A stability result for a certain third order differential equation

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Summary. - *The main object of this paper is to give sufficient conditions for the asymptotic stability (in the large) of the trivial solution $x = 0$ of the differential equation (1.7).*

1. In the paper [1] Ogurcov considered the equation

$$\dddot{x} + \alpha\ddot{x} + \varphi(x)\dot{x} + bx = 0 \tag{1.1}$$

in which α, b are constants and $\varphi(x)$ is a continuous function of x and proved (in [1; Theorem 3]) that every solution $x(t)$ of (1.1) satisfies

$$x(t) \to 0, \quad \dot{x}(t) \to 0, \quad \ddot{x}(t) \to 0 \tag{1.2}$$

as $t \to \infty$ provided that

$$\alpha > 0, \quad b > 0, \quad \alpha\varphi(x) > b \tag{1.3}$$

and that

$$\int_0^x \{\alpha\Phi(\xi) - b\xi\}\, d\xi \to \infty \text{ as } |x| \to \infty \tag{1.4}$$

where Φ is the function defined by

$$\Phi(x) = \int_0^x \varphi(\xi) d\xi. \tag{1.5}$$

The conditions (1.3) are natural generalizations of the well known Routh-Hurwitz conditions for the stability of solutions of third order differential equations with constant coefficients. A slight consideration of the restriction on φ in (1.3) shows quite clearly that if (1.3) holds then (1.4) is necessarily true so that an explicit restriction such as (1.4) becomes redundant if (1.3) is imposed on (1.1). This would however not be the case if the condition: $\alpha\varphi(x) > b$ were to be weakened to:

$$\alpha\Phi(x)/x > b \quad (x \neq 0)$$

as the latter restriction does not in general imply (1.4). In view of the fact that Ogurcov's arguments in his proof of [1; Theorem 3] are still valid, with only trivial changes, if (1.3) is replaced by

$$\alpha > 0, \quad b > 0, \quad \alpha\Phi(x)/x > b \quad (x \neq 0) \tag{1.6.}$$

one is inclined to feel that the hypotheses which Ogurcov anticipated in the formulation of his result were (1.4) and (1.6) rather than (1.3) and (1.4.). However even this stronger result appears to be unnecessarily restrictive on Φ especially when it is recalled that, for the equation

$$\dddot{x} + \alpha\ddot{x} + g(\dot{x}) + bx = 0,$$

in which the nonlinearity g now depends on $\dot{x}$ only, a stability result such as (1.2) holds (See [3]) subject only to the ROUTH-HURWITZ conditions:

$$\alpha > 0, \quad \alpha g(y)/y > b > 0 \quad (y \neq 0).$$

The original aim of the present paper was to present a parallel result for (1.1) by showing that the result (1.2) does hold subject to the conditions (1.6) alone; but it later developed that the arguments which I had prepared for this could be extended to the much more general equation

$$\dddot{x} + \alpha\ddot{x} + \varphi(x)\dot{x} + f(x) = 0 \tag{1.7}$$

in which f, not necessarily linear, is a differentiable function of x.

The stability result which emerged, and which I shall now prove, is the following:

THOREM. Let $f(0) = 0$ *and suppose that*

(i) $\alpha > 0, \quad f(x)/x > 0 \quad (x \neq 0)$,

(ii) $\varphi(x)$, $f'(x)$ *are continuous for all* x *and there is a constant* $\delta > 0$ *such that*

$$\alpha\varphi(x)/x > \delta \geq f'(x) \quad (x \neq 0).$$

Then every solution $x = x(t)$ of (1.7) *satisfies*

(1.8) $$x \to 0, \quad \dot{x} \to 0, \quad \ddot{x} \to \text{ as } t \to \infty$$

This extends also a previous theorem (See [2]) in which the stability result (1.8) was obtained subject to the additional requirement, analogous to (1.4), that

(1.9) $$\int_0^x \{\alpha\,\Phi(\xi) - \delta\xi\}\, d\xi \to \infty \text{ as } |\,x\,| \to \infty .$$

2. The initial steps in the proof of the theorem are as in [2; § 3]. Consider the system

(2.1) $$\dot{x} = y, \quad \dot{y} = z - \alpha y, \quad \dot{z} = -\,y\varphi(x) - f(x)$$

which is derived from (1.1) by setting $\dot{x} = y$ and $\dot{y} = z - \alpha y$. To prove the theorem it will suffice to show that every solution $(x,\ y,\ z)$ of (2.1) satisfies

2.2) $$x \to 0, \quad y \to 0, \quad z \to 0$$

as $t \to \infty$.

It is probably not so obvious from the system (2.1) that the uniqueness of the solutions of (2.1), for each appropriately assigned initial conditions, which will play an important role at some stage in the proof of the theorem, is in fact implied by our continuity conditions on φ and f'. Consider, however, the system

(2.3) $$\dot{x} = y, \ \dot{y} = z - \alpha y - \Phi(x), \ \dot{z} = -\,f(x),$$

which is obtained from (1.1) on setting $\dot{x} = y$ and $\dot{y} = -\,z - \alpha y - \Phi(x)$. There is no difficulty in arriving at the uniqueness property for the solutions of (2.3) for, since the nonlinear functions $\Phi(x)$, $f(x)$ have continuous first derivatives, the terms y, $z - \alpha y - \Phi(x)$ and $f(x)$ appearing on the right hand sides of the equations (2.3) are all Lipschitzian in x, y and z. The uniqueness of the solutions of (2.3) implies necessarily the uniqueness of the original equation (1.1) which, in turn implies that of the solutions of (2.1).

For the proof of (2.2) we shall make use of the function $V = V(x, y, z)$ caps which is defined by

(2.4) $$2V = 2\alpha\ F(x) + \delta\alpha^{-1}\left\{2\int_0^x \Phi(\xi)d\xi - 2x\Phi(x) + y^2\right\} + \Phi^2 + $$
$$+ z^2 + 2yf(x) + 2\{\Phi(x) - \delta\alpha^{-1}z\}\, x ,$$

where

$$F(x) = \int_0^x f(\xi) d\xi .$$

This function will be identified readily as the Lyapunov function V of [2] except that in place of functions g, h, G and H in [2] we have here φ, f, Φ and F respectively. Because of the suppression here for the condition (1.9) the present V is not necessarily unbounded for arbitrarily large values of $x^2 + y^2 + z^2$ as in [2], but the following properties of V(Lemmas 1 and 2 to follow) will suffice for our metheds.

LEMMA 1. - *Subject to the conditions of the theorem, the function V satisfies:*

$$(2.5) \qquad \begin{aligned} V(x, y, z) &= 0, \quad \text{for } x^2 + y^2 + z^2 = 0, \\ &> 0, \quad \textit{for } x^2 + y^2 z^2 \neq 0 . \end{aligned}$$

This result is in fact a part of the lemma in [2; § 3] and it was proved there using only conditions which are identical with those in the present theorem. Further details of its proof will therefore be omitted.

Next we have

LEMMA 2. - *Let* $(x(t), y(t), z(t))$ *be any solution of* (2.1). *Then*

$$(2.6) \quad \dot{V} \equiv \frac{d}{dt} V(x) t), y(t), z(t)) = - \{ [\delta - f'(x)] y^2 + \alpha^{-1} [\alpha \Phi(x) - \delta x] f(x) \}$$

This is in fact equation (4.1) of [2]. It can however be verified directly from (2.4) and (2.1), and it is unnecessary to supply here the details of the calculations.

The calculations in [2] show that, if the additional condition (1.8) is also imposed on Φ, then every solution $(x(t), y(t), z(t))$ of (2.1) is bounded for all sufficiently large t. This boundedness result, which played an important role in [2] is of course not availlable here, with (1.9) suppressed, but the following "partial" boundedness result will be quite adequate for our present needs.

LEMMA 3. - *Subject to the conditions of the theorem, any solution* $(x(t),$ $y(t), z(t))$ *of* (2.1) *satisfying*

$$(2.7) \qquad | x(t) | > 0 \quad \textit{for all} \quad t \geq t_0$$

is necessarily bounded for all $t \geq t_0$.

PROOF. - Assume the conditions of the theorem to be fulfilled. Let $(x(t), y(t)), z(t)$ be any solution of (2.1) which satisfies (2.7). Then, since $x(t)$ is continuous in t, either $x(t) > 0$ for all $t \geq t_0$ or $x(t) > 0$ for all $t \geq t_0$. We shall show that in either case there is a constant K, $0 < K < \infty$, such that

(2.8) $$| x(t) | \leq K, \ | y(t) | \leq K, \ | z(t) | \leq K, \ t \geq t_0 .$$

We take first the case

(2.9) $$x(t) > 0, \quad t \geq t_0 .$$

Consider the function $V(t) \equiv V(x(t), y(t), z(t))$.

Since

$$\delta - f'(x) \geq 0 \text{ and } \{\alpha\Phi(x) - \delta x\} f(x) \geq 0 \text{ for all } x,$$

by the hypotheses of the theorem, it is clear from (2.6) that $\dot{V} \leq 0$ for all t; and thus

(2.10) $$V(x(t), y(t), z(t)) \leq V(x(t_0), y(t_0), z(t_0)), \ t \geq t_0 .$$

Verify now that the expression (2.4) can be rearranged in the form:

$$2V = \{z + \Phi(x) - \delta\alpha^{-1}x\}^2 + \delta\alpha^{-1} \{y + \delta\alpha^{-1} f(x)\}^2 +$$
$$+ 2\alpha\delta^{-1} \int_0^x \{\delta - f'(\xi)\} f(\xi)\, d\xi + 2\delta\alpha^{-2} \int_0^x [\alpha\Phi(\xi) - \delta\xi]\, d\xi .$$

By the hypotheses of the theorem the two integrals here are both non-negative for all x. Hence

$$2V \geq \{z + \Phi(x) - \delta^{-1}x\}^2 + \delta\alpha^{-1} \{y + \alpha\delta^{-1} f(x)\}^2$$

for all x, y and z. On combining this with (2.10) we see at once that there are finite positive constants K_1, K_2 whose magnitudes depend on α, δ and $V(x(t_0), y(t_0), z(t_0))$ only such that

(2.11) $$| z(t) + \Phi(x(t)) - \delta\alpha^{-1}x(t) | \leq K_1 \quad t \geq t_0$$

(2.12) $$| y(t) + \alpha\delta^{-1} f(x(t)) | \leq K_2 \quad t \geq t_0 .$$

We take up now the term $z(t) + \Phi(x(t))$ appearing in (2.11). Since $\dot{x} = y$, $\dot{z} = - y\varphi(x) - f(x)$ it is evident that

$$\frac{d}{dt} [z(t) + \Phi(x(t))] = - f(x(t)) < 0, \quad t \geq t_0,$$

by (2.9), since $f(x) \operatorname{sgn} x > 0 \ (x \neq 0)$. Thus

(2.13) $$z(t) + \Phi(x(t)) \leq z(t_0) + \Phi(x(t_0)), \quad t \geq t_0 .$$

For any $\tau \leq t_0$ we shall now distinguish two cases:

$$z(\tau) + \Phi(x(\tau)) \geq 0, \tag{2.14}$$

$$z(\tau) + \Phi(x(\tau)) < 0. \tag{2.15}$$

In the case (2.14), (2.13) implies that

$$0 \leq z(\tau) + \Phi(x(\tau)) \leq z(t_0) + \Phi(z(t_0))$$

so that then, by (2.11),

$$|z(\tau)| \leq K_3 \equiv \alpha\delta^{-1}\{K_1 + |z(t_0)| + |\Phi(z(t_0))|\}.$$

To deal with the case (2.15) observe first that (2.11) implies that

$$-K_1 \leq \delta\alpha^{-1}x(t) - z(t) - \Phi(x(t)) \leq K_1, \quad t \geq t_0.$$

Thus, if (2.9) and (2.15) hold then

$$0 < \delta\alpha^{-1}x(\tau) - \{z(\tau) + \Phi(x(\tau))\} \leq K_1$$

and since each of the terms $\delta\alpha^{-1}x(\tau)$, $\{-z(\tau) - \Phi(x(\tau))\}$ is positive the last inequality necessarily implies that

$$0 < \delta\alpha^{-1}x(\tau) < K_1.$$

Thus in the case of (2.15) we have that

$$0 < x(\tau) \leq \alpha\rho^{-1}K_1;$$

and hence, on comparing our estimates of $x(\tau)$ for the cases (2.14) and (2.15), we see that

$$0 < x(t) \leq K_3, \quad t \geq t_0. \tag{2.16}$$

Since $\Phi(x)$ and $f(x)$ are continuous in x the boundedness result (2.16) together with inequalities (2.11) and (2.12) at once yields the boundedness of $y(t)$ and $z(t)$ for $t \leq t_0$. Thus in the case (2.9) the result (2.8) holds.

It remains now to tackle the case when

$$x(t) < 0, \quad t \geq t_0. \tag{2.17}$$

and to prove that (2.8) also holds. The proof is quite similas to that given for the case when (2.9) holds, and I shall only sketch the outlines.

In view of (2.11) and (2.12) it is obviously enough to verify the boundedness of $x(t)$ for all $t \geq t_0$. As before consider the function $z(t) + \Phi(x(t))$, but note that in view of (2.17), we now have that

$$z(t) + \Phi(x(t)) \geq z(t_0) + \Phi(x(t_0)), \quad t \geq t_0. \tag{2.18}$$

Suppose now that, at an arbitrary $\tau \geq t_0$, we have that

$$z(\tau) + \Phi(x(\tau)) \geq 0.$$

Then, by (2.18), $|z(\tau) + \Phi(x(\tau))|$ would be bounded by $|z(t_0)| + |\Phi(x(t_0))|$, so that, by (2.11),

$$|z(\tau)| \leq \alpha\delta^{-1}\{K_1 + |z(t_0)| + |\Phi(x(t_0))|\}.$$

Suppose, on the other hand, that

$$z(\tau) + \Phi(x(\tau)) > 0, \quad \tau \geq t_0.$$

Then, after representing (2.11) in the form:

$$-K_1 \leq z(\tau) + \Phi(x(\tau)) - \delta\alpha^{-1}x(\tau) \leq K_1$$

and then noting that the terms $(z + \Phi)$ and $-\delta\alpha^{-1}x$ here have the same signs, one arrives quite readily at the result:

$$|x(\tau)| < \alpha\delta^{-1}K_1.$$

Thus

$$|x(t)| \leq K_3, \quad t \geq t_0,$$

as before, and the proof of (2.8) in the case (2.17) is now complete.

3. PROOF OF THE THEOREM. We assume hence forth that all the conditions of the theorem are fulfilled and we turn now to prove that every solution (x, y, z) of (2.1) satisfies (2.2) as $t \to \infty$.

By a theorem of PLISS [3] the result (2.2) will indeed follow if it could be verified that:

(I) *the origin* $(0, 0, 0)$ *is a unique critical point of* (2.1) *which is stable in sense of Lyapunov;* and that

(II) *there is a plane* L *in the* (x, y, z)*- space with the following three properties:*

(P_1) *any trajectory of* (2.1) *which does not intersect* L *for all* $t \geq t_0$ *tends to the origin* $(0, 0, 0)$ *as* $t \to \infty$*;*

(P_2) *there exists a function* $W = W(x, y, z)$ *such that* $W(0, 0, 0) = 0$, $W(x, y, z) > 0$ *for all* $(x, y, z) \in L$ *such that* $x^2 + y^2 + z^2 \neq 0$, $W(x, y, z) \to \infty$ *as* $x^2 + y^2 + z^2 \to \infty$ *on* L;

(P_3) *if any trajectory* $\Gamma \equiv (x(t),\ y(t),\ z(t))$ *of* (2.1) *meets* L *at two points* Q_1, Q_2 *corresponding to the values* $t = t_1, t_2$ *with* $t_1 < t_2$, *then*

$$W(x(t_1),\ y(t_1),\ z(t_1)) > W(x(t_2),\ y(t_2),\ z(t_2))$$

where W *is the function in* (P_2).

Now since $f(x)$ vanishes only at $x = 0$ it is evident that $(0, 0, 0)$ is a unique critical point of the system (2.1). By Lemma 1 the function V defined by (1.4) is positive definite in x, y, z and by Lemma 2 the time derivative of $V(x, y, z)$ along solution paths of (2.1) is non negative, since

$$\delta - f'(x) \geq 0 \quad \text{and} \quad \{\alpha\Phi(x) - \delta x\} f(x) \geq 0$$

for all x. Hence the origin is stable in the sense of Lyupunov and the condition (I) is thereby verified.

To verify (II) we shall show that the plane $x = 0$ does have all the properties (P_i) ($i = 1, 2, 3$) if, for W, we take the function V defined by (2.4). Indeed let $(x(t),\ y(t),\ z(t))$ be any solution of (2.1) not intersecting the plane $x = 0$ for all $t \geq t_0$. Then by Lemma 3 this solution is bounded for all $t \geq t_0$. Hence by Theorem VIII [4, p. 66] and by Lemmas 1 and 2 this solution tends, as $t \to \infty$, to the largest invariant set, M say, contained in the locus $\dot{V} = 0$. But, by (2.6) this locus necessarily lies on the plane $x = 0$, since

$$\{\alpha\Phi(x) - \delta x\} f(x) > 0 \ (x \neq 0). \tag{3.1}$$

Thus, since the only trajectory of (2.1) lying in the plane $x = 0$ is the origin, we must have that $M \equiv (0, 0, 0)$ and the property (P_1) then follows.

The property (P_2) is almost immediate. For, by Lemma 1, $V(0, 0, 0) = 0$ and from the definition (2.4),

$$V(0,\ y)\ z) = \delta\alpha^{-1} y^2 + x^2$$

and this is strictly positive for $y^2 + z^2 \neq 0$ and tends to $+\infty$ as $y^2 + z^2 \to \infty$.

To come now to (P_3) let $(x(t),\ y(t),\ z(t))$ be any non trivial solution of (2.1) and let Γ denote the trajectory traced out by this solution in the $(x,\ y,\ z)$-space. Suppose that Γ denote meets the plane $x = 0$ at $t = t_1$ and $t = t_2$, $t_2 > t_1$. We have seen from (2.6) that, under our hypotheses on α, Φ and f,

$$V \leq -\alpha^{-1} [\alpha\Phi(x) - \delta x] f(x) \leq 0 \tag{3.2}$$

along Γ. Now, since $x(t_1) = 0 = x(t_2)$ with $t_2 > t_1$ it is evident that there is an interval $[\tau_1, \tau_2]$, with $t_1 < \tau_1 < \tau_2 < t_2$, such that

$$x(t) \neq 0 \quad \tau_1 \leq t \leq \tau_2 ; \tag{3.3}$$

for otherwise $x(t) = 0$, $t_1 \leq t \leq t_2$, and thus there is a nonempty subinterval of $[t_1, t_2]$ where $x \equiv 0 \equiv \dot{x} \equiv \ddot{x}$ and by uniqueness of solutions of (2.1) this would in turn imply that $(x(t), y(t), z(t)) \equiv (0, 0, 0)$ contrary to our choice of $(x(t), y(t), z(t))$. Hence (3.3) holds, and by (3.1) and (3.2) this leads to result:

$$V(x(t_1), y(t_1), z(t_1)) > V(x(t_2), y(t_2), z(t_2)),$$

thereby verifying (P_3). This concludes the proof of (2.2) and the theorem now follows.

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