concatenate K_1 and K_2, i.e., the result of merging is $K_1 \cup K_2$. When $K_1 \cup K_2$ is inconsistent, some inconsistency tolerant consequence relations, for example, the free consequence relation, could be used to deal with it.

Proposition 21 *Let K_1 and K_2 be two knowledge bases. Let K_{F-S-C} be the knowledge base obtained by merging K_1 and K_2 using the F-S-C merging operator. Then every free consequence of $K_1 \cup K_2$ can be inferred from K_{F-S-C}.*

Proposition 21 shows that K_{F-S-C} keeps all the free formulas unchanged, and combines all the subbases containing conflicting formulas using the maximum. By contrast, if we combine K_1 and K_2 by concatenation and deal with the inconsistency using the free consequence relation, then only free formulas are used and the conflicting formulas are ignored. Consequently, the converse of Proposition 21 is false.

Example 12 *Given two knowledge bases $K_1 = \{\phi, \neg\phi \vee \neg\psi\}$, $K_2 = \{\psi, \neg\phi \vee \delta, \psi \vee \delta\}$, the set of free formulas of $K_1 \cup K_2$ is $Free(K_1 \cup K_2) = \{\neg\phi \vee \delta, \psi \vee \delta\}$. Splitting K_1 and K_2 with regard to $Free(K_1 \cup K_2)$, we have $K_1 = K_{Con_1} \cup K_{Free_1}$ such that $K_{Con_1} = \{\phi, \neg\phi \vee \neg\psi\}$ and $K_{Free_1} = \emptyset$, and $K_2 = K_{Con_2} \cup K_{Free_2}$ such that $K_{Con_2} = \{\psi\}$ and $K_{Free_2} = \{\neg\phi \vee \delta, \psi \vee \delta\}$. We then have*

$$K_{Con} = \{\phi \vee \psi\} \text{ and } K_{Free} = \{\neg\phi \vee \delta, \psi \vee \delta\}.$$

Finally, $K_{F-S-C} = \{\phi \vee \psi, \neg\phi \vee \delta, \psi \vee \delta\}$. Clearly, $\phi \vee \psi$ cannot be inferred from $Free(K_1 \cup K_2)$.

Applicability to the merging of flat knowledge bases is a very important characteristic for the *F-S-C* merging operator. In the flat case, the *I-S-C* operator is reduced to the t-conorm based operator. The renormalization based merging operator is not applicable to flat knowledge bases because the resulting knowledge base is equivalent to a knowledge base with no information.

Proposition 22 *Let $E = \{K_1, ..., K_n\}$ be a set of knowledge bases. Let $K_{I-S-C}(E)$ and $K_{ct}(E)$ be the knowledge bases resulting from the I-S-C operator and t-conorm based operator respectively. Then $K_{I-S-C}(E) = K_{ct}(E) = \{\vee_{i=1}^{n} \phi_i : \phi_i \in K_i\}$. Let $K_{N,tn}(E)$ be the resulting knowledge base of a renormalization based operator. Then $K_{N,tn}(E) \equiv \top$.*

6.3.2 Comparison with other syntax-based merging operators

Let $\triangle_{F-S-C}$ denote our *F*-*S*-*C* merging operator. It is clear that for any knowledge profile E, $\triangle_{F-S-C}(E)$ is consistent, so $\triangle_{F-S-C}$ satisfies ($A1$). We also have the following properties for $\triangle_{F-S-C}$.

Proposition 23 *Let E be a knowledge profile. If E is consistent, then $\triangle_{F-S-C}(E) \equiv \bigwedge E$.*

Proposition 24 *If $K_1 \cup K_2$ is not consistent, then $\triangle_{F-S-C}(\{K_1\} \sqcup \{K_2\}) \not\models K_1$.*

Proposition 25 $\triangle_{F-S-C}(E_1) \cup \triangle_{F-S-C}(E_2) \models \triangle_{F-S-C}(E_1 \sqcup E_2)$.

Propositions 23, 24, 25 show that the operator $\triangle_{F-S-C}$ satisfies ($A2$), ($A4$), and ($A5$). However, our *F*-*S*-*C* merging operator does not satisfy all the other postulates in Definition 14.

Proposition 26 *The F-S-C merging operator $\triangle_{F-S-C}$ does not satisfy* ($A3$) *and* ($A6$) *in general.*

By Proposition 26, in the flat case, our *F*-*S*-*C* operator belongs to syntax-based or formula-based merging operators [2, 25]. Compared with model-based operators, formula-based ones are usually computationally more expensive (inference can be Π_2^p-complete) and satisfy fewer rationality postulates [28]. However, formula-based operators may outperform model-based ones *w.r.t* other criteria, such as strategy-proofness and discriminating power (see [19, 20, 21]). We now compare the *F*-*S*-*C* operator with other syntax-based merging operators. Let us first introduce a merging operator in [2].

Definition 22 *Let K be a knowledge base. A subset $M \subseteq K$ is called a maximal consistent subset of K if it satisfies the following conditions:*

1. $M \not\models \perp$,
2. $\forall M' \subseteq K$, *if* $M \subset M'$, *then* $M' \models \perp$.

The set of all maximal consistent subsets of K is denoted as $MAXCONS(K)$.

Definition 23 *Let E be a knowledge profile. The maximal-consistent-subsets (MCS for short) based merging operator, denoted Δ_{MC}, is defined as*

$$\Delta_{MC}(E) = \bigvee MAXCONS(\cup E).$$

It was shown in [25] that the MCS-based operator satisfies (A1), (A2), (A4) and (A5). However, it does not satisfy (A3) and (A6).

The resulting knowledge base of our operator is not comparable with that of the MCS-based operator in general.

Example 13 *Let $E = \{K_1, K_2\}$ contain two knowledge bases $K_1 = \{\phi, \neg\phi \vee \neg\psi, \varphi\}$ and $K_2 = \{\psi, , \neg\phi \vee \neg\varphi\}$. Since $\{\phi, \neg\phi \vee \neg\psi, \psi\}$ and $\{\phi, \neg\phi \vee \neg\varphi, \varphi\}$ are two minimal inconsistent subbases of $K_1 \cup K_2$, $Free(K_1 \cup K_2) = \emptyset$. Using the F-S-C merging operator, we get $K_{F-S-C} = \{\phi \vee \psi, \neg\phi \vee \neg\psi \vee \neg\varphi, \psi \vee \varphi\}$. In contrast, $K_1 \cup K_2$ contains five maximal consistent knowledge bases $B_1 = \{\phi, \psi, \varphi\}$, $B_2 = \{\phi, \psi, \neg\phi \vee \neg\varphi\}$, $B_3 = \{\phi, \varphi, \neg\phi \vee \neg\psi\}$, $B_4 = \{\phi, \neg\phi \vee \neg\psi, \neg\phi \vee \neg\varphi\}$, $B_5 = \{\psi, \varphi, \neg\phi \vee \neg\psi, \neg\phi \vee \neg\varphi\}$. So $\Delta_{MC}(E) = \bigvee_{i=1}^{5} B_i$. It is easy to check that $\psi \vee \varphi$ cannot be inferred from B_4, therefore, it is not inferred from $\Delta_{MC}(E)$. However, $\psi \vee \varphi \in K_{F-S-C}$, so it can be inferred from K_{F-S-C}. Conversely, $\phi \vee \varphi$ can be inferred from each B_i, so it is inferred from $\Delta_{MC}(E)$. However, $\phi \vee \varphi$ cannot be inferred from K_{F-S-C}.*

The MCS merging operator does not take into account the source of information in the combination process. That is, it simply conjoins the original knowledge bases first and then takes the disjunction of all the maximal consistent subsets as the resulting knowledge base. In [25], several selection functions were defined to choose from maximal consistent subsets those that best fit a "merging criteria". Generally, the merging operators based on selection functions do not satisfy (A3) and (A6). One exception is a merging operator called the intersection operator, which is defined as follows.

Definition 24 *[25] Let E be a knowledge set, M and K be two knowledge bases. Let us denote $dist_{\cap}(M,K) = |K \cap M|$ and $dist_{\cap}(M,E) = \Sigma_{K \in E} dist_{\cap}(M,K)$. Let $dist_{max} = max_{M_i \in MAXCONS(\cup E)}$ $(dist_{\cap}(M_i, E))$. Then the intersection operator is defined as*

$$\Delta^{\cap,\Sigma}(E) = \{M \in MAXCONS(\cup E) : dist_{\cap}(M,E) = dist_{max}\}.$$

The intersection operator selects those maximal consistent subsets that fit the knowledge bases on a maximum number of formulas. It was shown that operator $\Delta^{\cap,\Sigma}$ satisfies (A1),(A2),(A5),(A6). However, it does not satisfy (A3) and (A4) in general.

Let us go back to Example 13 again. It is easy to check that $dist_{max} = 4$ and $\Delta^{\cap,\Sigma}(E) = B_5$. So $\Delta^{\cap,\Sigma}(E) \models \phi$ for all $\phi \in K_{F-S-C}$. However, this conclusion is not always true.

Example 14 *Let $E = \{K_1, K_2\}$, where $K_1 = \{\phi, \varphi\}$ and $K_2 = \{\psi, \neg\phi \vee \neg\psi, \neg\varphi \vee \neg\delta, \delta\}$. $K_1 \cup K_2 = \{\phi, \varphi, \psi, \neg\phi \vee \neg\psi, \neg\varphi \vee \neg\delta, \delta\}$. There are nine maximal consistent subsets of $K_1 \cup K_2$, for example, $B_1 = \{\psi, \neg\phi \vee \neg\psi, \neg\varphi \vee \delta, \delta\}$ and $B_2 = \{\phi, \neg\phi \vee \neg\psi, \neg\varphi \vee \neg\delta, \delta\}$. It is easy to check that $dist_{\cap}(B_i, E) = 4$ for all i. So $dist_{max} = 4$ and $\Delta^{\cap,\Sigma}(E) = \bigvee_{i=1}^{9} B_i$. A formula can be inferred from $\Delta^{\cap,\Sigma}(E)$ iff it can be inferred from all B_i's. Next, we consider the F-S-C operator. The set of free formulas of $K_1 \cup K_2$ is $\emptyset$. So K_1 and K_2 are split into $K_{Con1} = \{\phi, \varphi\}$ and $K_{Free_1} = \emptyset$, and $K_{Con_2} = \{\psi, \neg\phi \vee \neg\psi, \neg\varphi \vee \neg\delta, \delta\}$ and $K_{Free_2} = \emptyset$. We then have $K_{Con} = \{\phi \vee \psi, \phi \vee \neg\varphi \vee \neg\delta, \phi \vee \delta, \varphi \vee \psi, \neg\phi \vee \neg\psi \vee \varphi, \varphi \vee \delta\}$ and $K_{Free} = \emptyset$. Thus, $K_{F-S-C} = \{\phi \vee \psi, \phi \vee \neg\varphi \vee \neg\delta, \phi \vee \delta, \varphi \vee \psi, \neg\phi \vee \neg\psi \vee \varphi, \varphi \vee \delta\}$. It is clear that $\phi \vee \neg\varphi \vee \neg\delta$ is not inferred from B_1 and $\psi \vee \varphi$ is not inferred from B_2, so both $\phi \vee \neg\varphi \vee \neg\delta$ and $\psi \vee \varphi$ cannot be inferred from $\Delta^{\cap,\Sigma}(E)$.*

From the analysis above we conclude that $\triangle_{F-S-C}$ satisfies most of the important postulates for merging operators in the flat case. It is a good alternative to syntax-based operators in classical logic. More importantly, $\triangle_{F-S-C}$ can be applied to merge uncertain knowledge bases in the framework of possibilistic logic, where the formulas in conflict are weakened to resolve inconsistency and the weights of free formulas are reinforced. So it provides a very good choice for dealing with the problem of combining sources of information, especially in the presence of uncertain and incomplete information.

7 Related Work

Possibilistic logic is closely related to Spohn's ordinal conditional functions [8, 16]. It was pointed out in [8] that a kappa distribution in ordinal conditional functions, which is a function from the set of interpretations to the set of integers or natural numbers, is related to a possibility distribution in possibilistic logic. The only difference is that we associate each interpretation ω with an integer $\kappa(\omega)$ in ordinal conditional functions, and the lower $\kappa(\omega)$, the more preferred it is. Much work has been done on merging sources of information in the framework of ordinal conditional functions [8, 33, 34, 35]. The possibilistic merging problem is related to the merging problem in ordinal conditional functions, which deals with the problem of aggregating kappa distributions (see [8]) or epistemic states (see [33]) using an aggregation operator. The syntactical representations of semantic merging operators are then obtained as in possibilistic logic (see Equation 6). Some of the problems we have identified in this paper for merging operators in possibilistic logic are also encountered by merging operators in ordinal conditional functions. In particular, a single operator is used to aggregate kappa distributions or epistemic states. In future work, we will consider applying the idea of the split-combination approach to merge sources of information in ordinal conditional functions.

In [32], some aggregation operators have been proposed to combine the *belief states* of a set of information sources, which are defined as modular, transitive relations over possible worlds. The authors also differentiate conflicting and conflict-free belief states in the context of merging. Our work differs from theirs in that our merging operators are defined on possibilistic knowledge bases, whilst their aggregation operators are defined on a set of belief states.

8 Conclusions and Further Work

In this paper we first proposed an incremental split-combination (*I-S-C*) merging operator which resolves inconsistency between individually consistent possibilistic knowledge bases. The *I-S-C* operator uses a weight obtained by an incremental algorithm to split each possibilistic knowledge base $\mathcal{B}_i$ into two subbases $\mathcal{C}_i$ and $\mathcal{D}_i$ and then combines the $\mathcal{C}_i$'s by a t-conorm based merging operator and the $\mathcal{D}_i$'s by a t-norm based merging operator. We proved that the resulting knowledge base of the *I-S-C* operator contains more original information than that of the t-conorm based operator. We then proposed another splitting method, called free-formula based splitting. The split-combination merging operator based on this splitting method, called the *F-S-C* operator, can be used to merge classical knowledge bases. We show that it satisfies most of the postulates for merging operators in propositional logic given in [24].

When discussing the logical properties of the *I-S-C* operator, we adapted the set of postulates for merging propositional knowledge bases in [24] to possibilistic logic. However, there is no possibilistic merging operator satisfying all the postulates, especially (P6). This raises a question: can we find a possibilistic merging operator which satisfies most of the postulates? This question will be discussed in a future paper. Another goal for future work is to propose more postulates for merging possibilistic knowledge bases.

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10 Appendix

Proposition 3 The resulting possibilistic knowledge base $\mathcal{B}_{Upper-S-C}$ of the *U-S-C* operator is consistent.

Proof. Suppose $\mathcal{B}_{Upper-S-C}$ is inconsistent, then we have $(\mathcal{C}\cup\mathcal{D})^*\models false$. By Equation 10, we have $\mathcal{C}_1^*\models\mathcal{C}^*$. Therefore $(\mathcal{C}_1\cup\mathcal{D})^*\models false$. However, we have assumed that $\mathcal{B}_1$ is consistent, so $\mathcal{C}_1^*$ must be consistent. Since $\mathcal{D}=\mathcal{D}_1\cup\mathcal{D}_2\cup\{(\phi_i\vee\psi_j, ct(\alpha_i,\beta_j))|(\phi_i,\alpha_i)\in\mathcal{D}_1\ and\ (\psi_j,\beta_j)\in\mathcal{D}_2\}$, $\mathcal{D}^*\equiv\mathcal{D}_1^*\cup\mathcal{D}_2^*$. Therefore, there must exist some formulas in $\mathcal{D}_1^*\cup\mathcal{D}_2^*$ which are in conflict with formulas in $\mathcal{C}^*$. This is a contradiction, because all formulas in $\mathcal{D}_1^* \cup \mathcal{D}_2^*$ are free in $\mathcal{B}_1^* \cup \mathcal{B}_2^*$. This completes our proof.

Proposition 4 Given two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$, suppose $\mathcal{B}_{Upper-S-C}$ and $\mathcal{B}_{I-S-C}$ are the merging results of $\mathcal{B}_1$ and $\mathcal{B}_2$ using the *U-S-C* operator and the *I-S-C* operator respectively, then we have

$$\mathcal{B}_{I-S-C} \vdash (\phi, \alpha), \quad for\ all\ (\phi, \alpha) \in \mathcal{B}_{Upper-S-C},$$

but not vice versa.

Proof. Suppose α_i is the splitting point obtained by Algorithm 1. It is clear that $\alpha_i \leq \alpha_m$, where $\alpha_m = Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$. Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are split by α_m into $\mathcal{B}_1 =< \mathcal{C}_1, \mathcal{D}_1 >$ and $\mathcal{B}_2 =< \mathcal{C}_2, \mathcal{D}_2 >$ respectively. Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are split by α_i into $\mathcal{B}_1 =< \mathcal{C}'_1, \mathcal{D}'_1 >$ and $\mathcal{B}_2 =< \mathcal{C}'_2, \mathcal{D}'_2 >$ respectively. We have $\mathcal{C}'_j \subseteq \mathcal{C}_j$ and $\mathcal{D}_j \subseteq \mathcal{D}'_j$, where $j = 1, 2$. Suppose $(\phi, \alpha) \in \mathcal{B}_{Upper-S-C}$, we need to consider two cases:

Case 1: $(\phi, \alpha) \in \mathcal{D}_1 \cup \mathcal{D}_2$, then we have $(\phi, \alpha) \in \mathcal{D}'_1 \cup \mathcal{D}'_2$. So $\mathcal{B}_{I-S-C} \vdash_\pi (\phi, \alpha)$.

Case 2: $(\phi, \alpha) = (\phi_1 \vee \phi_2, min(\alpha_1, \alpha_2))$, where $(\phi_1, \alpha_1) \in \mathcal{C}_1$ and $(\phi_2, \alpha_2) \in \mathcal{C}_2$. If (ϕ_1, α_1) or (ϕ_2, α_2) belong to $\mathcal{D}'_1 \cup \mathcal{D}'_2$, then we have $\mathcal{B}_{I-S-C} \vdash_\pi (\phi, \alpha)$. Otherwise, $(\phi_1, \alpha_1) \in \mathcal{C}'_1$ and $(\phi_2, \alpha_2) \in \mathcal{C}'_2$. Then $(\phi, \alpha) \in \mathcal{B}_{I-S-C}$, so $\mathcal{B}_{I-S-C} \vdash_\pi (\phi, \alpha)$.

$\mathcal{B}_{I-S-C}$ is not always equivalent to $\mathcal{B}_{Upper-S-C}$. By Example 4 and Example 6, we have $\mathcal{B}_{I-S-C} = \{(\phi \vee \delta, 0.9), (\neg\phi \vee \varphi, 0.8), (\neg\psi \vee \delta, 0.7), (\neg\phi \vee \psi, 0.6), (\delta \vee \varphi, 0.5), (\phi, 0.5)\}$ and $\mathcal{B}_{Upper-S-C} = \{(\phi \vee \delta,\ 0.9), (\neg\phi \vee \varphi,\ 0.8), (\neg\psi \vee \delta,\ 0.7), (\neg\phi \vee \psi \vee \delta \vee \varphi,\ 0.5), (\phi \vee \delta \vee \varphi,\ 0.5), (\phi \vee \psi,\ 0.3), (\neg\phi \vee \psi,\ 0.3)\}$. It is clear that $(\neg\phi \vee \psi, 0.6)$ in $\mathcal{B}_{I-S-C}$ cannot be inferred from $\mathcal{B}_{Upper-S-C}$.

Proposition 5 Given two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$, let $\mathcal{B}_{Upper-S-C}$ be the possibilistic knowledge base obtained by the *U-S-C* operator (which is based on a t-norm *tn* and a t-conorm *ct*) and $\mathcal{B}_{ct}$ be the resulting possibilistic knowledge base of merging using *ct*, then

$$\mathcal{B}_{Upper-S-C} \vdash_\pi (\phi, \alpha), \qquad \text{for all } (\phi, \alpha) \in \mathcal{B}_{ct} \tag{20}$$

Proof. By Equation 7, every formula in $\mathcal{B}_{dm}$ has the form $(\phi_i \vee \psi_j,\ min(\alpha_i,\ \beta_j))$, where $(\phi_i,\ \alpha) \in \mathcal{B}_1$ and $(\psi_j,\ \beta_j) \in \mathcal{B}_2$, so we consider four cases:

Case 1: $\alpha_i, \beta_j \leq Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, then we have $(\phi_i \vee \psi_j, min(\alpha_i, \beta_j)) \in \mathcal{B}_{Upper-S-C}$. So $\mathcal{B}_{Upper-S-C} \vdash_\pi (\phi_i \vee \psi_j, min(\alpha_i, \beta_j))$.

Case 2: $\alpha_i > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, and $\beta_j \leq Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, then $min(\alpha_i,\ \beta_j) = \beta_j$ and $\mathcal{B}_{Upper-S-C} \vdash_\pi (\phi_i, \alpha_i)$. Since $\phi_i \models (\phi_i \vee \psi_j)$ and $\alpha_i \geq \beta_j$, we have $\mathcal{B}_{Upper-S-C} \vdash_\pi (\phi_i \vee \psi_j, min(\alpha_i, \beta_j))$.

Case 3: $\alpha_i \leq Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, and $\beta_j > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, this case is a dual to case 2.

Case 4: $\alpha_i > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$ and $\beta_j > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, we can suppose $\alpha_i > \beta_j$, then $min(\alpha_i, \beta_j) = \beta_j$. Since $\phi_i \models (\phi_i \vee \psi_j)$, we have $\mathcal{B}_{Upper-S-C} \vdash_\pi (\phi_i \vee \psi_j, min(\alpha_i, \beta_j))$.

Proposition 6 Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two possibilistic knowledge bases. Suppose *tn* is the minimum and *ct* is an arbitrary t-conorm in Algorithm 1. Let γ be the splitting point obtained by Algorithm 1. Suppose $\gamma = Inc(\mathcal{B}_1 \cup \mathcal{B}_2)$, then $\mathcal{B}_{N,min} \subseteq \mathcal{B}_{I-S-C}$, but not vice versa.

Proof. Since $\gamma = Inc(\mathcal{B}_1 \cup \mathcal{B}_2)$ and $tn = min$, we have that $\mathcal{D}_1 \cup \mathcal{D}_2 = B_{N,min}$. Therefore, $\mathcal{B}_{N,min} \subseteq \mathcal{B}_{I-S-C}$. It is clear that the converse is false because $\mathcal{B}_{I-S-C} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{C}$ and $\mathcal{C}$ is not empty if $Inc(\mathcal{C}) > 0$.

Proposition 8 The *G-I-S-C* operator $\Delta_{G-I-S-C}$ satisfies (P1)-(P3). It does not satisfy (P4)-(P6) in general.

Proof. It is clear that (P1) holds.

(P2)If $\mathcal{B}_1 \cup ... \cup \mathcal{B}_n$ is consistent, then $Inc(\mathcal{B}) = 0$. So $Inc(\mathcal{B}) = \gamma_{n_1+1}$ and $l = n_1 + 1$. Then $\mathcal{C}_i = \emptyset$ and $\mathcal{D}_i = \mathcal{B}_i$ for $i = 1, 2$. So $\Delta_{G-I-S-C}(\mathcal{E}) = \Delta_{tn}(\mathcal{E})$, where $\Delta_{tn}(\mathcal{E})$ is the possibilistic knowledge base that results from merging $\mathcal{B}_i$ using the t-norm tn. By Equation 9, it is clear that (P2) holds.

(P3) Before the proof of (P3), we need two lemmas.

Lemma 3 *Suppose $\mathcal{E}_1 \equiv_s \mathcal{E}_2$, then $\cup\mathcal{E}_1 \equiv_s \cup\mathcal{E}_2$.*

Lemma 4 *Let $\mathcal{E}_1 = \{\mathcal{B}_1, ..., \mathcal{B}_n\}$ and $\mathcal{E}_2 = \{\mathcal{B}'_1, ..., \mathcal{B}'_n\}$, where $\mathcal{B}_i$ and $\mathcal{B}'_i$ are individually consistent. Suppose $\mathcal{E}_1 \equiv_s \mathcal{E}_2$, and $\mathcal{B}_i$ and $\mathcal{B}'_i$ $(i = 1, ..., n)$ are split w.r.t a weight γ into $\mathcal{B}_i = \langle \mathcal{C}_i, \mathcal{D}_i \rangle$ and $\mathcal{B}'_i = \langle \mathcal{C}'_i, \mathcal{D}'_i \rangle$ respectively, then $\mathcal{C}_i \equiv_s \mathcal{C}'_i$ and $\mathcal{D}_i \equiv_s \mathcal{D}'_i$.*

The proofs of Lemma 3 and Lemma 4 are clear from the definitions of equivalence of possibilistic knowledge profiles and possibilistic knowledge bases and so we do not provide them here.

We now continue the proof of (P3).

Suppose $\gamma^{\mathcal{E}_1}$ and $\gamma^{\mathcal{E}_2}$ are the splitting points of $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively obtained by Algorithm 2. We now prove that $\Delta_{G-I-S-C}(\mathcal{E}_1) \equiv_s \Delta_{G-I-S-C}(\mathcal{E}_2)$ by induction over the number of times t that the algorithm goes to Step 3. It is clear that if $\cup\mathcal{E}_1$ or $\cup\mathcal{E}_2$ is consistent, the conclusion holds. We assume $\cup\mathcal{E}_i$ $(i = 1, 2)$ are inconsistent.

1) When $t = 1$, by Lemma 3, $\cup\mathcal{E}_1 \equiv_s \cup\mathcal{E}_2$, we have $Inc(\cup\mathcal{E}_1) = Inc(\cup\mathcal{E}_2) = \gamma_k$ and then $l_{\mathcal{E}_1} = l_{\mathcal{E}_2} = k$. Suppose $\mathcal{B}^{\mathcal{E}_i}_{S-C}$ $(i = 1, 2)$ are resulting possibilistic knowledge bases of merging $\mathcal{E}_i$ by the *S-C* operator which uses γ_k as the splitting point. By Lemma 4, it is clear that $\mathcal{B}^{\mathcal{E}_1}_{S-C} \equiv_s \mathcal{B}^{\mathcal{E}_2}_{S-C}$. If $\mathcal{B}^{\mathcal{E}_1}_{S-C}$ (or $\mathcal{B}^{\mathcal{E}_2}_{S-C}$) is consistent, then the algorithm stops and we can conclude that $\Delta_{G-I-S-C}(\mathcal{E}_1) \equiv_s \Delta_{G-I-S-C}(\mathcal{E}_2)$. Otherwise, the Algorithm goes to Step 3 again.

2) Suppose $t = j$, where $j > 1$. Let γ_{l_1} and γ_{l_2} be the splitting point for $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively, $\mathcal{B}^{\mathcal{E}_1,j}_{S-C}$ and $\mathcal{B}^{\mathcal{E}_2,j}_{S-C}$ be the resulting possibilistic knowledge bases of merging $\mathcal{E}_1$ and $\mathcal{E}_2$ by the *S-C* operator which uses γ_{l_1} and γ_{l_2} as the splitting point respectively. Let us assume that $\gamma_{l_1} = \gamma_{l_2}$, $\mathcal{B}^{\mathcal{E}_1,j}_{S-C} \equiv_s \mathcal{B}^{\mathcal{E}_2,j}_{S-C}$, and $\mathcal{B}^{\mathcal{E}_i,j}_{S-C}$ $(i = 1, 2)$ are inconsistent. Since $\gamma_{l_1} = \gamma_{l_2}$, we have $l_1 = l_2$. Suppose $t = j + 1$. It is clear that $\gamma_{l_1-1} = \gamma_{l_2-1}$. Let $\mathcal{B}^{\mathcal{E}_1,j+1}_{S-C}$ and $\mathcal{B}^{\mathcal{E}_2,j+1}_{S-C}$ be the resulting possibilistic knowledge bases of merging $\mathcal{E}_1$ and $\mathcal{E}_2$ by the *S-C* operator which uses γ_{l_1-1} and γ_{l_2-1} as the splitting point respectively. We only need to prove that $\mathcal{B}^{\mathcal{E}_1,j+1}_{S-C} \equiv_s \mathcal{B}^{\mathcal{E}_2,j+1}_{S-C}$. This is easy to prove from Lemma 4. If $\mathcal{B}^{\mathcal{E}_1,j+1}_{S-C}$ is consistent, then the algorithm stops and we can conclude that $\Delta_{G-I-S-C}(\mathcal{E}_1) \equiv_s \Delta_{G-I-S-C}(\mathcal{E}_2)$. Otherwise, the Algorithm goes to Step 3.

By 1) and 2), it follows that $\Delta_{G-I-S-C}(\mathcal{E}_1) \equiv_s \Delta_{G-I-S-C}(\mathcal{E}_2)$.

(P4)-(P6): Let us consider the following counterexample.

Let $\mathcal{E}_1 = \{\mathcal{B}_1, \mathcal{B}_2\}$ and $\mathcal{E}_2 = \{\mathcal{B}_3, \mathcal{B}_4\}$, where $\mathcal{B}_1 = \{(\phi, 0.9), (\neg\psi, 0.7)\}$, $\mathcal{B}_2 = \{(\psi, 0.8), (\neg\phi, 0.6)\}$, $\mathcal{B}_3 = \{(\varphi, 0.4)\}$ and $\mathcal{B}_4 = \{(\chi, 0.4)\}$. Let $ct = max$ and $tn = min$. It is easy to check that $\Delta_{G-I-S-C}(\mathcal{E}_1) = \{(\phi, 0.9), (\neg\phi \vee \neg\psi, 0.6)\}$, $\Delta_{G-I-S-C}(\mathcal{E}_2) = \{(\varphi, 0.4), (\chi, 0.4)\}$. Since $\Delta_{G-I-S-C}$ $(\mathcal{E}_1) \cup \Delta_{G-I-S-C}(\mathcal{E}_2)$ is consistent, $\Delta_{G-I-S-C}(\{\Delta_{G-I-S-C}(\mathcal{E}_1)\}, \{\Delta_{G-I-S-C}(\mathcal{E}_2)\}) = \{(\phi, 0.9), (\neg\phi \vee \neg\psi, 0.6), (\varphi, 0.4), (\chi, 0.4)\}$. In contrast, we have $\Delta_{G-I-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2) = \{(\phi, 0.9), (\psi, 0.8), (\neg\phi \vee \neg\psi \vee \varphi \vee \chi, 0.4)\}$. $(\chi, 0.4) \in \Delta_{G-I-S-C}(\{\Delta_{G-I-S-C}(\mathcal{E}_1)\}, \{\Delta_{G-I-S-C}(\mathcal{E}_2)\})$, but it cannot be inferred from $\Delta_{G-I-S-C}$ $(\mathcal{E}_1 \sqcup \mathcal{E}_2)$. Conversely, $(\psi, 0.8) \in \Delta_{G-I-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2)$, but it cannot be inferred from $\Delta_{G-I-S-C}(\{\Delta_{G-I-S-C}(\mathcal{E}_1)\}, \{\Delta_{G-I-S-C}(\mathcal{E}_2)\})$. So (P4) and (P5) are not satisfied. We consider (P6). Since $\Delta_{G-I-S-C}(\mathcal{E}_1)^* \wedge \Delta_{G-I-S-C}(\mathcal{E}_2)^* \equiv \phi \wedge \neg\psi \wedge \varphi \wedge \chi$ and $(\Delta_{G-I-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2))^* \equiv \phi \wedge \psi \wedge (\varphi \vee \chi)$, so $\Delta_{G-I-S-C}(\mathcal{E}_1)^* \wedge \Delta_{G-I-S-C}(\mathcal{E}_2)^* \not\equiv (\Delta_{G-I-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2))^*$. However, $\Delta_{G-I-S-C}(\mathcal{E}_1) \cup \Delta_{G-I-S-C}(\mathcal{E}_2)$ is consistent, so (P6) is not satisfied.

Proposition 9 The t-conorm based merging operator Δ_{ct} satisfies (P1),(P3)-(P5). It does not satisfy (P2) and (P6) in general. The operator $\Delta_{N,tn}$ satisfies (P1)-(P3). It does not satisfy (P4)-(P6) in general.

Proof. We first consider operator Δ_{ct}.

It is clear that (P1) is satisfied by Δ_{ct}. We give a counterexample for (P2).

Example 15 *Let $\mathcal{E} = \{\mathcal{B}_1, \mathcal{B}_2\}$, where $\mathcal{B}_1 = \{(p, 0.7)\}$ and $\mathcal{B}_2 = \{(q, 0.9)\}$, where p, q are two propositional symbols. Let ct be a t-conorm, By Equation 10, $\Delta_{ct}(\mathcal{E}) = \{(p \vee q, \alpha)\}$, where $\alpha = tn(0.7, 0.9)$. It is easy to check that $\Delta_{ct}(\mathcal{E}) \not\vdash_\pi (p, 0.7)$, so $\nexists \beta \geq 0.7$ such that $\Delta_{ct}(\mathcal{E}) \not\vdash_\pi (p, \beta)$.*

(P3) holds because Δ_{ct} is semantically defined by aggregation of possibility distributions associated with $\mathcal{B}_i$.

(P4) and (P5): Let $\pi_{ct,\mathcal{E}_1}$, $\pi_{ct,\mathcal{E}_2}$ and $\pi_{ct,\mathcal{E}_1 \sqcup \mathcal{E}_2}$ be possibility distributions obtained by using ct to aggregate possibility distributions associated with possibilistic knowledge bases of $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_1 \sqcup \mathcal{E}_2$. Let π_{ct} be the possibility distribution obtained by aggregating $\pi_{ct,\mathcal{E}_1}$ and $\pi_{ct,\mathcal{E}_2}$ using ct. To prove $\Delta(\{\Delta_{ct}(\mathcal{E}_1)\}, \{\Delta_{ct}(\mathcal{E}_2)\}) \equiv_s \Delta(\mathcal{E}_1 \sqcup \mathcal{E}_2)$, we only need to prove $\pi_{ct}(\omega) = \pi_{ct,\mathcal{E}_1 \sqcup \mathcal{E}_2}(\omega)$, for all ω. This equation holds because ct is an associative and commutative operator.

It is clear that Δ_{ct} does not satisfy (P6).

Next we consider operator $\Delta_{N,tn}$.

(P1): By Equation 11, (P1) clearly holds.

(P2): If $\mathcal{B}_1 \cup ... \cup \mathcal{B}_n$ is consistent, $\Delta_{N,tn}(\mathcal{E}) = \Delta_{tn}(\mathcal{E})$, where $\Delta_{tn}(\mathcal{E})$ is the possibilistic knowledge base that results from merging $\mathcal{B}_i$ using the t-norm tn. By Equation 9, it is clear that (P2) holds.

(P3): By Equation 11, it is clear that (P3) holds due to $\Delta_{tn}(\mathcal{E}_1) \equiv_s \Delta_{tn}(\mathcal{E}_2)$.

(P4) and (P5): Let us consider the following counterexample.

Let $\mathcal{E}_1 = \{\mathcal{B}_1, \mathcal{B}_2\}$ and $\mathcal{E}_2 = \{\mathcal{B}_3, \mathcal{B}_4\}$, where $\mathcal{B}_1 = \{(\phi, 0.8), (\psi, 0.4), (\varphi, 0.4)\}$, $\mathcal{B}_2 = \{(\phi \rightarrow \neg\psi, 0.6)\}$, $\mathcal{B}_3 = \{(\varphi, 0.4)\}$ and $\mathcal{B}_4 = \{(\chi, 0.4)\}$. Let tn be the product operator. By Equation 9, we have

$$
\begin{aligned}
\Delta_{tn}(\mathcal{E}_1) &= \{(\phi, 0.8), (\neg\phi \vee \neg\psi \vee \varphi, 0.76), (\phi \rightarrow \neg\psi, 0.6), (\psi, 0.4), (\varphi, 0.4)\} \\
\Delta_{tn}(\mathcal{E}_2) &= \{(\varphi \vee \chi, 0.64), (\varphi, 0.4), (\chi, 0.4)\}.
\end{aligned}
$$

So

$$
\begin{aligned}
\Delta_{N,tn}(\mathcal{E}_1) &= \{(\phi, 0.8), (\neg\phi \vee \neg\psi \vee \varphi, 0.76), (\phi \rightarrow \neg\psi, 0.6)\} \\
\Delta_{N,tn}(\mathcal{E}_2) &= \{(\varphi \vee \chi, 0.64), (\varphi, 0.4), (\chi, 0.4)\}.
\end{aligned}
$$

Combining $\Delta_{N,tn}(\mathcal{E}_1)$ and $\Delta_{N,tn}(\mathcal{E}_2)$ by tn we get

$$
\begin{aligned}
\Delta_{tn}(\{\Delta_{N,tn}(\mathcal{E}_1)\}, \{\Delta_{N,tn}(\mathcal{E}_2)\}) \equiv_s \; & \{(\phi \vee \varphi \vee \chi, 0.93), (\neg\phi \vee \neg\psi \vee \varphi \vee \chi, 0.91), \\
& (\phi \vee \varphi, 0.88), (\phi \vee \chi, 0.88), (\neg\phi \vee \neg\psi \vee \chi, 0.86), \\
& (\neg\phi \vee \neg\psi \vee \varphi, 0.85), (\phi, 0.8), (\varphi \vee \chi, 0.64), \\
& (\phi \rightarrow \neg\psi, 0.6), (\varphi, 0.4), (\chi, 0.4)\}.
\end{aligned}
$$

Since $\Delta_{tn}(\{\Delta_{N,tn}(\mathcal{E}_1)\}, \{\Delta_{N,tn}(\mathcal{E}_2)\})$ is consistent, we have $\Delta_{N,tn}(\{\Delta_{N,tn}(\mathcal{E}_1)\}, \{\Delta_{N,tn}(\mathcal{E}_2)\}) = \Delta_{tn}(\{\Delta_{N,tn}(\mathcal{E}_1)\}, \{\Delta_{N,tn}(\mathcal{E}_2)\})$. By contrast, since $\mathcal{E}_1 \sqcup \mathcal{E}_2 = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\}$, we have

$$\begin{aligned}\Delta_{tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2) \quad \equiv_s \quad & \{(\phi \vee \varphi \vee \chi, 0.93), (\neg\phi \vee \neg\psi \vee \varphi \vee \chi, 0.91) \\ & (\phi \vee \varphi, 0.88), (\phi \vee \chi, 0.88), (\neg\phi \vee \neg\psi \vee \chi, 0.86), \\ & (\neg\phi \vee \neg\psi \vee \varphi, 0.85), (\phi, 0.8), (\varphi \vee \chi, 0.78), \\ & (\psi \vee \varphi \vee \chi, 0.78), (\neg\phi \vee \neg\psi \vee \neg\varphi, 0.76), (\varphi \vee \chi, 0.64), \\ & (\psi \vee \varphi, 0.64), (\psi \vee \chi, 0.64), (\varphi, 0.64), (\varphi \vee \psi, 0.64), \\ & (\phi \to \neg\psi, 0.6), (\varphi, 0.4), (\chi, 0.4), (\psi, 0.4)\}.\end{aligned}$$

The inconsistency degree of $\Delta_{tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2)$ is 0.4, so

$$\begin{aligned}\Delta_{N,tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2) \quad = \quad & \{(\phi \vee \varphi \vee \chi, 0.93), (\phi \vee \varphi, 0.88), (\phi \vee \chi, 0.88), \\ & (\neg\phi \vee \neg\psi \vee \chi, 0.86), (\neg\phi \vee \neg\psi \vee \varphi \vee \chi, 0.85), (\phi, 0.8), \\ & (\varphi \vee \chi, 0.78), (\psi \vee \varphi \vee \chi, 0.78), (\neg\phi \vee \neg\psi \vee \varphi, 0.76), \\ & (\neg\phi \vee \neg\psi \vee \chi, 0.76), (\neg\phi \vee \neg\psi \vee \neg\varphi, 0.76), (\varphi \vee \chi, 0.64), \\ & (\psi \vee \varphi, 0.64), (\psi \vee \chi, 0.64), (\varphi, 0.64), \\ & (\varphi \vee \psi, 0.64), (\phi \to \neg\psi, 0.6)\}.\end{aligned}$$

It is clear that $(\varphi, 0.4)$ in $\Delta_{N,tn}(\{\Delta_{N,tn}(\mathcal{E}_1)\}, \{\Delta_{N,tn}(\mathcal{E}_2)\})$ cannot be inferred from $\Delta_{N,tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2)$ and $(\varphi, 0.64)$ in $\Delta_{N,tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2)$ cannot be inferred from $\Delta_{N,tn}(\{\Delta_{N,tn}(\mathcal{E}_1)\}, \{\Delta_{N,tn}(\mathcal{E}_2)\})$.

(P6) Let us look at the counterexample in the proof of Proposition 8. Let $tn = min$. $\Delta_{N,tn}(\mathcal{E}_1) = \{(\phi, 0.9), (\psi, 0.8)\}$ and $\Delta_{N,tn}(\mathcal{E}_2) = \{(\varphi, 0.4), (\chi, 0.4)\}$. Clearly, $\Delta_{N,tn}(\mathcal{E}_1) \cup \Delta_{N,tn}(\mathcal{E}_2)$ is consistent. We also have $\Delta_{N,tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2) = \{(\phi, 0.9), (\psi, 0.8)\}$. It is clear that $\Delta_{N,tn}(\mathcal{E}_1)^* \wedge \Delta_{N,tn}(\mathcal{E}_2)^* \not\equiv (\Delta_{N,tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2))^*$.

Proposition 10 Generating a consistent possibilistic knowledge base by the *I-S-C* operator is in $F\Delta_2^p$ $(\mathcal{O}(n))$, where $\Delta_2^p(\mathcal{O}(n))$ denotes the set of decision problems decidable in polynomial time with no more than $\mathcal{O}(n)$ calls to an *NP* oracle, where n is the number of different valuations involved in $\mathcal{B}$ [22], and "F" in $F\Delta_2^p(\mathcal{O}(n))$ stands for *function* and is intended to turn a complexity class for decision problems into one for *search problems*, i.e., problems that have answers.

Proof. In Step 1, rearranging the weights of formulas in $\mathcal{B}$ is a *sort problem*, which can be solved in polynomial time. By Proposition 13 in [29], computing $Inc(\mathcal{B})$ is NP-hard and requires $\lceil log_2 n \rceil$ satisfiability checks. Steps 3, 4 and 5 can be carried out in polynomial time. Step 6 needs a satisfiability check, so it is NP-hard. Since $Inc(\mathcal{B})$ and $Free_{upp}(\mathcal{B})$ are lower and upper bounds respectively for the splitting point, we know that Steps 3-6 will be repeated at most k-m+1 times, where k and m are such that $a_k = Inc(\mathcal{B})$ and $a_m = Free_{upp}(\mathcal{B})$. Therefore, computation of the *I-S-C* operator needs at most k-m+$\lceil log_2 n \rceil$+1 satisfiability checks. This proves the proposition.

Proposition 11 Generating a consistent possibilistic knowledge base by a t-conorm based merging operator is in FP and generating a consistent possibilistic knowledge base by a renormalization based merging operator is in $F\Delta_2^p$.

Proof. To generate a possibilistic knowledge base by a t-conorm based merging operator, we only need to take the disjunctions of formulas from the individual possibilistic knowledge bases and compute the weights associated with the disjunctions. Both computations can be done in polynomial time.

The renormalization based merging operator is computed in two steps. The first step is to combine the original possibilistic knowledge bases using a t-norm. This step can be done in polynomial time. In the second step, we need to compute the inconsistency degree of the possibilistic knowledge base obtained by the first step. Then only those possibilistic formulas whose weights are greater than the inconsistency degree are kept in the resulting possibilistic knowledge base. By Proposition 13 in [29], computing the inconsistency degree of a possibilistic knowledge base is NP-hard and requires $\lceil log_2 n \rceil$ satisfiability checks, where n is the number of different valuations involved in $\mathcal{B}$. Therefore, computation of renormalization based operator needs at most $\lceil log_2 n \rceil$ satisfiability checks. This proves the proposition.

Proposition 14 Let $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, ..., n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, ..., m\}$ be two possibilistic knowledge bases and let $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ be their respective associated possibility distributions. We have $\pi_{\mathcal{B}_{I-S-C}}(\omega) = \pi_{I-S-C}(\omega)$, for all ω.

Proof. Let us compare Algorithm 1 and Algorithm 3 step by step.

Step 1 and Step 2: both algorithms have the same Step 1 and Step 2;

Step 3: by Proposition 12, we have $\pi_{11} = \pi_{\mathcal{C}_1}$ and $\pi_{12} = \pi_{\mathcal{D}_1}$, and $\pi_{21} = \pi_{\mathcal{C}_2}$ and $\pi_{22} = \pi_{\mathcal{D}_2}$;

Step 4: by Equation 6 and discussions in Section 2, we have $\pi_1 = \pi_{\mathcal{C}}$ and $\pi_2 = \pi_{\mathcal{D}}$;

Step 5: by Equation 6 and discussions in Section 2, we have $\pi_{I-S-C} = \pi_{\mathcal{C}\cup\mathcal{D}}$, that is, $\pi_{I-S-C} = \pi_{\mathcal{B}_{I-S-C}}$;

Step 6: Since $\pi_{I-S-C} = \pi_{\mathcal{B}_{I-S-C}}$, π_{I-S-C} is subnormal if and only if $\mathcal{B}_{I-S-C}$ is inconsistent.

By the comparison above, we can infer that Algorithm 1 and Algorithm 3 output the same splitting point γ. Therefore, by Proposition 12, it is clear that $\pi_{\mathcal{B}_{I-S-C}} = \pi_{I-S-C}$.

Proposition 15 Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two possibilistic knowledge bases and let $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ be their respective associated possibility distributions. Let π_{I-S-C} be the resulting possibility distribution of merging by the semantic *I-S-C* operator and π_{ct} be the resulting possibility distribution by the t-conorm, then π_{I-S-C} is more specific than π_{ct}, that is $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$ for all $\omega \in \Omega$.

Proof. Let ω be an arbitrary possible world. By Definition 17, we have $\pi_{I-S-C}(\omega) = min(ct(\pi_{11}(\omega), \pi_{21}(\omega)), tn(\pi_{12}(\omega), \pi_{22}(\omega)))$. By contrast, $\pi_{ct}(\omega) = ct(\pi_{\mathcal{B}_1}(\omega), \pi_{\mathcal{B}_2}(\omega))$. Since $\pi_{\mathcal{B}_1}(\omega) = min(\pi_{11}(\omega), \pi_{12}(\omega))$ and $\pi_{\mathcal{B}_2}(\omega) = min(\pi_{21}(\omega), \pi_{22}(\omega))$, we have $\pi_{ct}(\omega) = ct(min(\pi_{11}(\omega), \pi_{12}(\omega)), min(\pi_{21}(\omega), \pi_{22}(\omega)))$. We now prove that $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$. Since $tn(\pi_{12}(\omega), \pi_{22}(\omega)) \leq min(\pi_{12}(\omega), \pi_{22}(\omega))$, $\pi_{I-S-C}(\omega) \leq min(ct(\pi_{11}(\omega), \pi_{21}(\omega)), min(\pi_{12}(\omega), \pi_{22}(\omega)))$. We consider the following two cases:

Case 1: Suppose $min(\pi_{12}(\omega), \pi_{22}(\omega)) \geq ct(\pi_{11}(\omega), \pi_{21}(\omega))$. Then $\pi_{I-S-C} \leq ct(\pi_{11}(\omega), \pi_{21}(\omega))$. Since $ct(\pi_{11}(\omega), \pi_{21}(\omega)) \geq max(\pi_{11}(\omega), \pi_{21}(\omega))$, we have $min(\pi_{12}(\omega), \pi_{22}(\omega)) \geq max(\pi_{11}(\omega), \pi_{21}(\omega))$. So $\pi_{11}(\omega) \leq min(\pi_{12}(\omega), \pi_{22}(\omega))$ and $\pi_{21}(\omega) \leq min(\pi_{12}(\omega), \pi_{22}(\omega))$. We then have $\pi_{ct}(\omega) = ct(\pi_{11}(\omega), \pi_{21}(\omega))$. Therefore, $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$.

Case 2: Suppose $min(\pi_{12}(\omega), \pi_{22}(\omega)) < ct(\pi_{11}(\omega), \pi_{21}(\omega))$. Then $\pi_{I-S-C} \leq min(\pi_{12}(\omega), \pi_{22}(\omega))$. We can prove that $min(\pi_{12}(\omega), \pi_{22}(\omega)) \leq \pi_{ct}(\omega)$. Suppose either $\pi_{11}(\omega) \geq \pi_{12}(\omega)$ or $\pi_{21}(\omega) \geq \pi_{22}(\omega)$, then $\pi_{ct}(\omega) \geq min(\pi_{12}(\omega), \pi_{22}(\omega))$. So $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$. Otherwise, suppose $\pi_{11}(\omega) < \pi_{12}(\omega)$ and $\pi_{21}(\omega) < \pi_{22}(\omega)$, then $\pi_{ct}(\omega) = ct(\pi_{11}(\omega), \pi_{21}(\omega))$. By assumption, we have $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$.

According to Case 1 and Case 2, we can conclude that $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$. This completes the proof.

Proposition 16 Given two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$, if $\mathcal{B}_{F-S-C}$ is the possibilistic knowledge base obtained by the *F-S-C* method and $\mathcal{B}_{Upper-S-C}$ is the possibilistic knowledge base obtained by the *U-S-C* method, then

$$\mathcal{B}_{F-S-C} \vdash_\pi (\phi,\ \alpha), \quad \text{for all } (\phi,\ \alpha) \in \mathcal{B}_{Upper-S-C}, \tag{21}$$

but not vice verse.

Proof. Let $(\varphi,\delta)\in\mathcal{B}_{Upper-S-C}$, since $\mathcal{B}_{Upper-S-C}=\mathcal{C}\cup\mathcal{D}$, where $\mathcal{C}$ is obtained by Equation 12 and $\mathcal{D}$ is obtained by Equation 13, we have $(\varphi,\delta)\in\mathcal{C}\cup\mathcal{D}$. On the one hand, suppose $(\varphi,\delta)\in\mathcal{D}$, then $(\varphi,\delta)\in\mathcal{D}_1\cup\mathcal{D}_2$. By Lemma 1, $\mathcal{D}_1\subseteq\mathcal{D}_{Free_1}$ and $\mathcal{D}_2\subseteq\mathcal{D}_{Free_2}$. So $(\varphi,\delta)\in\mathcal{D}_{Free_1}\cup\mathcal{D}_{Free_2}=\mathcal{D}_{Free}$. Since $\mathcal{B}_{F-S-C}=\mathcal{C}_{Con}\cup\mathcal{D}_{Free}$, we have $(\varphi,\delta)\in\mathcal{B}_{F-S-C}$, so $\mathcal{B}_{F-S-C}\vdash_\pi(\varphi,\delta)$. On the other hand, suppose $(\varphi,\delta)\in\mathcal{C}$, then (φ,δ) has the form $(\phi_i\vee\psi_j,min(\alpha_i,\beta_j))$, where $(\phi_i,\alpha_i)\in\mathcal{C}_1$ and $(\psi_j,\beta_j)\in\mathcal{C}_2$. By Lemma 1, $\mathcal{C}_{Con_1}\subseteq\mathcal{C}_1$ and $\mathcal{C}_{Con_2}\subseteq\mathcal{C}_2$. We consider the following two cases:

Case 1: $(\phi_i,\alpha_i)\in\mathcal{C}_{Con_1}$ and $(\psi_j,\delta_j)\in\mathcal{C}_{Con_2}$. In this case, we have $(\varphi,\delta)=(\phi_i\vee\psi_j,min(\alpha_i,\beta_j))\in\mathcal{C}_{Con}$. So $(\varphi,\delta)\in\mathcal{B}_{F-S-C}$ and $\mathcal{B}_{F-S-C}\vdash_\pi(\varphi,\delta)$.

Case 2: $(\phi_i,\alpha_i)\notin\mathcal{C}_{Con_1}$ or $(\psi_j,\beta_j)\notin\mathcal{C}_{Con_2}$. Assume $(\psi_j,\beta_j)\notin\mathcal{C}_{Con_2}$(for $(\phi_i,\alpha_i)\notin\mathcal{C}_{Con_1}$, the proof is similar). In this case, $(\psi_j,\beta_j)\in\mathcal{D}_{Free}$, so $(\psi_j,\beta_j)\in\mathcal{B}_{F-S-C}$. Since $\psi_j\models\varphi$, and $\beta_j\geq min(\alpha_i,\beta_j)=\delta$, we have $\mathcal{B}_{F-S-C}\vdash_\pi(\varphi,\delta)$.

Conversely, let us look at Example 8 again. $Free_{upper}(\mathcal{B}_1\cup\mathcal{B}_2)=0.7$. So $\mathcal{B}_1$ is split into $\mathcal{C}_1=\{(\phi,0.7),(\psi,0.7)\}$ and $\mathcal{D}_1=\emptyset$, and $\mathcal{B}_2$ is split into $\mathcal{C}_2=\{(\neg\phi,0.6),(\psi,0.4)\}$ and $\mathcal{D}_2=\emptyset$. So $\mathcal{B}_{Upper-S-C}=\{(\phi\vee\psi,0.4),(\neg\phi\vee\psi,0.6),(\psi,0.4)\}$. By Example 11, $\mathcal{B}_{F-S-C}=\{(\psi,0.88)\}$. So every possibilistic formula in $\mathcal{B}_{Upper-S-C}$ can be inferred from $\mathcal{B}_{F-S-C}$. By contrast, $(\psi,0.88)$ in $\mathcal{B}_{F-S-C}$ cannot be inferred from $\mathcal{B}_{Upper-S-C}$.

Proposition 17 The *F-S-C* operator $\triangle_{F-S-C}$ satisfies (P1), (P2), (P4). It does not satisfy (P3), (P5) and (P6) in general.

Proof. By Lemma 2, (P1) is satisfied. (P2) is satisfied because the *F-S-C* operator is equivalent to the t-norm based operator when $\mathcal{B}_1\cup...\cup\mathcal{B}_n$ is consistent. (P3) is falsified. Let us look at a counterexample.

Example 16 *Let* $\mathcal{E}_1=\{\mathcal{B}_1,\mathcal{B}_2\}$, *where* $\mathcal{B}_1=\{(\phi,0.6),(\psi,0.6)\}$ *and* $\mathcal{B}_2=\{(\neg\psi,0.9)\}$. *Let* $\mathcal{E}_2=\{\mathcal{B}_3,\mathcal{B}_4\}$, *where* $\mathcal{B}_3=\{(\phi\wedge\psi,0.6)\}$ *and* $\mathcal{B}_4=\{(\neg\psi,0.9)\}$. *It is clear that* $\mathcal{E}_1\equiv_s\mathcal{E}_2$. *Suppose* $ct=max$ *and* $tn=min$. *We have* $\Delta_{F-S-C}(\mathcal{E}_1)=\{(\phi,0.6)\}$ *and* $\Delta_{F-S-C}(\mathcal{E}_2)=\{(\phi\vee\neg\psi,0.6)\}$. *So* $\Delta_{F-S-C}(\mathcal{E}_1)\not\equiv_s\Delta_{F-S-C}(\mathcal{E}_2)$.

(P4): Let $\mathcal{E}_1=\{\mathcal{B}_1,...,\mathcal{B}_m\}$ and $\mathcal{E}_2=\{\mathcal{B}_{m+1},...,\mathcal{B}_n\}$. Let $\mathcal{B}=\mathcal{B}_1\cup...\cup\mathcal{B}_n$. Suppose $\mathcal{B}_i$ $(i=1,...,n)$ are split *w.r.t* $Free(\mathcal{B})$ into $\mathcal{C}_i$ and $\mathcal{D}_i$. Let $\mathcal{C}$ and $\mathcal{D}$ be the resulting possibilistic knowledge bases of merging $\mathcal{C}_i$ by a t-conorm ct (its dual t-norm is tn) and $\mathcal{D}_i$ by a t-norm tn' (its dual t-conorm is ct'). By Equation 10 and Equation 9, we have $\mathcal{C}=\{(\phi_1\vee...\vee\phi_n,tn(\alpha_1,...,\alpha_n)):(\phi_i,\alpha_i)\in\mathcal{C}_i\}$ and $\mathcal{D}=\mathcal{D}_1\cup...\cup\mathcal{D}_n\cup\{(D_j,1-tn'(x_1,...,x_n)):j=2,...,n\}\}$ (see Proposition 2 for the definition of D_j). Then $\Delta_{F-S-C}(\mathcal{E}_1\sqcup\mathcal{E}_2)=\mathcal{C}\cup\mathcal{D}$. Next, suppose $\mathcal{B}_i$ $(i=1,...,m)$ are split *w.r.t* $Free(\mathcal{B}_1\cup...\cup\mathcal{B}_m)$ into $\mathcal{C}'_i$ and $\mathcal{D}'_i$, and $\mathcal{B}_i$ $(i=m+1,...,n)$ are split *w.r.t* $Free(\mathcal{B}_{m+1}\cup...\cup\mathcal{B}_n)$ into $\mathcal{C}'_i$ and $\mathcal{D}'_i$. It is clear that $\mathcal{C}'_i\subseteq\mathcal{C}_i$ and $\mathcal{D}_i\subseteq\mathcal{D}'_i$. Let $\mathcal{C}'$ and $\mathcal{D}'$ be the resulting possibilistic knowledge bases of merging $\mathcal{C}'_i$ $(i=1,...,m)$ by ct and $\mathcal{D}'_i$ $(i=1,...,m)$ by tn', and $\mathcal{C}''$ and $\mathcal{D}''$ be the resulting possibilistic knowledge bases of merging $\mathcal{C}'_i$ $(i=m+1,...,n)$ by ct and $\mathcal{D}'_i$ $(i=m+1,...,n)$ by tn'. By Equations 10 and 9, we have $\mathcal{C}'=\{(\phi_1\vee...\vee\phi_m,tn(\alpha_1,...,\alpha_m)):(\phi_i,\alpha_i)\in\mathcal{C}'_i,i=1,...,m\}$ and $\mathcal{D}'=\mathcal{D}'_1\cup...\cup\mathcal{D}'_m\cup\{(D'_j,1-tn'(x_1,...,x_m)):j=2,...,m\}\}$, and $\mathcal{C}''=\{(\phi_{m+1}\vee...\vee\phi_n,tn(\alpha_{m+1},...,\alpha_n)):(\phi_i,\alpha_i)\in\mathcal{C}'_i,i=m+1,...,n\}$ and $\mathcal{D}''=\mathcal{D}'_{m+1}\cup...\cup\mathcal{D}'_n\cup\{(D''_j,1-tn'(x_1,...,x_n)):j=2,...,n\}\}$. By Definition 19, $\Delta_{F-S-C}(\mathcal{E}_1)=\mathcal{C}'\cup\mathcal{D}'$ and $\Delta_{F-S-C}(\mathcal{E}_2)=\mathcal{C}''\cup\mathcal{D}''$. Suppose $\Delta_{F-S-C}(\mathcal{E}_1)$ is split *w.r.t* $Free(\Delta_{F-S-C}(\mathcal{E}_1)\cup\Delta_{F-S-C}(\mathcal{E}_2))$ into $\mathcal{C}_{\mathcal{E}_1}$ and $\mathcal{D}_{\mathcal{E}_1}$, and $\Delta_{F-S-C}(\mathcal{E}_2)$ is split *w.r.t* $Free(\Delta_{F-S-C}(\mathcal{E}_1)\cup\Delta_{F-S-C}(\mathcal{E}_2))$ into $\mathcal{C}_{\mathcal{E}_2}$ and $\mathcal{D}_{\mathcal{E}_2}$. Let $\mathcal{C}_{\mathcal{E}_1,\mathcal{E}_2}$ and $\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2}$ be obtained by merging $\mathcal{C}_{\mathcal{E}_1}$ and $\mathcal{C}_{\mathcal{E}_2}$ by ct and merging $\mathcal{D}_{\mathcal{E}_1}$ and $\mathcal{D}_{\mathcal{E}_2}$ by tn respectively. By Equations 10 and 9, we have $\mathcal{C}_{\mathcal{E}_1,\mathcal{E}_2}=\{(\phi_i\vee\psi_j,tn(\alpha_i,\beta_j)):(\phi_i,\alpha_i)\in\mathcal{C}_{\mathcal{E}_1}\ and\ (\psi_j,\beta_j)\in\mathcal{C}_{\mathcal{E}_2}\}$ and $\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2}=\mathcal{D}_{\mathcal{E}_1}\cup\mathcal{D}_{\mathcal{E}_2}\cup\{(\phi_i\vee\psi_j,ct'(\alpha_i,\beta_j):(\phi_i,\alpha_i)\in\mathcal{D}_{\mathcal{E}_1}\ and\ (\psi_j,\beta_j)\in\mathcal{D}_{\mathcal{E}_2}\}\}$. So $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\},\{\Delta_{F-S-C}(\mathcal{E}_2)\})=\mathcal{C}_{\mathcal{E}_1,\mathcal{E}_2}\cup\mathcal{D}_{\mathcal{E}_1,\mathcal{E}_2}$. We now prove that for every $(\phi,\alpha)\in\Delta_{F-S-C}(\mathcal{E}_1\sqcup\mathcal{E}_2)$, $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\},\{\Delta_{F-S-C}(\mathcal{E}_2)\})\vdash_\pi(\phi,\alpha)$.

Suppose $(\phi,\alpha)\in\mathcal{D}$, then we have the following two cases:

Case 1: Suppose $(\phi,\alpha)\in\mathcal{D}_1\cup...\cup\mathcal{D}_n$, without loss of generality, we assume that $(\phi,\alpha)\in\mathcal{D}_i$ for some $i\leq m$. Since $\mathcal{D}_i\subseteq\mathcal{D}'$ and $\mathcal{D}'\subseteq\Delta_{F-S-C}(\mathcal{E}_1)$, $(\phi,\alpha)\in\Delta_{F-S-C}(\mathcal{E}_1)$. We must have $(\phi,\alpha)\in Free(\Delta_{F-S-C}(\mathcal{E}_1)\cup\Delta_{F-S-C}(\mathcal{E}_2))$. Otherwise, there exist $\mathcal{C}'_s\subseteq\mathcal{C}'$, $\mathcal{D}'_s\subseteq\mathcal{D}'$, $\mathcal{C}''_s\subseteq\mathcal{C}''$ and $\mathcal{D}''_s\subseteq\mathcal{D}''$, such that $(\phi,\alpha)\in\mathcal{D}'_s$ and $(\mathcal{C}'_s\cup\mathcal{D}'_s\cup\mathcal{D}''_s\cup\mathcal{D}''_s)^*$ is a minimal inconsistent subbase of $(\Delta_{F-S-C}(\mathcal{E}_1)\cup\Delta_{F-S-C}(\mathcal{E}_2))^*$. It follows that (ϕ,α) is in conflict in $\mathcal{B}$. So $(\phi,\alpha)\in\mathcal{C}_i$ for some i, which is a contradiction. Therefore, $(\phi,\alpha)\in\mathcal{D}_{\mathcal{E}_1}$ and so $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\},\{\Delta_{F-S-C}(\mathcal{E}_2)\})\vdash_\pi(\phi,\alpha)$.

Case 2: Suppose $(\phi,\alpha)=(D_j,1-tn'(x_1,...,x_n))$ for some $k\geq 2$. That is, $(\phi,\alpha)=(\phi_{i_1}\vee...\vee\phi_{i_k},ct'(\alpha_{i_1},...,\alpha_{i_k}))$, where $(\phi_{i_j},\alpha_{i_j})\in\mathcal{D}_{i_j}$ and $i_j\in\{1,...,n\}$. Without loss of generality, we assume that $i_1<...<i_l<m<...<i_k$. According to the discussions in Case 1, we have $(\phi_{i_1}\vee...\vee\phi_{i_l},ct'(\alpha_{i_1},...\alpha_{i_l}))\in\mathcal{D}'$ and $(\phi_{i_{l+1}}\vee...\vee\phi_{i_k},ct'(\alpha_{i_{l+1}},...,\alpha_{i_k}))\in\mathcal{D}''$. So

$(\phi_{i_1}\vee...\vee\phi_{i_k},ct'(ct'(\alpha_{i_1},...\alpha_{i_l}),ct'(\alpha_{i_{l+1}},...,\alpha_{i_k}))\in\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\},\{\Delta_{F-S-C}(\mathcal{E}_2)\})$.

Then $(\phi_{i_1}\vee...\vee\phi_{i_k},ct'(\alpha_{i_1},...,\alpha_{i_k}))\in\Delta_{F-S-C}$ $(\{\Delta_{F-S-C}(\mathcal{E}_1)\},\{\Delta_{F-S-C}(\mathcal{E}_2)\})$. Therefore, we have $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\},\{\Delta_{F-S-C}(\mathcal{E}_2)\})\vdash_\pi(\phi,\alpha)$.

Suppose $(\phi,\alpha)\in\mathcal{C}$, then $(\phi,\alpha)=(\phi_1\vee...\vee\phi_n,tn(\alpha_1,...,\alpha_n))$, where $(\phi_i,\alpha_i)\in\mathcal{C}_i$ for all i. Suppose $(\phi_i,\alpha_i)\in\mathcal{C}'_i$ for all i, then $(\phi_1\vee...\vee\phi_m,tn(\alpha_1,...,\alpha_m))\in\mathcal{C}'$ and $(\phi_{m+1}\vee...\vee\phi_n,tn(\alpha_{m+1},...,\alpha_n))\in\mathcal{C}''$. If $(\phi_1\vee...\vee\phi_m,tn(\alpha_1,...,\alpha_m))\in\mathcal{D}_{\mathcal{E}_1}$ or $(\phi_{m+1}\vee...\vee\phi_n,tn(\alpha_{m+1},...,\alpha_n))\in\mathcal{D}_{\mathcal{E}_2}$, then it is clear that $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\},\{\Delta_{F-S-C}(\mathcal{E}_2)\})\vdash_\pi(\phi,\alpha)$. Otherwise, we have $(\phi_1\vee..\vee\phi_n,tn(tn(\alpha_{m+1},...,\alpha_n),tn(\alpha_{m+1},...,\alpha_n)))\in\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\},\{\Delta_{F-S-C}(\mathcal{E}_2)\})$. So Δ_{F-S-C} $(\{\Delta_{F-S-C}(\mathcal{E}_1)\},\{\Delta_{F-S-C}(\mathcal{E}_2)\})\vdash_\pi(\phi,\alpha)$. Suppose, without loss of generality, $(\phi_i,\alpha_i)\in\mathcal{D}'_i$ for some $i<m$ and $(\phi_j,\alpha_j)\in\mathcal{C}'_i$ for all $j\neq i$. Then $(\phi_i,\alpha_i)\in\mathcal{D}'$, $(\phi_1\vee...\vee\phi_{i-1}\vee\phi_{i+1}\vee..\vee\phi_m,tn(\alpha_1,...,\alpha_{i-1},\alpha_{i+1},...,\alpha_m))\in\mathcal{C}'$ and $(\phi_{m+1}\vee...\vee\phi_n,tn(\alpha_1,...,\alpha_n))\in\mathcal{C}''$. So $\Delta_{F-S-C}(\{\Delta_{F-S-C}$ $(\mathcal{E}_1)\},\{\Delta_{F-S-C}(\mathcal{E}_2)\})\vdash_\pi(\phi,\alpha)$.

Therefore, we have proven that $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\},\{\Delta_{F-S-C}(\mathcal{E}_2)\})\vdash_\pi(\phi,\alpha)$.

(P5): Let us look at a counterexample.

Let $\mathcal{E}_1=\{\mathcal{B}_1,\mathcal{B}_2\}$ and $\mathcal{E}_2=\{\mathcal{B}_3,\mathcal{B}_4\}$, where $\mathcal{B}_1=\{(\phi,0.8),(\varphi,0.4)\}$, $\mathcal{B}_2=\{(\psi,0.4)\}$, $\mathcal{B}_3=\{(\neg\phi\vee\neg\psi,0.6)\}$ and $\mathcal{B}_4=\{(\varphi,0.6)\}$. Suppose the t-norm is the product operator and the t-conorm is the probabilistic sum. We then have

$$\begin{aligned}\Delta_{F-S-C}(\mathcal{E}_1)&=\{(\phi\vee\psi,0.88),(\phi,0.8),(\psi\vee\varphi,0.64),(\varphi,0.4),(\psi,0.4)\}\\ \Delta_{F-S-C}(\mathcal{E}_2)&=\{(\neg\phi\vee\neg\psi\vee\varphi,0.84),(\neg\phi\vee\neg\psi,0.6),(\varphi,0.6)\}.\end{aligned}$$

$\Delta_{F-S-C}(\mathcal{E}_1)$ is split $w.r.t$ $Free(\Delta_{F-S-C}(\mathcal{E}_1)\cup\Delta_{F-S-C}(\mathcal{E}_2))$ into $\mathcal{C}_{\mathcal{E}_1}=\{(\phi,0.8),(\psi,0.4)\}$ and $\mathcal{D}_{\mathcal{E}_1}=\{(\phi\vee\psi,0.88),(\psi\vee\varphi,0.64),(\varphi,0.4)\}$, and $\Delta_{F-S-C}(\mathcal{E}_2)$ is split $w.r.t$ $Free(\Delta_{F-S-C}(\mathcal{E}_1)\cup\Delta_{F-S-C}(\mathcal{E}_2))$ into $\mathcal{C}_{\mathcal{E}_2}=\{(\neg\phi\vee\neg\psi,0.6)\}$ and $\mathcal{D}_{\mathcal{E}_2}=\{(\neg\phi\vee\neg\psi\vee\varphi,0.84),(\varphi,0.6)\}$. So $(\psi\vee\varphi,0.856)\in\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\},\ \{\Delta_{F-S-C}(\mathcal{E}_2)\})$. By contrast, $\mathcal{E}_1\sqcup\mathcal{E}_2=\{\mathcal{B}_1,...,\mathcal{B}_4\}$. Let $\mathcal{B}=\mathcal{B}_1\cup...\cup\mathcal{B}_4$. $\mathcal{B}_i$ $(i=1,2,3,4)$ are split $w.r.t$ $Free(\mathcal{B})$ into $\mathcal{C}_1=\{(\phi,0.8)\}$ and $\mathcal{D}_1=\{(\varphi,0.4)\}$, $\mathcal{C}_2=\{(\psi,0.4)\}$ and $\mathcal{D}_2=\emptyset$, $\mathcal{C}_3=\{(\neg\phi\vee\neg\psi,0.6)\}$ and $\mathcal{D}_3=\emptyset$, and $\mathcal{C}_4=\emptyset$ and $\mathcal{D}_4=\{(\varphi,0.6)\}$. Combining $\mathcal{C}_i$ using the probabilistic sum we get $\mathcal{C}=\emptyset$, and combining $\mathcal{D}_i$ using the product operator we get $\mathcal{D}=\{(\varphi,0.76),(\varphi,0.6),(\varphi,0.4)\}$, which is equivalent to $\{(\varphi,0.76)\}$. So $\Delta_{F-S-C}(\mathcal{E}_1\sqcup\mathcal{E}_2)\equiv_s\{(\varphi,0.76)\}$. It is clear that $\Delta_{F-S-C}(\mathcal{E}_1\sqcup\mathcal{E}_2)\not\vdash_\pi(\psi\vee\varphi,0.856)$. Therefore, $\Delta_{F-S-C}(\mathcal{E}_1\sqcup\mathcal{E}_2)\not\vdash_\pi\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\},\{\Delta_{F-S-C}(\mathcal{E}_2)\})$.

(P6): We have the following counterexample.

Let $\mathcal{E}_1=\{\mathcal{B}_1,\mathcal{B}_2\}$ and $\mathcal{E}_2=\{\mathcal{B}_3,\mathcal{B}_4\}$, where $\mathcal{B}_1=\{(\phi,0.8),(\varphi,0.4)\}$, $\mathcal{B}_2=\{(\psi,0.4)\}$, $\mathcal{B}_3=\{(\neg\phi\wedge\neg\psi,0.6)\}$ and $\mathcal{B}_4=\{(\psi,0.6)\}$. Suppose the t-norm is the product operator and the t-conorm is the probabilistic sum. We then have $\Delta_{F-S-C}(\mathcal{E}_1)=\{(\phi\vee\psi,0.88),(\phi,0.8),(\psi\vee\varphi,0.64),(\varphi,0.4),$

$(\psi, 0.4)\}$ and $\Delta_{F-S-C}(\mathcal{E}_2) = \{(\neg\phi \vee \psi, 0.36)\}$. It is clear that $\Delta_{F-S-C}(\mathcal{E}_1) \cup \Delta_{F-S-C}(\mathcal{E}_2)$ is consistent. However, we have $\Delta_{F-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2) = \{(\varphi, 0.4)\}$. Clearly, $\Delta_{F-S-C}(\mathcal{E}_1)^* \wedge \Delta_{F-S-C}(\mathcal{E}_2)^* \not\equiv (\Delta_{F-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2))^*$. So Δ_{F-S-C} does not satisfy (P6).

Proposition 18 $\triangle_{F-S-C}$ satisfies (P7). The *G-I-S-C* operator, t-conorm based operator, and renormalization based operator do not satisfy (P7).

Proof. It is clear that $\triangle_{F-S-C}$ satisfies (P7). Clearly, the t-conorm based operator, and the renormalization based operator do not satisfy (P7). To show that the *G-I-S-C* operator does not satisfy (P7), we consider the following counterexample. Let $\mathcal{B}_1 = \{(\phi, 0.8), (\psi, 0.7)\}$ and $\mathcal{B}_2 = \{(\neg\psi, 0.6), (\varphi, 0.5)\}$. It is easy to check that $\mathcal{B}_{G-I-S-C} = \mathcal{B}_1$. $(\varphi, 0.5)$ is a free formula in $\mathcal{B}_1 \cup \mathcal{B}_2$, however, it is deleted after merging.

Proposition 19 Generating a knowledge base by the *F-S-C* method is $F\Sigma_2^p$-complete.

Proof. It has been proven in [13] that the computation of all the free formulas and conflicting formulas is Σ_2^p-complete. So the splitting of $\mathcal{B}_1 \cup \mathcal{B}_2$ is Σ_2^p-complete. The combination step can be done in polynomial time. The proposition follows immediately.

Proposition 21 Let K_1 and K_2 be two knowledge bases. Let K_{F-S-C} be the knowledge base obtained by merging K_1 and K_2 using the *F-S-C* merging operator. Then every free consequence of $K_1 \cup K_2$ can be inferred from K_{F-S-C}.

Proof. By Proposition 20, $K_{F-S-C} = K_{Free_1} \cup K_{Free_2} \cup \{\phi \vee \psi | \phi \in K_{Con_1}, \psi \in K_{Con_2}\}$. Since $K_{Free_1} \cup K_{Free_2} = Free(K_1 \cup K_2)$, we have

$$K_{F-S-C} \;=\; Free(K_1 \cup K_2) \cup \{\phi \vee \psi | \phi \in K_{Con_1}, \psi \in K_{Con_2}\}.$$

So $Free(K_1 \cup K_2) \subseteq K_{F-S-C}$. If φ is a *free consequence* of $K_1 \cup K_1$, then $Free(K_1 \cup K_2) \models \varphi$. Therefore, $K_{F-S-C} \models \varphi$.

Proposition 24 If $K_1 \cup K_2$ is not consistent, then $\triangle_{F-S-C}(K_1 \sqcup K_2) \not\models K_1$.

Proof. Suppose $\triangle_{F-S-C}(K_1 \sqcup K_2) \models K_1$, then we have $\triangle_{F-S-C}(K_1 \sqcup K_2) \models K_{Con_1}$, which is equivalent to $K_{Free_1} \cup K_{Free_2} \cup K_{Con} \models K_{Con_1}$, where $K_{Con} = \{\phi \vee \psi : \phi \in K_{Con_1}, \psi \in K_{Con_2}\}$. Since $K_{Con_2} \models K_{Con}$, $K_{Free_1} \cup K_{Free_2} \cup K_{Con_2} \models K_{Con_1}$. However, $K_{Free_1} \cup K_{Free_2} \cup K_{Con_2}$ is consistent, so $K_{Free_1} \cup K_{Free_2} \cup K_{Con_2} \cup K_{Con_1}$ is consistent, which contradicts the assumption.

Proposition 25 $\triangle_{F-S-C}(E_1) \cup \triangle_{F-S-C}(E_2) \models \triangle_{F-S-C}(E_1 \sqcup E_2)$

Proof. If $\triangle_{F-S-C}(E_1) \cup \triangle_{F-S-C}(E_2)$ is inconsistent, the conclusion clearly holds. So we assume that it is consistent. Let $K' = \cup(E_1)$, $K'' = \cup(E_2)$ and $K = \cup(E_1 \sqcup E_2)$. Suppose K', K'' and K are split as K'_{Con} and K'_{Free}, K''_{Con} and K''_{Free}, and K_{Con} and K_{Free} respectively. It is clear that $K'_{Con} \cup K''_{Con} \subseteq K_{Con}$ and $K_{Free} \subseteq K'_{Free} \cup K''_{Free}$. Suppose $E_1 = \{K_{11}, ..., K_{1k}\}$, $E_2 = \{K_{21}, ..., K_{2m}\}$, and $E = \{K_1, ..., K_n\}$, where $n = k + m$. So K_{1i} in E_1 are split into $K_{1i,Con} = K'_{Con} \cap K_{1i}$ and $K_{1i,Free} = K'_{Free} \cap K_{1i}$, K_{2j} in E_2 are split into $K_{2j,Con} = K''_{Con} \cap K_{2j}$ and $K_{2j,Free} = K''_{Free} \cap K_{2j}$, and K_k in E are split into $K_{k,Con} = K_{Con} \cap K_k$ and $K_{k,Free} = K_{Free} \cap K_k$. So $\triangle_{F-S-C}(E_1) = \vee_i(K_{1i,Con}) \cup K'_{Free}$[4], $\triangle_{F-S-C}(E_2) = \vee_j(K_{2j,Con}) \cup K''_{Free}$, and $\triangle_{F-S-C}(E_1 \sqcup E_2) = \vee_j(K_{k,Con}) \cup K_{Free}$. We now need to prove $\vee_i(K_{1i,Con}) \cup \vee_j(K_{2j,Con}) \cup K'_{Free} \cup K''_{Free} \models \vee_k(K_{k,Con}) \cup K_{Free}$. Since $K_{Free} \subseteq K'_{Free} \cup K''_{Free}$, it is clear that every formula in K_{Free} can be inferred from $\vee_i(K_{1i,Con}) \cup \vee_j(K_{2j,Con}) \cup K'_{Free} \cup K''_{Free}$. For any formula $\phi \in \vee_k(K_{k,Con})$ such that $\phi = \phi_1 \vee ... \vee \phi_n$, if there exists a ϕ_i such that $\phi_i \in K'_{Free}$ or $\phi_i \in K''_{Free}$, then it is clear ϕ can be inferred from $\vee_i(K_{1i,Con}) \cup \vee_j(K_{2j,Con}) \cup K'_{Free} \cup K''_{Free}$. Otherwise, there must exist a formula $\psi \in \vee_i(K_{1i,Con})$

[4] For simplicity, given a set of knowledge bases $\{K_i : i = 1, ..., n\}$, we use $\vee_i(K_i)$ to denote all the disjunctions among K_i, i.e, $\vee_i(K_i) = \{\phi_1 \vee ... \vee \phi_n : \phi_i \in K_i\}$.

and a formula $\psi' \in \vee_j(K_{2j,Con})$ such that $\phi = \psi \vee \psi'$. So ϕ is inferred from $\vee_i(K_{1i,Con}) \cup \vee_j(K_{2j,Con}) \cup K'_{Free} \cup K''_{Free}$. Therefore, every formula in $\vee_k(K_{k,Con}) \cup K_{Free}$ can be inferred from $\vee_i(K_{1i,Con}) \cup \vee_j(K_{2j,Con}) \cup K'_{Free} \cup K''_{Free}$.

Proposition 26 The *F-S-C* merging operator $\triangle_{F-S-C}$ does not satisfy ($A3$) and ($A6$) in general.

Proof. For ($A3$), let us consider the following counterexample:

$$E_1 = \{\{\phi, \psi\}, \{\neg\phi\}\}, \quad E_2 = \{\{\phi \wedge \psi\}, \{\neg\phi\}\}.$$

It is clear $E_1 \leftrightarrow E_2$. However, $\triangle_{F-S-C}(E_1) = \{\psi\}$, whilst $\triangle_{F-S-C}(E_2) = \{\neg\phi \vee \psi\}$. So ($A3$) does not hold.

For (A6), let us consider the following counterexample:

$$E_1 = \{\{\phi \wedge \neg\psi\}, \{\psi\}\}, \quad E_2 = \{\{\phi \rightarrow \neg\psi\}, \{\gamma\}\}.$$

We have $\Delta_{F-S-C}(E_1) = \{\phi \vee \psi\}$ and $\Delta_{F-S-C}(E_2) = \{\phi \rightarrow \neg\psi, \gamma\}$. It is clear that $\Delta_{F-S-C}(E_1)$ and $\Delta_{F-S-C}(E_2)$ are consistent. So $\Delta_{F-S-C}(E_1) \cup \Delta_{F-S-C}(E_2) = \{\phi \vee \psi, \phi \rightarrow \neg\psi, \gamma\}$. However, $E_1 \sqcup E_2 = \{\{\phi \wedge \neg\psi\}, \{\psi\}, \{\phi \rightarrow \neg\psi\}, \{\gamma\}\}$, so $\Delta_{F-S-C}(E_1 \sqcup E_2) = \{\gamma\}$. $\phi \vee \psi$ cannot be inferred from $\Delta_{F-S-C}(E_1 \sqcup E_2)$, so (A6) is not satisfied.

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